Tr. J. of Physics
23 (1999) , 357 – 377.

International Workshop:
"Magnetic Activity in Cool Stars"
(ed. O. Demircan)

Photometric Imaging of Starspots, Techniques and Reliability

Zeki EKER
King Saud University, College of Science
Department of Astronomy, Riyadh, SAUDI ARABIA

Received 15 September 1998

Abstract

Historical development of the starspot hypothesis and the unsolved problems associated with it are presented. The "non–uniqueness" attributed to the spot solutions is a major problem which could still discredit current efforts of starspot modeling by photometric or spectroscopic data. The basics of the starspot photometry (direct and inverse photometric problem, and the error analysis of the inverse problem) are presented. Since an analytical formulation exists to compute synthetic light curves from the physical parameters of spots, and recomputing original parameters analytically from the synthetic curves are also possible, the starspot hypothesis is a consistent physical problem. According to the error analysis presented, that is, according to the propagated uncertainties from the synthetic curves to the recomputed parameters, most of the problems of the current modeling techniques originate from the insufficient accuracy of the observed data.

1. Introduction

The first variable stars discovered are novas and supernovas. Relaying on Chinese records, variable star history can go back as much as 2000 years [1]. The first variable star which was not nova or supernova, Omicron Ceti=Mira, was discovered by Johannes Goldsmid = David Fabricius in 1596. About seventy years later (1667), Ismael Boulliau established a period of the variability of Omicron Ceti as 333 days and proposed the very first physical model to explain with a model one hemisphere of the star is darker than the other. Eclipsing variability, ALGOL was discovered in 1669 by Geminiaro Montanari. However, during these early centuries (17th, 18th, and 19th) starspots were invoked to explain the variability of almost all variable stars. Spots were used to explain even novas and supernovas. Apparently, those models were inspired by sunspots because a live example of sunspots were known since Galileo(1564-1642).

However, the uniqueness problem in the starspot hypothesis has been noticed and spots lost popularity by the turn of the century. Basically, there were two reasons for this. 1) radial velocity variations on the spectra of β Lyra (eclipsing) and δ Cep (cepheid) were discovered and recorded by Belepolsky (1893). Both systems are considered eclipsing. But, an alternative explanation with a pulsation mechanism was also available. Actually, binary-pulsation debate continued until Shapley (1914)[2]. 2) Theoretical arguments against starspot hypothesis have been published [3] independently by two authors Bruns (1882) and Russell (1906) which are a) starspots cannot explain every kind of variability; b) any shape of light curve, unless there is no discontinuity, can be explained by arbitrary spots. Consequently, starspot hypothesis was abandoned until 1950.

The starspots were rejected in this period because any light curve shape could be explained by eclipses, radial or non-radial pulsations, obscuring by circumstellar material or by the combination of similar physical events. Along with the fact that no spotted star had been found in this period, according to Hall [1], the uniqueness problem, nevertheless, probably contributed to the starspot's fall from grace. Similarly, Vogt [4] thinks that the uniqueness problem is one of the reasons why the starspot hypothesis has been slow to achieve widespread acceptance. After being abandoned half a century, starspots were used first time by Kron [5, 6, 7] to explain the out of eclipse variations of the light curves of AR Lac, RS CVn, RT And and YY Gem. However, works of Kron were ignored for another 20 years.

First quantitative formulations of photometric imaging were presented in early seventies [8, 9, 10, 11]. Those models were using a single spot with a uniform temperature. However, a single spot is not sufficient to explain observed asymmetry in the light curves, so that, two-spot models became standard [12, 13, 14, 15, 16]. During revival of starspot hypothesis at this time, starspots were included in the models because other kinds of mechanisms (pulsation, eclipse, absorption by circumstellar material etc.) failed to explain some irregular light curves.

Kopal (1982) was not happy with this development. He was arguing "Starspot hypothesis is too simple, unphysical to explain light variability" and insisting "Unless an alternative check on their existence and distribution independent of the light curves, observed light curves cannot produce unique information"[17]. But, Kopal's voice did not receive proper attention. Development of models with spots continued not only in photometry but also in spectroscopy. Using line profiles of neutral metals, the Doppler imaging technique [18] was introduced by Vogt and Penrod (1983) and maximum entropy [19] method applied into Doppler imaging by Vogt, Penrod and Hadzes (1987). Saar and Neff (1990) suggested a technique to observe TiO bands for searching starspots [20]. Recently, Neff, O'Neal and Saar (1995) succeded in determining a unique spot temperature from the TiO spectra of II Peg [21].

In the classical field of photometry, new techniques like ILOT (Information Limit Optimization technique, [22]) and models with the least squares [23, 24, 25] are introduced. Obviously, the most of the astronomers today do not deny existence of starspots because numerous models with spots are already published. Despite the fact that many starspot models were published without serious objections, these publications are not free from several basic problems which could still discredit the starspot hypothesis. This presentation aims to study unsolved problems of starspot hypothesis within the scope of the basics of the starspot photometry.

2. Unsolved problems

The unsolved problems of the starspot hypothesis could be itemized as following:

- Non–uniqueness of spot solutions are frequently announced [1, 4, 12, 17, 19, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38]. Many modellers from every field (photometric and spectroscopic) and every technique apparently confirms the non-uniqueness of spot solutions.
- Indeterminacy of spot latitudes usually complained in the photometric solutions [10, 12, 14, 15, 22, 23, 27, 33, 36, 37, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53].
- Because (the item above) a spot latitude is a notoriously hard parameter to ascertain, many modellers preferred to use fixed latitudes [10, 15, 23, 39, 41, 50, 52, 53]. Latitude fixing occurs in photometric models only.
- Unphysical modeling [54, 55, 56, 57, 58, 59, 60, 61, 62, 63]. Hall's group uses down half of the sine curves to construct a synthetic light curve to fit the observed data rather than using a synthetic light curve which is computed from real physical parameters of a spot such as size, shape, location and temperature.
- The distribution of spots on stellar surfaces according to published models are not realistic [64, 65]. Almost all of the techniques appeared in the literature produce distributions at which spots are concentrated on the hemisphere with the visible pole. Therefore, most published models violate solar analogy as well as the physics associated with the phenomena because there is no physical reason to break distribution symmetry between the hemispheres which are divided by the equator. It may be possible to explain why published models show a different distribution characteristics than sunspots. But, explaining the cause of a problem is not a solution.

Except the last item above preceded three items are directly related to the first problem "non-uniqueness". This is because, if a latitude is fixed since it is a notoriously hard parameter to attain, the fixed latitude imply non-uniqueness since a fixed latitude is a free choice among the all or a limited range of different latitude values which could all be possible solutions. On the other hand, the Hall's group prefers non-physical modeling because non-uniqueness of spot solutions are generally believed , thus, why bother to compute synthetic light curves from the physical parameters. This would be meaningless anyhow.

The uniqueness problem, therefore, is the most important source of negative credits about the starspot hypothesis. The problem is serious because if infinite number of solutions with various distributions exist as many authors claim, there cannot be a true solution or a solution which can be close to the true solution within the error limits. Thus, the problem of uniqueness is directly related to believability of the models and to the trust towards the modeling efforts. Therefore, it can be said that starspot models does not have a positive credit except to produce synthetic curves within the observed uncertainties.

The problem of uniqueness, however, is described differently in the published literature. This fact is clear within the following comments:

> "Within the assumption that starspot groups are single and circular in outline, this approach in fact produce unique results by covering the entire range of possible combinations of relevant variables. However, solutions are still nonunique to the extent that other spot shapes (e.g. elongated) or multiple spot solutions may well exist." [4]
>
> "Casting the problem in matrix form, IR=D, where I and D are image and data vectors and R is the transfer matrix between the two, the solution for the image vector is given by $I = DR^{-1}$. In practice, this cannot be done. This is because, R to be invertable, not only must the matrix be square, but also rows of R must be independent.", "Even if R is invertable, the reconstructed image always inherit an uncertainty, therefore, it will not be unique." [19].

Thus, the word "uniqueness" does not have a unique meaning among the many published models. Therefore, one needs to analyze the uniqueness problem first. This study, however, does not intend to clarify this problem now, but at least aims to answer following questions in order to show scientific consistancy of starspot hypothesis within the basics of starspot photometry.

- Does a spot with a specific size, temperature, shape and location have a unique signature on the light curves?
- Is it possible to recover physical spot chracteristics (parameters) from light curves?

3. Basics of starspot photometry

The basics of starspot photometry would be a direct answer to the questions above. However, the problem can be examined in two steps; 1) direct photometric problem, 2) inverse photometric problem.

Direct photometric problem is defined as the problem of computing maculation wave function, that is the synthetic light curves, from the real physical parameters of spots like sizes, shapes, locations and temperatures. The direct photometric problem, however, cannot provide a direct solution for the spot parameters. One needs to do many trials until producing a curve fitting the observations within their uncertainties. Trial and error methods, however, cannot assure uniqueness of the solutions unless all combinations of the relevant parameters are tried. On the other hand, the problem appears ill posed because the number of discrete spots are unknown. Note that the number of minima in the light curves only indicate a minimum number not the true number.

The inverse photometric problem is defined as the problem of deducing real spot parameters from the light curves directly without trials. Although, the true number of spots are unknown, by the help of the inverse photometric problem for a single spot, the consistency of the starspot hypothesis can be proven.

3.1. Direct photometric problem

A light loss due to cool surface spots in the magnitude scale can be expressed as :

$$\Delta m_\lambda(\phi) = -2.5 \log \frac{F_s + F_p}{F} \tag{1}$$

where F_s and F_p are the fluxes received from the spotted regions and the rest of the photosphere respectively. F is the reference flux representing a spotfree disk. $F_s + F_p$ varies as the star rotates because projected areas of the spots on the disk vary. Thus $F_s + F_p$ is phase dependent. Equation (1) could be used for any number of spots with any shape. Then

$$F_s = \sum_{n=1}^{n} \int_{\omega_n} I_s(\theta)\, d\omega \tag{2}$$

where n represents the number of visible spots. ω_n is the surface area of the spot n. The surface intensity of the spots are symbolized by I_s. The differential solid angle is $d\omega$ and the foreshortening angle is θ. The flux which is received from the rest of the photosphere (F_p) , then, can be computed by

$$F_p = F - \sum_{n=1}^{n} \int_{\omega_n} I_p(\theta)\, d\omega \tag{3}$$

where imaginary photospheric flux over spotted regions is subtracted from the flux of a spotfree disk. For the sake of simplicity, let us assume a single circular spot. Then, equation (2) becomes

$$F_s = \frac{R^2}{d^2} \int_0^{2\pi} \int_0^{r} I_s(\theta) \cos\theta \sin r dr d\varphi \tag{4}$$

where $I_s(\theta)\cos\theta$ represents component of surface intensity in the direction to the line of sight. $\sin r dr d\varphi$ is the differential solid angle at which r stands for an angular distance of the differential surface element from the center of the spot. Thus, if one locates the spot at the center of the disk (r changes to θ) and extends the integral limits to cover whole disk, by replacing spot intensity $I_s(\theta)$ with the photospheric intensity $I_p(\theta)$, the spotfree flux can be computed

$$F = \frac{R^2}{d^2}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} I_p(\theta)\cos\theta\sin\theta d\theta d\varphi \tag{5}$$

By adding (2) and (3) for the single spot case,

$$F_s + F_p = F + \frac{R^2}{d^2}\int_0^{2\pi}\int_0^{r}[I_s(\theta) - I_p(\theta)]\cos\theta\sin r dr d\varphi \tag{6}$$

the total flux from the spot and the rest of the photosphere can be computed. If the term in the square parentheses in the integral is replaced by

$$I_s(\theta) - I_p(\theta) = (\frac{I_s(\theta)}{I_p(\theta)} - 1)I_p(\theta) = (\alpha - 1)I_p(\theta) \tag{7}$$

where α, spot to photosphere intensity ratio, which is the temperature parameter, could be defined as following

$$\alpha_\lambda = \frac{I_s(\theta)}{I_p(\theta)} = \frac{B_\lambda(T_s)}{B_\lambda(T_p)} = \frac{e^{\frac{hc}{\lambda k T_p}} - 1}{e^{\frac{hc}{\lambda k T_s}} - 1} \tag{8}$$

which shows the temperature and wavelength dependence of α clearly.

Spot to photosphere intensity ration (α) is expected to vary over the surface of the spot as if at the sunspots. However, an average value of α may represent an average spot temperature, thus, α can be assumed constant over a spot. But, the spot location with respect to the center of the disk is changing due to the rotation. One has to consider variations of α at each location. Nevertheless, according to the observations of sunspots, there is no clear θ dependence of α. Thus, at least for the first approximation α can be taken as a constant that is independent of the θ. This allows α to come out of the integral. Then the equation (6) becomes

$$F_s + F_p = F + \frac{R^2}{d^2}(\alpha - 1)\int_0^{2\pi}\int_0^{r} I_p(\theta)\cos\theta\sin r dr d\varphi \tag{9}$$

At the final step of computing the right hand side of equation (1), one needs to know the limb darkening law (photospheric intensity distribution over the surface of the disk). Unfortunately, the exact form of the limb darkening law is unknown. Commonly used linear and quadratic forms are just approximations from the direction dependence of emerging flux which is computed from model atmospheres. For different practices, therefore, the solutions of equation (5) and (9) will be presented in three different limb darkening cases. First, the solutions of equation (5) according to:

1. case n (no limb darkening case): $I_p(\theta) = I_0$

$$F = \frac{\pi R^2}{d^2} I_0 \tag{10}$$

2. case ℓ (linear limb darkening case) : $I_p(\theta) = I_0[1 - U(1 - \cos\theta)]$

$$F = \frac{\pi R^2}{d^2} I_0 (1 - \frac{U}{3}) \tag{11}$$

3. case q (quadratic limb darkening case) : $I_p(\theta) = I_0[1 - U_1(1 - \cos\theta) + U_2(1 - \cos\theta)^2]$

$$F = \frac{\pi R^2}{d^2} I_0 (1 - \frac{U_1}{3} + \frac{U_2}{6}) \tag{12}$$

where I_0 is emerging intensity from the photosphere into the radial direction. U is linear, and U_1 and U_2 are quadratic limb darkening coefficients. Substituting proper F and proper limb darkening into equation (9), the logarithmic term of equation (1) can be expressed in the same fashion:

For the case n (no limb darkening):

$$\frac{F_s + F_p}{F} = 1 + \frac{\alpha - 1}{\pi} I_n \tag{13}$$

For the case ℓ (linear limb darkening) :

$$\frac{F_s + F_p}{F} = 1 + \frac{\alpha - 1}{\pi(1 - \frac{U}{3})} [(1 - U) I_n + U I_\ell] \tag{14}$$

For the case q (quadratic limb darkening) :

$$\frac{F_s + F_p}{F} = 1 + \frac{\alpha - 1}{\pi(1 - \frac{U_1}{3} + \frac{U_2}{6})} [(1 - U_1 + U_2) I_n + (U_1 - 2U_2) I_\ell + U_2 I_q] \tag{15}$$

where

$$I_n = \int_0^{2\pi} \int_0^r \cos\theta \sin r dr d\varphi \tag{16}$$

$$I_\ell = \int_0^{2\pi} \int_0^r \cos^2\theta \sin r dr d\varphi \tag{17}$$

$$I_q = \int_0^{2\pi} \int_0^r \cos^3\theta \sin r dr d\varphi \tag{18}$$

Notice that the high level cases are reducible to the low level cases. For example if U_2 is zero, the quadratic case is not different than the linear case. Similarly, if limb darkening coefficients are zero ($U_2 = U_1 = 0$ and $U = 0$), computing the right hand side of equation (1) for the quadratic or linear cases reduces to the no limb darkening case. Therefore, if an higher order polynomial would satisfy surface intensity distribution on the disk, the present formulation can easily be adopted.

Finally, expressing θ in terms of r and φ.

$$\cos\theta = \cos\theta_0 \cos r + \sin\theta_0 \sin r \cos\varphi \tag{19}$$

where θ_0 represents position of the spot center with respect to the disk center; r is the angular distance of the differential surface element from the spot center; φ is the azimuth angle; the following solutions for I_n, I_ℓ and I_q can be obtained after integration.

$$I_n = \pi \sin^2 r \cos\theta_0 \tag{20}$$

$$I_\ell = \frac{2\pi}{3}(1 - \cos^3 r) - \pi \cos r \sin^2 r \sin^2\theta_0 \tag{21}$$

$$I_q = \frac{\pi}{2}(1 - \cos^4 r)\cos^3\theta_0 + \frac{3\pi}{4}\sin^4 r \cos\theta_0 \sin^2\theta_0 \tag{22}$$

where r represents the angular radius of the circular spot. However, one should notice that these integrals are true under the condition $\theta_0 \leq (\pi/2 - r)$. On the other hand, they must attain a value zero if $\theta_0 \geq (\pi/2 + r)$ which is the case the spot is totally hidden behind the star. Computing above integrals at the crossover phases (partial visibility at the edge of the disk) needs a special treatment. Solutions for partial visibility can be found in the appendix of the article by Eker[66].

Using θ_0 and r, one can compute Δm which is a single value corresponding to a phase. To produce a synthetic light curve, however, θ_0 of each spot must be computed at all phases of a full rotation. Consequently, cordinates of the spots (centers) in a corotating frame (λ, β) and the inclination of the rotation axis (i) also enter in the computation as spot parameters. Thus, for computing θ_0

$$\cos\theta_0 = \cos i \sin\beta + \sin i \cos\beta \cos(\phi - \lambda) \tag{23}$$

where λ and β are longitude and latitude of a spot center and ϕ represent rotational phases, $\phi = \frac{2\pi}{P}(t - t_0)$, where P is the rotational period and t_0 is an arbitrary time to start phases. Note: $\beta = 0$ locates the spot on the equator and $i = 0$ means axis of rotation is parallel to the line of sight (pole view).

3.2. Inverse photometric problem

According to Eker, the difference between the synthetic curves, which are computed by the linear and the quadratic cases from the tabulated limb darkening coefficients, decreases towards the longer wavelengths and higher Teff values [66]. Eker found that the linear law is sufficient at the V band for the effective temperature range 4000 to 6020 K. That is, using the linear law or quadratic law do not make a noticeable difference on the synthetic curves at the bands from visual to infrared. Therefore, the most of the published starspot models are satisfied with the linear limb darkening law. Consequently, equation (14) expressing the right hand side of the equation (1) will be used in the inverse problem studied in this study. Eliminating $\frac{F_s+F_p}{F}$ between (1) and (14), following can be written

$$\frac{10^{-\frac{\Delta m_\lambda(\phi)}{2.5}}-1}{\alpha_\lambda-1}=\frac{1}{\pi(1-\frac{U_\lambda}{3})}[(1-U_\lambda)I_n(\phi)+U_\lambda I_\ell(\phi)] \tag{24}$$

Theoretically, this equation is true for every band (U, B, V, R, I etc.). However, it has been argued that it is more suitable for the bands at visual or longer wavelengths. Let us assume we use two bands V and R. Then, if it is for the V band,

$$\frac{10^{-\frac{\Delta m_V(\phi)}{2.5}}-1}{\alpha_V-1}=\frac{1}{\pi(1-\frac{U_V}{3})}[(1-U_V)I_n(\phi)+U_V I_\ell(\phi)] \tag{25}$$

and for the R band,

$$\frac{10^{-\frac{\Delta m_R(\phi)}{2.5}}-1}{\alpha_R-1}=\frac{1}{\pi(1-\frac{U_R}{3})}[(1-U_R)I_n(\phi)+U_R I_\ell(\phi)] \tag{26}$$

The equations (25) and (26) can be combined into a matrix format

$$\left|\begin{array}{c} B_V(\phi) \\ B_R(\phi) \end{array}\right| = \left|\begin{array}{cc} a & b \\ c & d \end{array}\right| \left|\begin{array}{c} I_n(\phi) \\ I_\ell(\phi) \end{array}\right| \tag{27}$$

that is, B = T I, where the entries of matrix B are

$$B_V(\phi)=\frac{10^{-\frac{\Delta m_V(\phi)}{2.5}}-1}{\alpha_V-1} \quad , \quad B_R(\phi)=\frac{10^{-\frac{\Delta m_R(\phi)}{2.5}}-1}{\alpha_R-1}$$

and the entries of the matrix T are

$$a=\frac{1-U_V}{\pi(1-\frac{U_V}{3})} \quad , \quad b=\frac{U_V}{\pi(1-\frac{U_V}{3})}$$
$$c=\frac{1-U_R}{\pi(1-\frac{U_R}{3})} \quad , \quad d=\frac{U_R}{\pi(1-\frac{U_R}{3})}$$

The transformation matrix T is invertable; then $I=T^{-1}$ B which is

$$\left|\begin{array}{c} I_n(\phi) \\ I_\ell(\phi) \end{array}\right| = \frac{1}{ad-bc}\left|\begin{array}{cc} d & -b \\ -c & a \end{array}\right| \left|\begin{array}{c} B_V(\phi) \\ B_R(\phi) \end{array}\right| \tag{28}$$

in the open form.

The equation (28) decouples the geometric parameters from the wavelength dependent temperature effects. To deduce phase dependent geometric functions $I_n(\phi)$ and $I_\ell(\phi)$ from the two light curves, one needs a priory knowledge of limb darkening coefficients, temperatures of the spot and photosphere only. The unspotted magnitude is the brightness level which the light variations are referenced. The unspotted magnitude is known to be an ambiguous parameter observationally. However, for the synthetic curves, especially for a single spot, it is a known parameter. So that, with four parameters (U_V, U_R, T_p and T_s), the light curves at the two color bands can be transformed to the two functions $I_n(\phi)$ and $I_\ell(\phi)$ by the equation (28). Notice that, as if equation (1) in the direct problem, equation (28) is valid for any number of spots with any shape. However, to proceed we assume applying it to the synthetic light curves of a single circular spot with a uniform temperature.

The first parameters to recover from the light curves is the longitude of the spot. The maximum effect of a spot occurs when it is on the central meridian. Thus, determining the minimum phase is equivalent to determining the longitude of the spot. For most of the studies, either the zero phase or zero longitude is arbitrary. Therefore, let $\phi_{min} = 0$, as if the phases start at the light minimum. Then

$$\frac{[I_n(0)]^2 - [I_n(\phi)]^2}{\pi[I_\ell(0) - I_\ell(\phi)]} = \frac{\sin^2 r}{\cos r} = X \tag{29}$$

helps to compute the size of the spot.

Equation above requires that the transformation matrix (28) must be applied at two phases; zero and any other arbitrary phase ϕ. However, one has to be careful not to choose a phase at which the spot is partially visible or hidden. See Eker [65] how to avoid this problem.

Since the value of X is known, from equation (29), it can be written

$$\cos^2 r + X \cos r - 1 = 0 \tag{30}$$

which gives a solution for the size,

$$\cos r = -\frac{X}{2} + \frac{1}{2}\sqrt{X^2 + 4} \tag{31}$$

Notice that, a quadratic equation gives two solutions, but one of the solutions is discarded since it implies a spot which is bigger than the visible hemisphere with $\cos r < 0$. The value of X is positive. That is, $0 < X < \infty$ so (31) is valid to give a unique value within $0 < r < \pi/2$.

Another advantage of present formulation is that the inclination of rotation axis (i) is not required to perform the transformation from the light curves to the functions $I_n(\phi)$ and $I_\ell(\phi)$. The i can be recovered together with β from the light curves. Because $\cos\theta_0 = \sin(i + \beta)$ at the minimum phase, from (20),

$$\sin(i + \beta) = \frac{I_n(0)}{\pi \sin^2 r} \tag{32}$$

can be written to compute $(i + \beta)$. Moreover, the ratio $I_n(0)/I_n(\phi)$ supplies following relation between i and β.

$$\tan i = C \ \tan\beta \tag{33}$$

where

$$C = \frac{I_n(0) - I_n(\phi)}{I_n(\phi) - I_n(0)\cos\theta}$$

which is also a known quantity.

The two equations and the two unknowns, that is, (32) and (33) provide following solutions for i and β.

$$\tan i = \frac{\mp(1-\cos\phi)\sqrt{(\pi\sin^2 r)^2 - I_n^2(0)} + \sqrt{(1-\cos\phi)^2(\pi\sin^2 r)^2 - [I_n(0)(1+\cos\phi) - 2I_n(\phi)]^2}}{2\left[I_n(\phi) - I_n(0)\cos\theta\right]} \tag{34}$$

$$\tan\beta = \frac{\mp(1-\cos\phi)\sqrt{(\pi\sin^2 r)^2 - I_n^2(0)} + \sqrt{(1-\cos\phi)^2(\pi\sin^2 r)^2 - [I_n(0)(1+\cos\phi) - 2I_n(\phi)]^2}}{2\left[I_n(0) - I_n(\phi)\right]} \tag{35}$$

Since there can be two values of (i and β) within $0 < (i+\beta) < \pi$ to satisfy equation (32), there must be two solutions. The two solutions for i and β are clear because the numerators in (34) and (35) are double valued. These two solutions are called pairs by Eker [65,66]. The pairs are not independent. That is, if one of the pairs is known, the second pair can be computed by the pairing rules [65,66]. Suppose (i_1, β_1) and (i_2, β_2), where i_1 and i_2 from (34) and β_1 and β_2 from (35), represent pairs. According to Eker [65] the pairing rules are :

1) if $A = i + \beta$, $A_1 = A_2$ or $\pi - A_1 = A_2$

2) $i_1 + |\beta_2| = \frac{\pi}{2}$ and $i_2 + |\beta_1| = \frac{\pi}{2}$

3) Latitude in both pairs must carry same sign. That is if $\beta_1 < 0$ then $\beta_2 < 0$

Since the projected area is maximum when the spot is on the central meridian, $I_n(0) > I_n(\phi)$ and $I_\ell(0) > I_\ell(\phi)$ in all cases. If $I_n(\phi) > I_n(0)\cos\phi$ the equations (34) and (35) give two pairs as $i_1 > 0$, $i_2 > 0$ and $\beta_1 > 0$; $\beta_2 > 0$. But one has to be careful for the following special cases:

Case 1) if $I_n(\phi) < I_n(0)\cos\phi$

This case happen when the latitude of the spot is negative. In this case, equations (34) and (35) provide one of the pair with opposite sign. That is, $i < 0$ and $\beta > 0$. Therefore, one has to reverse the signs of i and β for the pair with the negative inclination. Always remember that i_1 and i_2 must be positive and the signs of β_1 and β_2 must be same both pairs.

Case 2) if $I_n(0) = \pi\sin^2 r$

This case provide a unique solution since the first square root terms in (34) and (35) becomes zero. In this case the numerators are no more double valued.

Case 3) if $I_n(\phi) = I_n(0)\cos\phi$

This case gives triple solution (3 pairs). The pairs are $(i, 0)$, $(\pi/2, \beta)$ and $(\pi/2, -\beta)$ where i and β can be computed by

$$\sin i = \frac{I_n(0)}{\pi \sin^2 r} \quad , \; and \qquad \cos(\mp\beta) = \frac{I_n(0)}{\pi \sin^2 r} \tag{36}$$

Case 4) if $I_n(0) = \pi \sin^2 r$, and $I_n(\phi) = I_n(0)\cos\phi$

This case is a combination of the cases 2 and 3. Therefore, it implies a unique solution with a definite pair which is $(\pi/2, 0)$ where $i = \pi/2$ and $\beta = 0$.

Mathematical non–unique solutions of i and β are caused by spherical symmetry of a circular spot on a spherical star. That is, the spot may produce same projected areas as the star rotates only at two combination of i and β given by the pairing rules. If i and β are chosen freely to produce a light curve, as long as the other spot parameters (size, longitude and temperature), stays the same, $i_2 = \frac{\pi}{2} - |\beta|$, and $\beta_2 = \frac{\pi}{2} - i$ can also produce same light curve if the spot shape is circular. However, if one uses two or more circular spots or any number of spots (even one) with a shape other than a circle, the synthetic light curves must be unique because the spherical symmetry works only with a single circular spot. See the detailed discussion in Eker[65].

Although there is no special formula to compute it, the spot temperature is another parameter which is supplied by the two light curves. The formulas (29), (34) and (35) which are given to compute r, i and β appear to have ϕ (phase) dependence. However, anyone would agree those parameters must be independent of ϕ. But the truth is that those parameters become independent of ϕ only at the correct spot temperature. Therefore, using different values of ϕ at the equations (29), (34) and (35), various r, i and β values could be determined for each value of ϕ. Then, repeating this process for various T_s values in the range $0 < T_s < T_p$, the correct T_s can be found when all $r(\phi)$, $i(\phi)$ and $\beta(\phi)$ becomes independent of ϕ. Eker [65] gives two methods to compute the spot temperature.

3.3. Error Analysis

Most spot solutions today are obtained by trial and error [25] methods. More advanced techniques like the least squares [23, 24] and ILOT [22] also exist. Some curve fitting techniques like the least squares, the maximum entropy and the ILOT may support the standard error estimates of derived parameters. But, those uncertainties are not the true uncertainties to relate the recovered image to the image of virtually existing spots. Those uncertainties do not indicate more than the effect of scatter of the data, or an estimation of how well the geometrical model constrain the related parameters. Studying only the case of two-spot solutions, Kovari and Bartus [67] investigated the effect of photometric accuracy for determining spot parameters. Model dependence of their solutions are clear. So that, they were only able to test the reliability of two-spot modeling technique.

Although solutions for multiple spots are not yet available by the inverse method, the formulas presented for a single spot allows one to propagate observed uncertainties up to the derived parameters. Error propagation even for a single spot would be a direct way to examine the reliability of the reconstructed images as well as the reliability of the light curves supplying them.

3.3.1. Uncertainty of the projected spot areas

The transformation matrix (equation 28) can be written in a different form to show the input parameters openly.

$$I_n(\phi) = \frac{\pi U_R(1-\frac{U_V}{3})(10^{-\frac{\Delta m_V(\phi)}{2.5}}-1)}{(U_R-U_V)(\alpha_V-1)} - \frac{\pi U_V(1-\frac{U_R}{3})(10^{-\frac{\Delta m_R(\phi)}{2.5}}-1)}{(U_R-U_V)(\alpha_R-1)} \quad (37)$$

$$I_\ell(\phi) = -\frac{\pi(1-U_R)(1-\frac{U_V}{3})(10^{-\frac{\Delta m_V(\phi)}{2.5}}-1)}{(U_R-U_V)(\alpha_V-1)} + \frac{\pi(1-U_V)(1-\frac{U_R}{3})(10^{-\frac{\Delta m_R(\phi)}{2.5}}-1)}{(U_R-U_V)(\alpha_R-1)} \quad (38)$$

where $I_n(\phi)$ is the projection areas of the spots on the disk and $I_\ell(\phi)$ is the auxliary function which exist since liear limb darkening assumed.

The precision and accuracy of these two functions can be determined by the accuracy of the input parameters (U_V, U_R, δm_V, δm_R, α_V and α_R). Except α_V and α_R, the uncertainty of the other parameters can be considered independent and random. The uncertainty of an α involves effective wavelength of the observation bands, photospheric and spot temperatures by the equation (8). For now let us assume T_p and effective wavelengths are known accurately, that is, with a negligible error. Thus, in the first approximation, the uncertainties of α_V and α_R can be ignored because varying T_s in the range $0 < T_s < T_p$, it is possible to hit a value having negligible uncertainty. Consequently, four independent uncertainties are left to propagate. Then, the uncertainties of above functions can be computed as

$$\delta I_n(\phi) = \sqrt{(\frac{\partial I_n(\phi)}{\partial U_V}\delta U_V)^2 + (\frac{\partial I_n(\phi)}{\partial U_R}\delta U_R)^2 + (\frac{\partial I_n(\phi)}{\partial m_V}\delta m_V)^2 + (\frac{\partial I_n(\phi)}{\partial m_R}\delta m_R)^2} \quad (39)$$

$$\delta I_\ell(\phi) = \sqrt{(\frac{\partial I_\ell(\phi)}{\partial U_V}\delta U_V)^2 + (\frac{\partial I_\ell(\phi)}{\partial U_R}\delta U_R)^2 + (\frac{\partial I_\ell(\phi)}{\partial m_V}\delta m_V)^2 + (\frac{\partial I_\ell(\phi)}{\partial m_R}\delta m_R)^2} \quad (40)$$

where (37) and (38) permit partial derivatives which is needed in above relations. Notice that up to here no assumptions has been made about the surface spots. Therefore, the equation (39) gives the uncertainty of projected areas of virtually existing spots. In order to continue to compute uncertainty of the size, let us assume or use the light curves at two color bands of a single circular spot.

3.3.2. Uncertainty of the size

The equation (31) allows the uncertainty of the spot size can be computed by

$$\delta r = \left|\frac{dr}{dX}\right|\delta X \quad (41)$$

where

$$\frac{dr}{dX} = \sqrt{\frac{\cos r}{X(X^2+4)}} \tag{42}$$

On the other hand, the uncertainty of X in (41) comes from

$$\delta X = \left|\frac{\partial X}{\partial I_n(0)}\right| \delta I_n(0) + \left|\frac{\partial X}{\partial I_n(\phi)}\right| \delta I_n(\phi) + \left|\frac{\partial X}{\partial I_\ell(0)}\right| \delta I_\ell(0) + \left|\frac{\partial X}{\partial I_\ell(\phi)}\right| \delta I_\ell(\phi) \tag{43}$$

The partial derivatives can be taken from (29) and the uncertainties of I_n and I_ℓ at the phases zero and ϕ can be computed by (39) and (40). Since the input uncertainties for computing δX are not independent and random anymore, the propagated uncertainty by (43) represent an upper limit. Since (41) involves (43), the uncertainty of the size is also an upper limit.

3.3.3. Uncertainity of $(i + \beta)$

Using partial derivatives from (34) and (35) and involved uncertainties δr, $\delta I_n(0)$ and $\delta I_n(\phi)$, original uncertainties $[\delta U_V, \delta U_R, \delta m_V(\phi)$ and $\delta m_R(\phi)]$ can be propagated to compute the uncertainty of i and β in a similar way. Eker [68] describes two methods to compute them. But, for the sake of brevity, only the uncertainty of $(i + \beta)$ will be described here. Defining $A = i + \beta$, the equation (32) becomes

$$\sin A = \frac{I_n(0)}{\pi \sin^2 r} \tag{44}$$

Consequently, for the relative error of sin A, it can be written

$$\frac{\delta \sin A}{\sin A} = \frac{\delta I_n(0)}{I_n(0)} + 2\,\frac{\delta \sin r}{\sin r} \tag{45}$$

The relative error of $\sin r$ can be expressed in terms of the relative error of $\cos r$. If $y^2 + z^2 = 1$ ($y = \sin r$, $z = \cos r$), then $\frac{dy}{y} = -\frac{z^2}{y^2}\frac{dz}{z}$, so that $\frac{\delta \sin r}{\sin r} = -\frac{\cos^2 r}{\sin^2 r}\frac{\delta \cos r}{\cos r}$. Using this in (45), it becomes

$$\frac{\delta \sin A}{\sin A} = \frac{\delta I_n(0)}{I_n(0)} + 2\,\frac{1}{\tan^2 r}\,\frac{\delta \cos r}{\cos r} \tag{46}$$

Finally the relative error of $\cos r$ can be expressed in terms of X. From (31), $\frac{d \cos r}{\cos r} = \frac{-dx}{\sqrt{X^2+4}}$. Substituting this into (46)

$$\frac{\delta \sin A}{\sin A} = \frac{\delta I_n(0)}{I_n(0)} + \frac{2\ \delta X}{\tan^2 r \sqrt{X^2+4}} \tag{47}$$

is a straight forward equation to compute relative error of $sinA$ by the quantities formerly calculated. Then, absolute error of A

$$\delta A = (\tan A)M \tag{48}$$

where $M = \frac{\delta \sin A}{\sin A}$ represents the relative error of $\sin A$ which is the right hand side of equation (47).

The uncertainty of A has a special importance because it represents roughly the lower limits of the uncertainties of i and β. For example if i is known, then $\beta = A - i$ (from the definition $A = i + \beta$). Consequently the uncertainty of the latitude

$$\delta\beta = \delta A + \delta i \tag{49}$$

If the uncertainty of i is zero, the uncertainty of β must be equal to the uncertainty of A. Otherwise, the uncertainty of β becomes bigger than δA.

3.3.4. Uncertainty of the spot temperature

If equation (24) is solved for α_λ

$$\alpha_\lambda = \frac{\pi(1 - \frac{U_\lambda}{3})(10^{-\frac{\Delta m_\lambda}{2.5}} - 1)}{(1 - U_\lambda)I_n + U_\lambda I_\ell} + 1 \tag{50}$$

then, the uncertainty of each α in two bands can be computed by

$$\delta\alpha_\lambda = \left|\frac{\partial\alpha_\lambda}{\partial U_\lambda}\right|\delta U_\lambda + \left|\frac{\partial\alpha_\lambda}{\partial m_\lambda}\right|\delta m_\lambda + \left|\frac{\partial\alpha_\lambda}{\partial I_n}\right|\delta I_n + \left|\frac{\partial\alpha_\lambda}{\partial I_\ell}\right|\delta I_\ell \tag{51}$$

then, using the definition $\alpha_\lambda = \frac{B_\lambda(T_s)}{B_\lambda(T_p)}$, the relative uncertainty of the spot temperature is

$$\frac{\delta T_s}{T_s} = \frac{\lambda k T_s}{hc}(1 - e^{-\frac{hc}{\lambda k T_s}})\frac{\delta\alpha_\lambda}{\alpha_\lambda} \tag{52}$$

3.3.5. Uncertainty of longitude

The longitude of a spot is an independent parameter. Its value and uncertainty can be deduced directly from the measurements of the minimum phase on the light curves. Therefore, no special formula are presented for them.

4. Applications

Using the analytical formulas of the inverse photometric problem, Eker [65] recovered the original model parameters from pre-generated synthetic light curves of nine models with various spot configurations. Table 1 displays the system parameters and Table 2 gives the model parameters of those nine models.

Table 3 gives the input parameters used by Eker [65] which are significant up to six decimal digits. The solution parameters including the projected area $I_n(\phi)$ and A are

Table 1. System Parameters

Band	Wavelength (Å)	U	T_p (K)
V	5500	0.75	4820[a]
R	7000	0.61	...

[a] Single circular spot with uniform temperature

Table 2. Model Parameters: $T_p - T_s = 1300$ K (all models); longitude (λ): π (all models)

	Models								
parameters	1	2	3	4	5	6	7	8	9
Inclination (i)(deg.)	90	90	75	75	75	60	60	60	60
Latitude (β) (deg.)	18	0	0	-8.5	30	15	31	81	-30
Size (r) (deg.)	19.09	19.09	23.65	23.65	20	20	21	21	21

Table 3. Input Parameters

	Models							
parameters	1	2	3	4	5&6	7	8	9
$\Delta m_V(0)$ (mag.)	0.127851	0.139971	0.202129	0.183956	0.144345	0.169581	0.076659	0.053656
$\Delta m_R(0)$ (mag.)	0.110888	0.120449	0.174463	0.160124	0.124837	0.145773	0.070662	0.050993
Phase (ϕ)	45	45	45	45	45	45	45	30
$\Delta m_V(\phi)$ (mag.)	0.070545	0.076641	0.111011	0.099247	0.086860	0.109417	0.069122	0.038807
$\Delta m_R(\phi)$ (mag.)	0.064471	0.069529	0.101015	0.091182	0.078549	0.097755	0.064289	0.037846

Table 4. Solution Parameters(recovered $\mp$ uncertainty)

parameters	Models							
$\mp$uncertainty	1	2	3	4	5&6	7	8	9
$I_n(0)$ ($R_\star^2$)	0.31959	0.33603	0.48833	0.46362	0.35498	0.40347	0.25392	0.20174
$\mp\delta I_n(0)$	$\mp$0.000017	$\mp$0.000017	$\mp$0.000016	$\mp$0.000016	$\mp$0.000015	$\mp$0.000016	$\mp$0.000018	$\mp$0.000018
Size (r) (deg)	19.089	19.089	23.651	23.649	20.002	21.001	21.014	21.003
$\mp\delta r$	$\mp$0.0045	$\mp$0.0042	$\mp$0.0034	$\mp$0.0036	$\mp$0.0048	$\mp$0.005	$\mp$0.032	$\mp$0.014
A($i + \beta$) (deg)	72.013	90	74.99	66.506	74.957	89.328	38.944	29.993
$\mp\delta A$	$\mp$0.0897	$\mp$0.027	$\mp$0.0658	$\mp$0.0422	$\mp$0.1083	$\mp$2.4036	$\mp$0.1376	$\mp$0.0457
i_1 (deg)	72	90	75	74.99	59.98	59.67	8.99	59.84
$\mp\delta i_1$	$\mp$0.146	$\mp$0.027	$\mp$0.1085	$\mp$0.1994	$\mp$0.0914	$\mp$1.227	$\mp$0.0541	$\mp$13.95
i_2 (deg)	89.99	....	89.99	81.51	75.03	60.33	60.05	60.15
$\mp\delta i_2$	$\mp$0.564		$\mp$0.0428	$\mp$0.157	$\mp$0.0549	$\mp$1.196	$\mp$0.1146	$\mp$13.91
β_1 (deg)	0.01	0	-0.01	-8.49	14.97	29.67	29.95	-29.85
$\mp\delta\beta_1$	$\mp$0.0564	$\mp$0.027	$\mp$0.0428	$\mp$0.157	$\mp$0.0549	$\mp$1.196	$\mp$0.1146	$\mp$13.91
β_2 (deg)	18	....	-15	-15.01	30.02	30.33	81.01	-30.16
$\mp\delta\beta_2$	$\mp$0.146		$\mp$0.1085	$\mp$0.1994	$\mp$0.0914	$\mp$1.227	$\mp$0.0541	$\mp$13.95
$\mp\delta\alpha_V$	$\mp$0.00003	$\mp$0.000027	$\mp$0.000018	$\mp$0.00002	$\mp$0.000026	$\mp$0.000022	$\mp$0.00005	$\mp$0.000073
$\mp\delta T_s$ (Kelvin)	$\mp$0.104	$\mp$0.0949	$\mp$0.064	$\mp$0.0708	$\mp$0.0918	$\mp$0.0774	$\mp$0.177	$\mp$0.226
$\mp M$ (rad.)	$\mp$0.00059	$\mp$0.000475	$\mp$0.000308	$\mp$0.000321	$\mp$0.000508	$\mp$0.000492	$\mp$0.0029	$\mp$0.00138
$\mp M$ (deg.)	$\mp$0.029	$\mp$0.027	$\mp$0.017	$\mp$0.018	$\mp$0.029	$\mp$0.028	$\mp$0.17	$\mp$0.079

Table 5. Uncetainty of the solution parameters if $\delta m_V = \delta m_R = 0.005$ mag., $\delta U_V = \delta R_R = 0.005$ mag.

	Models							
∓uncertainty	1	2	3	4	5&6	7	8	9
$\mp\delta I_n(0)$	0.0853	0.0846	0.0804	0.0814	0.0842	0.0826	0.0888	0.0905
$\mp\delta r$(deg)	22.57	21.07	17.27	17.92	24.04	24.83	159.7	71.07
$\mp\delta A$(deg)	448.96	136.20	239.96	211.45	541.95	12034	688.22	228.53
$\mp\delta i_1$(deg)	731	136.2	543	998	457	6147	270	69768
$\mp\delta i_2$(deg)	282		214	786	274	5989	573	69536
$\mp\delta\beta_1$(deg)	282	136.2	214	786	274	5989	573	69536
$\mp\delta\beta_2$(deg)	731		543	998	457	6147	270	69768
$\mp\delta\alpha_V$	0.1543	0.1417	0.0972	0.1060	0.1368	0.1168	0.2523	0.3674
$\mp\delta T_s$ (K)	544	500	343	344	482	412	890	1295
$\mp M$ (rad.)	2.544	2.377	1.541	1.604	2.542	2.463	14.863	6.910
$\mp M$ (deg.)	145.76	136.20	88.32	91.91	145.65	141.10	851.58	395.92

presented in the order of derivation in Table 4. The uncertainties, which are recorded below the recovered parameters in Table 4, are computed by assuming that the input parameters (Table 3) are significant to six decimal digits in the magnitude scale ($\delta m_V \approx \delta m_R \approx 10^{-6}$ mag.). The limb darkening coefficients were assumed to have no error. The uncertainty of α_V and the corresponding uncertainty of T_s, and the relative error of $\sin A$ which is defined as M are also appended to the Table 4.

It is clear from the Table 4 that the parameters are accurately predicted. Comparing original and recovered parameters, it is also clear that the propagated uncertainties are in the correct order. Unfortunately, the real light curves are not that accurate. In order to imitate realistic uncertainties, that is, assuming $\delta m_V \approx \delta m_R \approx 0.005$ mag and $\delta U_V \approx \delta U_R \approx 0.005$, the propagated errors are recomputed and recorded in Table 5.

According to Table 5, the system inclination (i), the spot latitude (β) and $A = i + \beta$ would not be reliable at all since their uncertainties are too big. Even the spot sizes are uncertain as much as themselves. The uncertainty of the projected areas are the smallest but relatively speaking they are about 20 to 25% and even larger for the models 8 and 9.

Considering the fact that those uncertainties are upper limits, the actual errors would be smaller. But still, the most accurate light curves of today's technology do not appear good for photometric imaging except maybe in determining projected areas and spot temperatures. Table 5 suggests about ∓500 K uncertainty for the recovered spot temperature which is not good but acceptable. The latitude (β) is the most uncertain parameter. Even if the M values are considered in Table 5, that is, even if i is known precisely, the uncertainty of β would be greater than the allowed limits of β which are $(0 < \beta < 90)$.

5. Conclusions

1. Photometric imaging of starspots is a consistent physical problem. A spot with specific parameters must have unique effects on the light curves. Oherwise it would never be possible to recompute spot parameters analytically from the light curves.

2. However, accuracy of the current earth-based photometric data (observations) is insufficient for a successful photometric imaging. Among the parameters, the biggest uncertainty is associated with the spot latitudes. Thus, today's most reliable earth based observations with $\delta m_\lambda \approx 0.005$ mag accuracy is not sufficiently accurate to locate even for a single spot on the latitude scale.

$$\delta\beta > 90\,\mathrm{deg}$$

3. Problems such as indeterminacy of spot latitudes, latitude fixing and related non-uniqueness ambiguities could now be atributed to the insufficient accuracy of the data.

4. Polar spots which is not observed on the sun but claimed by various models could be a problem also associated with the uncertainty of the data.

5. Uncertainty of the limb darkening coefficients is negligible beside the uncertainty of the brightness measurements. That is, uncertainty of the brightness dominates.

6. $\delta m_\lambda \approx 0.0001$ mag or better accuracy is needed for a reliable photometric imaging.

References

[1] D. S. Hall, *I.A.P.P.P.,Comn.,* **no: 54** (1994) 1.

[2] H. M. Shapley, *ApJ,***40** (1914) 448.

[3] H. N. Russell, *ApJ,***24** (1906) 1.

[4] S. S. Vogt, *ApJ,***250** (1981) 327.

[5] G. E. Kron, *PASP,***59** (1947) 261.

[6] G. E. Kron, *AJ,***55** (1950) 69.

[7] G. E. Kron, *ApJ,***115** (1952) 301.

[8] D. S. Hall, *PASP,***84** (1972) 323.

[9] B. W. Bopp, D. S. Evans, *MNRAS,***164** (1973) 343.

[10] C. A. O. Torrez, S. Ferraz Mello, *A&A,***27** (1973) 231.

[11] E. Budding, *Ap&SS,***48** (1977) 207.

[12] B. W. Bopp, P. V. Noah, *PASP,***92** (1980) 717.

[13] J. D. Dorren, M. J. Siah, E. F. Guinan, G. P. McCook, *AJ,***86** (1981) 592.

[14] C. H. Poe, J. A. Eaton, *ApJ,***289** (1985) 644.

[15] J. D. Dorren, *ApJ,***320** (1987) 756.

[16] J. Lodenquai, J. McTavish, *AJ,***96** (1988) 741.

[17] Z. Kopal, *Ap&SS,***87** (1982) 149.

[18] S. S. Vogt, G. D. Penrod, *PASP,***95** (1983) 565.

[19] S. S. Vogt, G. D. Penrod, A. P. Hatzes, *ApJ,***321** (1987) 496.

[20] S. H. Saar, J. E. Neff, Cool Stars, Stellar Systems, and the Sun, ed. G. Wallerstein, ASP Conf. Ser. Vol. 9, (San Fransisko:ASP), 13.

[21] J. E. Neff, D. O'Neal, S. H. Saar, *ApJ,***452** (1995) 879.

[22] E. Budding, M. Zeilik, *ApJ,***319** (1987) 827.

[23] Y. W. Kang, R.E. Wilson, *ApJ,***97** (1989) 848.

[24] R. G. Samec, W. Su, D. Terrell, D. P. Hube, *AJ,***106** (1993) 318.

[25] R. E. Wilson, *PASP,***106** (1994) 921.

[26] D. P. Huenemoerder, L. W. Ramsey, *ApJ,***319** (1987) 392.

[27] J. Neff, Active Close Binaries, ed. C. Ibanoglu (Dordrecht: Kluwer), 809.

[28] C. V. S. R. Sarma, P. Vivekananda Rao, M. B. K. Sarma, *J. Astrophys. Astron.,***12** (1991) 49.

[29] G. J. Anders, D. W. Coates, K. Thompson, *Proc. Astron. Soc. Australia,***10** (1992) 33.

[30] R. C. Dempsey, B. W. Bopp, K. G. Strassmeier, A. F. Granados, G. W. Henry, D. S. Hall, *ApJ,***392** (1992) 187.

[31] D. P. Huenemoerder, L. W. Ramsey, D. L. Buzasi, H. L. Nations, *ApJ,***404** (1993) 316.

[32] C. Maceroni, F. van't Veer, *A&A,***277** (1993) 515.

[33] L. Patkos, A. Hempelmann, *A&A,***292** (1994) 119.

[34] N. M. Elias, A. Quirrenbach, A. Witzel, C. E. Naundorf, R. Wegner, E. F. Guinan, G. P. McCook, *ApJ,***439** (1995) 983.

[35] P. B. Byrne, P. M. Panagi, A. C. Lanzafame, S. Avgoloupis, D. P. Huenemoerder, D. Kilkeny, F. Marang, K. P. Panov, G. Roberts, J. H. Seriadakis, F. van Wyk, *A&A,***299** (1995) 115.

[36] A. V. Raveendran, S. Mohin, *A&A,***301** (1995) 788.

[37] A. Hempelmann, A. P. Hatzes, M. Kurster, L. Patkos, *A&A,***317** (1997) 125.

[38] K. Olah, Zs. Kovari, J. Bartus, K. G. Strassmeier, D. S. Hall, G. W. Henry, *A&A,***321** (1997) 811.

[39] J. R. Stauffer, J. D. Dorren, J. L. Africano, *AJ,***91** (1986) 1443.

[40] E. Budding, M. Zeilik, *Ap&SS,***232** (1995) 355.

[41] M. Zeilik, D. A. Cox, C. De Blasi, M. Rhodes, E. Budding, *ApJ,***345** (1989) 991.

[42] M. Zeilik, M. Ledlow, M. Rhodes, M. J. Arevalo, E. Budding, *ApJ,***354** (1990) 352.

[43] M. Zeilik, S. Gordon, E. Jaderlund, M. Ledlow, D. L. Summers, P. A. Heckert, E. Budding, T. S. Banks, *ApJ,***421** (1994) 303.

[44] K. G. Strassmeier, *ApJ,***348** (1990) 682.

[45] T. Banks, E. Budding, *Ap&SS,***167** (1990) 221.

[46] T. Banks, *Proc. Astron. Soc. Australia,***9** (1991) 148.

[47] T. Banks, P. M. Kilmartin, E. Budding, *Ap&SS,***183** (1991) 309.

[48] K. G. Strassmeier, K. Olah, *A&A,***259** (1992) 595.

[49] G. J. Anders, D. W. Coates, K. Thompson, *Proc. Astron. Soc. Australia,***10** (1992) 33.

[50] J. A. Eaton, G. W. Henry, C. Bell, A. Okorogu, *AJ,***106** (1993) 1181.

[51] K. Olah, E. Budding, H. I. Kim, P. B. Etzel, *A&A,***291** (1994) 110.

[52] P. A. Heckert, J. I. Ordway, *AJ,***109** (1990) 2169.

[53] D. L. Summers, P. A. Heckert, *AJ,***113** (1997) 2242.

[54] D. S. Hall, G. W. Henry, J. R. Sowell, *AJ,***99** (1990) 396.

[55] D. S. Hall, S. E. Gessner, H. C. Lines, R. D. Lines, *AJ,***100** (1990) 2017.

[56] D. S. Hall, F. C. Fekel, G. W. Henry, W. S. Barksdale, *AJ,***102** (1991) 1808.

[57] D. S. Hall, F. C. Fekel, G. W. Henry, J. A. Eaton, W. S. Barksdale, V. Dadonas, Z. Eker, P. Kalv, C. R. Chambliss, R. E. Fried, G. L. Fortier, H. J. Landis, H. P. Louth, H. D. Powell, T. G. McFaul, R. Miles, P. Nielsen, T. R. Renner, S. P. Robb, D. M. Slauson, H. J. Stelzer, R. Wasson, J. E. Wood, *AJ,***109** (1995) 1277.

[58] D. S. Hall, G. W. Henry, *AJ,***104** (1992) 1936.

[59] F. C. Fekel, J. C. Browning, G. W. Henry, M. D. Morton, D. S. Hall, *AJ,***105** (1993) 2265.

[60] G. W. Henry, J. A. Eaton, J. Hemer, D. S. Hall, *ApJ Supl.,***97** (1995) 513.

[61] L. J. Crews, D. S. Hall, G. W. Henry, R. D. Lines, H. C. Lines, R. E. Fried, *AJ,***109** (1995) 1346.

[62] J. A. Eaton, G. W. Henry, F. C. Fekel, *ApJ,***462** (1996) 888.

[63] A. B. Kaye, D. S. Hall, L. R. Patterson, H. D. Nations, P. A. Heckert, D. L. Summers, *AJ,***111** (1996) 1322.

[64] Z. Eker, *ApJ,***445** (1995) 526.

[65] Z. Eker, *ApJ,***473** (1996) 388.

[66] Z. Eker, *ApJ,***420** (1994) 373.

[67] Zs. Kovari, J. Bartus *A&A,***323** (1997) 801.

[68] Z. Eker, Accepted for publication in *ApJ,*(1998).