Tr. J. of Physics
22 (1998) , 83 – 106.

Rotation in Stars

Halil KIRBIYIK
Middle East Technical University,
Physics Department, 06531, Ankara - TURKEY

Received 18.06.1996

Abstract

In this study the subject of rotation in stars and its effects have been considered. First the historical development of the subject is reviewed. After discussing the observational aspects of rotation, the effect of rotation on evolution has been investigated. The velocity of rotational meridional circulation currents and related time scales were estimated. Mixing induced by circulation currents and the turbulence which may appear as a result of such a mixing have been studied. Order of magnitude calculations have shown that rotational mixing may occur in early type stars and thus contributes to the diffusion of chemical elements. Thus, it is argued that such a rotational mixing can explain the abnormal abundances observed in the atmospheres of OBN stars. In addition, it has been concluded that the same argument may be extended to explain the observed abundances in the atmosphere of Supernova 1987A.

1. Introduction

Studies about stellar rotation begin with Johannes Goldschmidth's observations of Sunspots with a refracting telescope in 1611. In the years following Galileo Galilei sunspot observations were also done by various investigators in England and Germany.

Galilei found that a spot traverses the solar disk in 14 days. German priest C. Scheiner, showed as a result of his observations, that the apparent rotation period of the Sun was 27 days. He observationally found that rotation period had a latitudinal dependence, and velocity decreased closer to the polar regions.

In the 1800's amateur astronomers Richard Carrington and German astronomer Gustav Sporer observationally showed that the Sun did not rotate uniformly, but its rotation period was minimum in the equatorial region, and it gradually increased toward the polar region.

In 1842 the Austrian physicist Christian Doppler discovered the so called "Doppler effect". Johann Zöllner developed a new spectroscope in 1871 and thought that the rotation of the Sun could be observed with it. Walter S. Adams and George E. Hale from the Mount Wilson Solar Observatory were the first to achieve this.

In 1877, it was suggested that axial rotation caused broadening in some absorption lines [1]. Herman Vogel did not accept this at first but in 1898 he changed his mind and supported the idea that stellar rotation caused broadening in absorption lines.

Axial rotation in stars was first discovered in 1909. Frank Schlesinger did this discovery by observing δ Librae. After one year, he observed the same effect in λ Tauri. Richard Rositer observed the similar phenomena in β Lyrae. These were binary stars.

The subject of axial rotation in single stars was systematically studied between 1930 and 1934 by O. Struve, Christian T.Elvey and Miss. Christine Westgate. They found that O, B, A and F-spectral type stars were fast rotating [2,3]. The subject was ignored until 1949, when the subject became popular with Arne Slettebak [4]. Following him, the subject of stellar rotation was always in astronomers' agenda and has been used to explain many observational findings.

Involvement of rotation in fluids was considered early on by such investigators as I.Newton, C.Huygens, R.Decartes, A.C.Clairaut, C.Maclaurin, M.P.S.Laplace, A.M.Legendre, K.Jacobi, H.Poincare and M.Liapunov. However, fundamental studies on rotating stars were done in the first half of the twentieth century, with considerable work done in the last few decades [25-26,29-30,38-40,50,73-75].

At the beginning of the 20 th century it was known that stars had a centrally condensed gaseous structures. E.A.Milne [5] constructed a star model in radiative equilibrium in 1923. S.Chandrasekhar did similar study for polytropes in 1933.

In 1920's the most important work on stellar rotation was done by Hugo von Zeipel(6). In 1924, for a chemically homogeneous, uniformly rotating star in radiative equilibrium he found that the nuclear energy generation rate at any point inside the star was given as

$$\in_n = \text{cons} \cdot (1 - \Omega^2/2\pi G\rho)$$

Here, G=gravitation constant and ρ is the density at the corresponding point.

It is clear that $\in_n$ becomes infinite at the surface of the star. This is not realistic. Heinreich Vogt [7] and A.E.Eddington [8], in 1924, starting with this fact showed that a uniformly rotating star could not totally be in radiative equilibrium. It was suggested that temprature and pressure could not be constant on equipotential surfaces and thus meridional circulation currents should result. The first important work related to this subject was done by Eddington [9] in 1929.

2. Observations and its Implications

Stellar rotation is detected by measuring Doppler broadening in absorption lines. Turbulence also causes broadening, but broadening due to axial rotation can be distinguished. We can observationally measure only $V \sin i$ in single rotating stars. Here V is rotation velocity and i is the angle between the rotation axis and the line of sight (Figure-1). It is not possible to say something about the sense of rotation axis. For that matter, one needs statistical information.The following can be said about the sense of rotation axis:

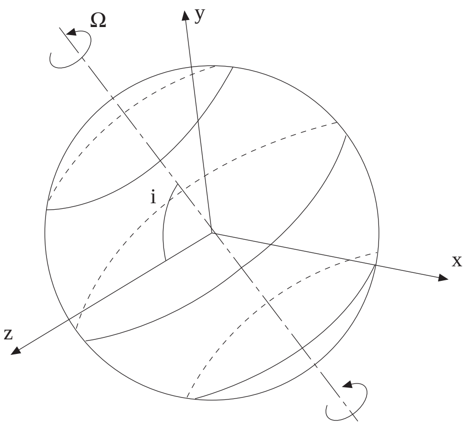

Figure 1.

i) Stars remember the angular momentum vector,
ii) Rotation axes are distributed randomly in space.
Let us consider the first case:
The rotation axis of the Galaxy is perpendicular to the plane of the Galaxy. Let us consider the rotation of a gas nebula with radius R_c. The angular momentum per unit mass is

$$h \cong (V_2 - V_1)R_c/2$$

Here, $V_2 - V_1$ is the difference between linear velocities corresponding to the closest and the farhest points with respect to the centre of the Galaxy. From the differential rotation of the Galaxy it is known that $V \cong 10$km sec^{-1} kpc^{-1}. Since $V_2 - V_1 \cong 10(2R_c)$ km sec^{-1} kpc^{-1}, one gets $h \cong 10R_c^2$. This gives the initial angular momentum of the gas nebula. If a star is formed from such a nebula then $h_f \cong V_{\rm rot}R_f$, with "f" standing for final, and by substituting the value of h given above one gets $V_{\rm rot} = 10R_c^2/R_*$ km sec^{-1} kpc^{-1}. Typically, $R_* \cong 10^{11}$ cm$\cong 10^{-7}$ pc $= 10^{-10}$ kpc, and by taking $R_c = 1$ pc we obtain $V_{\rm rot} \cong 10^5$ km sec^{-1}, which is comparable to the speed of light. This is not possible.

On the other hand, when observations are considered it is seen that $V \sin i$ values do not correlate with the galactic coordinates. We expect V sin i correlated with the galactic latidues, but are not. The first hypothesis implies that the axis of a binary system is also perpendicular to the galactic plane. However, planes of visual binaries are randomly distributed. Thus, because of velocity calculations done above and the observational evidences mentioned, the first case can be excluded. The second case is more likely. For this, we need statistical information.

We notice the accumulation of data especially after 1949 [10,11,12,13]. Utilizing these data the variation of the mean of $V \sin i$ with respect to the spectral type for single stars is shown in Figure 2 [14]. In Figure 2, the curve shaded with short lines show the position of the mainsequence stars [15], while the curve shaded with longer lines show the position of stars of luminosity class III and IV [10]. As seen, rotation becomes faster toward the early type stars. This fact was already noticed in 1930's. There is a distinct difference between the fast rotators and slow rotators. From Figure 2, It is clear that the early type giants rotate more slowly compared to the mainsequence stars of the same spectral type, but it is the opposite in late A and F type stars. This result may be related to evolution. F-type giants are evolved B and A-type mainsequence stars. Naturally they rotate faster than the slow rotating mainsequence stars. Note that the change in the graph occurs around F0-spectral type stars. This may be related to the envelopes of late type stars. The presence of the envelope may lead to stellar wind and thus causing a loss in angular momentum.

As it has already ben discussed in the above paragraph that there is a sharp division between fast and slow rotating stars. Late type stars rotate slowly. The reason may be the presence of convective envelopes. Convective envelope means we are likely to have stellar wind. It is now accepted that the sun as well as stars of different types lose mass in varying proportions and such a loss causes a loss in angular momentum which in turn affects the evolution of rotating stars [76-82]. Rotation only enhances the wind mass loss. Another process which contributes to the rotational slow-down is magnetic breaking. As it is known, stars have magnetic field. Stars with convective envelopes have solar like flares on their surfaces. With flares, charged particles are ejected and they interact with the magnetic field. As a consequence of such a mechanism, a corotating particle cloud is formed around the star. Those which can move far enough from the surface of the star will leave the system. Thus a certain amount of angular momentum is removed from the star. This is known as magnetic breaking. This is thought to be another mechanism slowing down the rotation of stars [40,79,83,84,85].

Olson [16] did observations on the rotation of close binary stars. Most of the star systems that he observed were eclipsing binaries and thus the inclination of the orbit was known. Assuming that the rotation axis was perpendicular to the plane of the orbit the real equatorial velocity V_ε was found. We see the plotting of the equatorial velocity of close binary stars against spectral type in Figure 3 [16]. In the same Figure the graph of the mean velocity of single stars on the mainsequence is also given. The rotation velocities of the components , in close binary systems, when compared with single stars of the same spectral type are low. This is clearly seen in the graph. Hence it can be concluded that the effect of the component is such that it slows down the star. This was first noted by Kreiken[86] .

3. Rotation and Evolution

As we have noticed in part II, early type stars rotate faster than the late type stars. It is easy to observe them because they require high dispersion. Such properties make early type stars interesting.

The effect of rotation on stellar structure has been known for sometime. Most of the old studies have been summarized in Jean's book, "Astronomy and Cosmogony" [17]. Investigations on the stability and structure of uniformly rotating fluid masses have continued and has become the subject of present day studies [18,19]. Milne (5) built the first detailed model for slowly rotating star in pure radiative equilibrium. Chandrasekhar and Lebovitz [87] discussed the bifurcation along sequences of uniformly rotating fluid masses. It must be noted that the basic concepts which are the fundamentals of the contemporary researches on rotating stars were developed in the first half of the twentieth century. However it will be noticed in the following paragraphs or chapters of this work that the theory of rotating stars has greatly advanced in the second half of the twentieth century, especially in the last few decades.

The effect of rotation on stellar structure and evolution was studied by many scientists in great detail in the second half of the 20 th century. Nevertheless, the reason for the subject's becoming popular is the supernova explosion in Large Magellanic cloud(LMC) in 1987. It is known that there were a B-type supergiant, known as Sanduleak-69202, in the place of SN 1987A. There has been some difficulties in explaining some observations related to 1987A. Thus, rotation has been included in an attempt to explain these observations.

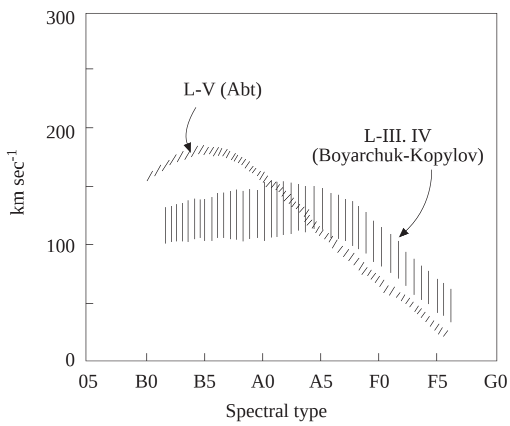

Figure 2. The graph of the mean velocities versus spectral type for single stars

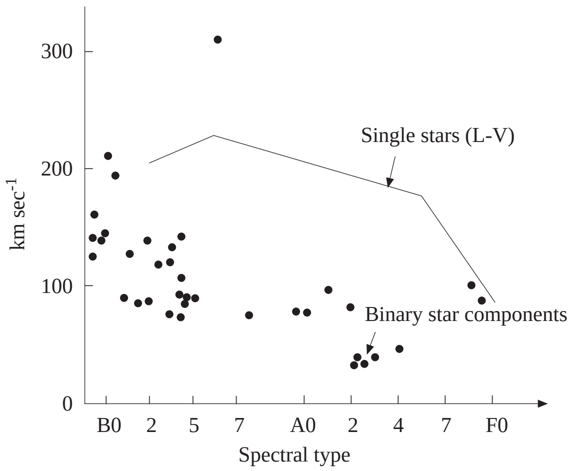

Figure 3. The distribution of close binary stars (dots).The mean velocity graph for the mainsequence single stars(continuous line)

From IUE observations of supernova 1987A, high values for N/C and N/O ratios in the slowly expanding circumstellar matter, have been found [20]. Besides, the ratio of He/H has also been found to be high. Observations imply that there was an unexpected chemical structure in the envelope of the star before explosion took place.

The above mentioned observations shoul be explained and the following questions should be answered.

i) The explosion occurred in the blue region of the HR diagram. This contradicts the standard theory of evolution. This is one of the questions to be answered.

ii) Has the B-spectral type Sanduleak 69202 star ever become a red supergiant in its evolutionary life time?

There has been three types of explanations to the question “why did explosion occur in blue region?”:

i) With the poor metal abundance in the Large Magellanic Cloud[21]: Having less metal in chemical composition avoids the star’s evolution toward the red giant region in the HR diagran.

ii) With the high rate of mass loss [22]:If the star has a high rate of mass loss, it first evolves toward the red region in the HR diagram.

iii) With the effect of convection on the evolution of the supergiants.

The above explanations have not been found to be sufficient for the observations. For example, it has been known that there existed some ssupergiants in the LMC which have the same luminosity as that of Sanduleak 69202. On the other hand the presence of ahydrogen envelope of 10 M_o conflicts with the high rate of mass loss necessary for returninig to the blue region.

Abnormal abundances was thought to be explained by convective mixing, but was unsuccesful [23].

In explaining the observations Weiss and his colleages have suggested that rotation plays a role in producing abnormal chemical structure via circulation currents by rotation. The suggestion is that circulation currents carry elements, yielded in the core as a result of thermonuclearr reactions, to the surface and thus abundance observations can be explained. Saio et al. [24] have infact tried to explain the observed chemical abundances of supernova 1987A via fast rotation and resulted mixing.

After the 1960's much work has been done on the effect of rotation on stellar structure [25, 26, 27, 28, 29, 30]. In these works rigidly rotating mainsequence star models were obtained and found that the mainsequence in the HR diagram was moved to the right [Figure 4]. On the other hand Kippenhahn and his

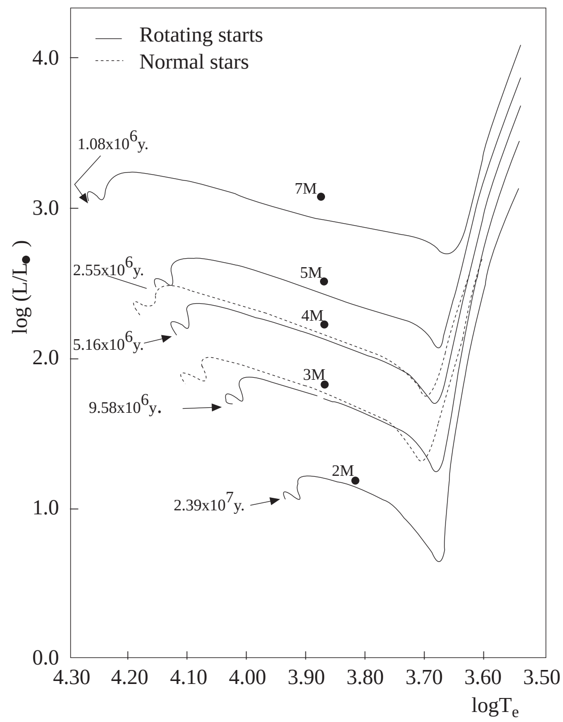

Figure 4. The effect of rotation on the positions and evolutionary tracks of some model stars of different masses in the HR diagram. The point at the end of the evolutionary track indicates the position of the star on the mainsequence and the corresponding times indicate the time to arrive on the mainsequence

colleagues [31], by assuming angular momentum is conserved locally in radiative regions and globally in convective regions, have shown that the inner regions of the star rotate faster and the envelope rotates slower. Crampin and Hoyle [32] pointed out that matter ejected as a consequence of rotation form a disk around the equatorial plane. Additional information on rotating stars and the details of the above mentioned works can be found in Mestel [33], Kraft [34], Slettabak [14] and in "Annual Review of Astronomy and Astrophysics, vol. 10" [35]. J.-L. Tassaul [36] gives a complete review of rotation in stars in his book called "Theory of Stellar Rotation".

In the years following 1970, differential rotation attracted more attention. The reason was that the argument which was developed by Kippenhahn and colleagues could lead to the formation of binary stars. Such a scenario was suggested by Roxbourgh [37] in 1966. However it was demonstrated that unless inner regions rotated very fast compared to the surface regions, the effect of rotation on the mainsequence stars was minute. Endal and Sofia [38, 39, 40] studied the effect of rotation on the evolution of stars in the post-mainsequence stage. In the last decade or so, work on rotating stars has been going on intensively [41-45].

As it is known, stars remain on the mainsequence for a long time. During this period, there might be some mechanism which slows down the fast rotating core. For example, this could be done by meriodinal circulation. perhaps, that is why the effect of rotation is not significant on the mainsequence. However, rotation becomes significant in the post-mainsequence stages.

The effect of rotation on stars in the pre-mainsequence stage has been studied in great dtail in [46-51].

It has been shown that rotation changes the structure of stars. During evolution both structure and the initially given angular momentum distribution evolve. However, in view of the observations of supernova 1987A, the nature of circulation currents and their interaction with evolution deserve more attention.

The effect of rotation on the structure and evolution of stars can be treated in three main groups:

i) Rotation reduces internal pressure. This is due to the centripetal force. The reduced pressure in the core makes the star to evolve differently. Assuming that the distribution of angular momentum is spherically symmetric and the star is spherical, the mean centripetal force is found to be $\Omega^2\, r\, \sin^2\theta$, and the hydrostatic equilibrium equation becomes

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho + \frac{2}{3}\Omega^2 r\rho \tag{1}$$

or

$$\frac{dP}{dM} = -\frac{GM_r}{4\pi r^4} + \frac{\Omega^2}{6\pi r} \tag{2}$$

where P is pressure, ρ is density, M is mass, r is the distance to the centre of the star and Ω is the angular velocity. Such a change in the structure induces some effects in hydrogen burning time in the core. As a consequence, burning takes place slowly, increasing the burning time by about 3 % [31, 52].

ii) Rotation reduces luminosity, and changes the shape of the star. It distorts the sphericity and creates an oblateness. The evolutionary effect of this is not significant, but it changes the spectrum of the star. Fast rotating stars appear to be cooler.

The star becomes oblate and equipotential surfaces are no longer spherical anymore.

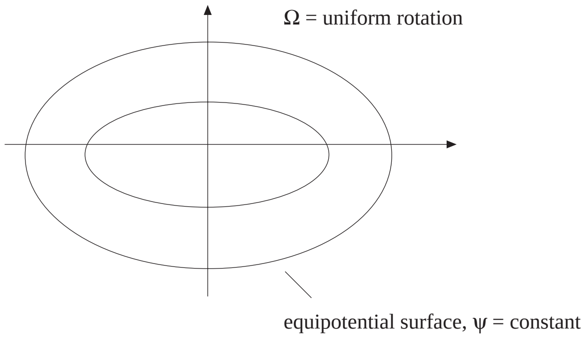

Figure 5.

Since the star is in hydrostatic equilibrium, P and ρ are constants on the equipotential surfaces. Thus we can write that $P = P(\Psi), \rho = \rho(\Psi)$ and $T = T(\Psi)$. The hydrostatic equilibrium eqution is

$$\frac{1}{\rho}\nabla P = \nabla\Psi. \tag{3}$$

If energy is transported by radiative processes then the expression for the flux of energy is

$$F = -\frac{16}{3}\frac{\sigma}{\kappa}\frac{T^3}{\rho}\nabla T,$$

or

$$F = -f(\Psi)\nabla\Psi = -f(\Psi)g, \tag{4}$$

where

$$f(\Psi) = -\frac{16}{3}\frac{\sigma}{\kappa}\frac{T^3}{\rho}\frac{dT}{d\Psi}$$

and

$$g = -\nabla\Psi.$$

Equation (4) shows that the surface flux varies directly with the gravity on the equipotential surfaces. Ψ in equations (3) and (4) represent the total potential ($\Psi =$ gravitational potential + rotational potential). Hence, the outgoing radiation flux at the surface of the star is

$$F_{\rm sur} \approx g_{\rm eff}. \tag{5}$$

This relation was first found by Von Zeipel and it is known as "the law of gravity darkening". The expression for the effective temperature is

$$T_{\rm eff} = (F_{\rm sur}/\sigma)^{1/4}. \tag{6}$$

The star is flatter in polar regions. Hence, $g_{\rm eff}$ will be greater in these regions. As seen in equation (6) this would imply greater effective temperature.

iii) The most significant effect of rotation is the transport of angular momentum and consequently having a mixing in chemical elements in certain regions of the star. This effect is the result of rotational instabilities.

As standard evolutionary star models have become insufficient, new physics which has not been considered so far, has been taken into account. Recently the physics related to the interaction between evolution and rotation and rotational instabilities have been included at least partially if not completely. Quite an amount of work and advancement have been done on the subject in the last decade [53-56).

In an axially rotating star, the tansport of energy is not solely done by radiation but it is also done partially by meridional circulation. The radiative equilibrium is

$$\underline{\nabla} \cdot \underline{F} = \rho \in_n, \tag{7}$$

where $\in_n =$ nuclear energy generation rate. Let us assume that the above condition is satisfied in an axially rotating star. Then we can write that

$$\begin{aligned} \rho \in_n &= \underline{\nabla} \cdot \underline{F} = -\underline{\nabla} \cdot (f(\Psi)\nabla\Psi)\underline{\nabla}^2\Psi - \frac{df}{d\Psi}|\nabla\Psi|^2 && (8) \\ \nabla^2\Psi &= \nabla^2\Phi + \nabla^2\vartheta \quad (\vartheta \text{ rotational potential}) && (9) \end{aligned}$$

If Ω is taken constant, the rotational potential is

$$\vartheta = \int_0^{\varpi} \Omega^2 \varpi' d\omega' = \frac{1}{2}\Omega^2\varpi^2, \tag{10}$$

where $\varpi = r\sin\theta$. From equation (9), we then have

$$\begin{aligned} \nabla^2\Psi &= \nabla^2\Phi + \nabla^2\vartheta = -4\pi G\rho + \underline{\nabla} \cdot (\Omega^2\varpi) \\ \nabla^2\Psi &= -4\pi G\rho + \frac{1}{\varpi}\frac{d}{d\varpi}(\Omega^2\varpi^2). && (11) \end{aligned}$$

Equation (8) can be written as

$$\rho \in_n = f(\Psi)(4\pi G\rho - \frac{1}{\varpi}\frac{d}{d\varpi}(\Omega^2\varpi^2) - \frac{df}{d\Psi}|\nabla\Psi|^2 \tag{12}$$

Let us choose Ω such that

$$\frac{1}{\varpi}\frac{d}{d\varpi}(\Omega^2\varpi^2) = \text{const.} = 2C_1$$

and hence

$$\Omega^2 = C_1 + \frac{C_2}{\varpi^2}. \tag{13}$$

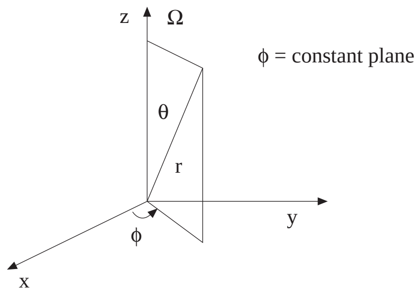

Figure 6. Φ constant and it corresponds to a meridional plane

In equation (12), everything but $|\nabla\Psi|^2$ is constant on the equipotential surfaces. $|\nabla\Psi|^2$ varies from equator to the polar regions, because the equipotential surfaces are not parallel. This is possible only when $df/d\Psi = 0$ which in turn implies $f = constant = C$. Thus from eqution (12) we get,

$$\in_n = 4\pi GC(1 - \frac{C_1}{2\pi G\rho}), \tag{14}$$

At the surface $\rho \to 0$ and $\in_n \to -\infty$. This is not realistic, because $\in_n$ vanishes in all but central regions of the star. Equation (14) is known as the von Zeipel paradox. It can be concluded from equation (14) that the radiative equilibrium condition can not be used rigidly in axially rotating stars.

Vogt[7] and Eddington [8,9] made use of the von Zeipel paradox [6]. By using it they have shown that if the star is not in rigid radiative equilibrium, there occurs temperature and pressure differences on the equipotential surfaces which lead to meriodinal circulations. That is why the phenomenon is known as the Eddington-Vogt circulations in literature. The velocity of Eddington-Vogt circulation was first determined by Sweet [57]. Many other workers followed him [58-63]. The importance of circulation currents in evolution can be estimated by studying the related time scales. It is therefore necessary to determine roughly the magnitudes of velocities of circulation currents and the related time scales.

Thermal equilibrium in axially rotating stars is maintained by the circulation currents set in meridional plane. Circulation currents transport heat energy. If Q is the transported heat energy and F is the flux, then the thermal equilibrium equation is

$$\rho \in_n = -\underline{\nabla} \cdot \underline{F} = \rho \frac{dQ}{dt}. \tag{15}$$

For steady state,

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \underline{v} \cdot \underline{\nabla} Q = \underline{v} \cdot T \underline{\nabla} S = T \underline{v} \cdot \underline{\nabla} S,$$

where S is entropy, and v is the circulation velocity. Also,

$$\frac{dQ}{dt} = \frac{du}{dt} + P\frac{dv}{dt} = c_v \frac{dT}{dt} - \frac{P}{\rho^2}\frac{d\rho}{dt},$$

where u is internal energy, which we can rewrite as

$$\rho \frac{dQ}{dt} = c_v(\rho \frac{dT}{dt} - \frac{P}{\rho}\frac{d\rho}{dt}),$$

using the relation $d/dt = \partial/\partial t + \underline{v} \cdot \underline{\nabla}$, and for steady state,

$$\begin{aligned} \rho \frac{dQ}{dt} &= c_v(\rho \frac{dT}{d\Psi}\frac{d\Psi}{dt} - \frac{P}{\rho}\frac{d\rho}{d\Psi}\frac{d\Psi}{dt}) \\ &= c_v(\rho \frac{dT}{d\Psi} - \frac{P}{\rho}\frac{d\rho}{d\Psi})\underline{v} \cdot \underline{\nabla}\Psi. \end{aligned} \tag{16}$$

If the energy generation is taken to be zero ($\in_n = 0$), equation (15) becomes

$$-\underline{\nabla} \cdot \underline{F} = c_v(\rho \frac{dT}{d\Psi} - \frac{P}{\rho}\frac{d\rho}{d\Psi})\underline{v} \cdot \underline{\nabla}\Psi, \tag{17}$$

Using $P/\rho = (\mathcal{R}/\mu)T = (c_p - c_v)T = c_v(\gamma - 1)T$,and eliminating P and $d\rho/d\Psi$ equation (17) takes the following form:

$$-\underline{\nabla} \cdot \underline{F} = \rho(\frac{\gamma}{\gamma - 1}\frac{\mathcal{R}}{\mu}\frac{dT}{d\Psi} - 1)\underline{v} \cdot \underline{\Delta}\Psi. \tag{18}$$

Here P, ρ, T, c_v, c_p, and μ have their usual meanings. $\gamma = c_p/c_v$ and $\mathcal{R}$ is the gas constant.

Let us try to estimate the magnitude of v from equation (18):

$$\underline{\nabla} \cdot \underline{F} \approx \alpha \frac{1}{R} \frac{L}{R^2} = \frac{\alpha L}{R^3}$$

with $\alpha = F_c/F_g =$ centripetal force/gravitational force.

$$\begin{aligned} \nabla \Psi &\cong \frac{GM}{R^2} \\ \rho &\cong \frac{M}{R^3} \end{aligned}$$

and the quantity in the paranthesis can be taken about equal to 1. Using these approximate values, one gets

$$\begin{aligned} v &\approx \alpha \frac{L}{R^3} \frac{R^5}{GM^2} = \alpha \frac{LR^2}{GM^2} \\ v &\approx \alpha \frac{LR^2}{GM^2}. \end{aligned} \tag{19}$$

Here L, R and M are the luminosity, radius and mass of the star respectively. G is the gravitational constant. For the Sun $\alpha = 2.5.10^{-5}$ and thus

$$v_0 \approx 2.10^{-9} \mathrm{cmsec}^{-1}.$$

It is clear from equation (19) that the velocity of circulation currents depends on the gross properties (L,R,M) of the star and its rotation. The associated time scale for the circulation is

$$t_{cir} \approx R/v \approx (GM^2/R)/\alpha L = t_{KH}/\alpha, \tag{20}$$

where t_{KH} is the Kelvin-Helmholtz time scale. It is defined as

$$t_{KH} = \text{Gravitational energy/Luminosity} = (GM^2/R)/L. \tag{21}$$

Since the parameter α is less than one, it is obvious in equation (20) that

$$t_{cir} > t_{KH}, \tag{22}$$

This result shows that circulation currents are not important in the thermal time scale.

Now, let us find the evolutionary time scale and compare it with the circulation time scale. For simplicity, consider the evolutionary time as the time which corresponds to the burning of 20 % of the total mass of a pure hydrogen star. In converting hydrogen into helium, 6×10^{18} ergs energy is released per one gram of hydrogen converted into helium. Thus,

$$t_{ev} \approx 6 \cdot 10^{18}(M/5)/L$$

or

$$t_{ev} \approx 10^{18}M/L.$$

If we combine this with equation (20) we get,

$$t_{cir}10^{-18}(GM/\alpha R)t_{ev}. \tag{23}$$

If it is evaluated for the Sun for which $\alpha = 2.10^{-5}$ one gets,

$$t_{cir,0} \approx 100t_{ev}. \tag{24}$$

We can conclude from this result that the meridional circulation currents in the Sun will not affect its evolution, because the circulation time scale is 100 times larger than the evolutionary time.

However, if we repeat the same approximate calculation for an early type star, for example for a B-type star, we obtain a different result. For early type stars observations tell us that they have equatorial velocity around 200 km sec^{-1} and $(GM/R)^{1/2} \approx 600$ km sec^{-1}. Thus we find that $\alpha = 0.1$. When α is substituted in equation (23), we have for the circulation time scale

$$t_{cir} \approx 3.610^{-2}t_{ev} \tag{25}$$

This means that circulation is important in B-type stars, because its time scale is much shorter than the evolutionary time. Hence we should expect mixing as a consequence of rotation in early type stars. This has infact been shown by Eryurt et al [64] that, in a $20M_0$ LMC star, chemical elements are mixed via rotational circulation. In regions of the star where circulation currents are in operation the star will have a homegeneous chemical composition. But, we should remember that there is an opposing agent against this. When chemical composition varies along radius, i.e, if a radial chemical composition gradient develops (μ-gradient, and $\nabla_\mu > 0$), a current which is called μ- current, opposing to rotational circulation might develop. This effect chokes the rotational circulation currents [65].

5. Axial Rotatin and Mixing

In the previous section, it has been stated that when a chemical composition gradient develops in a star it induces an effect which opposes meridional circulation currents. However at the begining of the core evolution when a chemical composition gradient (μ- gradient) was not yet formed, a chemical mixing might occur. Mixing might continue to operate even if we have an excess of chemical composition gradient. Since angular momentum is conserved, it is natural to expect a differential rotation in the star [68-70]. Differential rotation, on equipotential surfaces, is unstable (shear unstable), and turbulent motion occurs. This motion is mainly horizontal, but it finally becomes three dimensional. The diffusion of chemical elements will be implemented by the component of the turbulent motion in radial direction [54].

For the turbulence to survive, it is necessary to have a continuous energy input. From the classical turbulent theory and dimensional arguments [71] the above mentioned input energy which is the energy consumed by the turbulence is

$$\in_t \approx \frac{v^3}{l} \text{ erg sec}^{-1}. \tag{26}$$

Here, v=velocity and l is the dimension of the energy transporting eddy. The source of this energy is the thermal imbalance, which results from axial rotation. Thus thermal imbalance supplies the energy necessary for meridional circulation currents and turbulence.

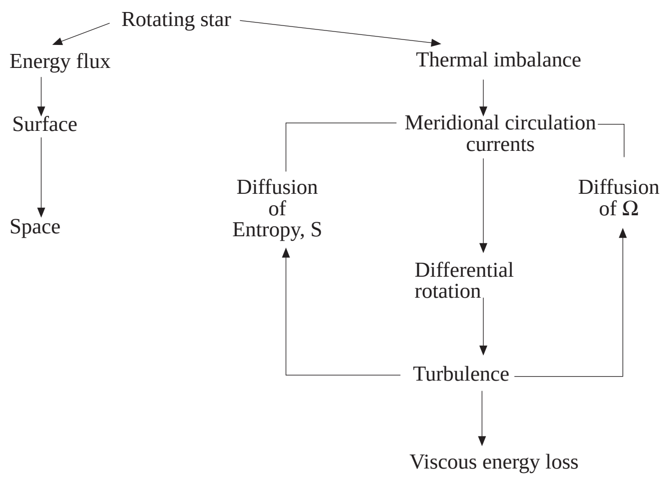

Figure 7. The consumption of the energy of an axially rotating star. S is the entropy and Ω is the angular velocity of rotation.

Energy consumption rate per unit mass which is given in eqution (26) must be less than the source energy which is supplied by thermal imbalance, because the consumption of the whole source energy by turbulence is not realistic. Hence we have the following inequality to hold:

$$\frac{v^3}{l} < K\Omega^2. \tag{27}$$

Here, $K\Omega^2$ is the thermal energy per unit mass per unit time which was given in equation (15) and it is equal to dQ/dt. K represents the thermal diffusion and Ω is the angular velocity. If we express K explicitly, it is,

$$K = \frac{4}{3}\frac{ac}{\kappa\rho^2}\frac{T^3}{c_p}, \tag{28}$$

where κ is the opacity, a is the radiation constant and c is the speed of light. The specific heat at constant pressure is

$$c_p = \frac{P\delta}{\nabla_{ad}\rho T} \qquad \text{and } \delta = (\frac{\partial \ln \rho}{\partial \ln T})_P.$$

There is a second condition associated with equation (26). For the shear instability to be continuous we need to impose the following condition related to the critical Reynolds number,

$$vl > R_c \approx \frac{vl}{\nu} \approx 10^3, \tag{29}$$

where $\nu =$ kinematic viscosity and Rc is the critical Reynolds number.

The efficiency coefficient for converting energy resulted from the thermal imbalance into turbulent energy is defined by

$$\eta = \in_t / K\Omega^2 \tag{30}$$

We can finally write the expression for transporting matter via turbulent motion in radial direction as

$$D = K\eta, \tag{31}$$

where D is the diffusion coefficient.

If chemical composition gradient is taken into account, the efficiency coefficient becomes,

$$\eta \approx \frac{\Omega^2 r}{g}\frac{H_p}{r}(\nabla_{ad} - \nabla_{rad} + \nabla_\mu)^{-1}, \tag{32}$$

and hence,

$$D = K\frac{\Omega^2 r}{g}\frac{H_p}{r}(\nabla_{ad} - \nabla_{rad} + \nabla_\mu)^{-1}, \tag{33}$$

The Reynolds number associated with the radial diffusion is [72]

$$R_e = \frac{K}{\nu}\frac{\Omega^2 r}{g}\frac{H_p}{r}(\nabla_{ad} - \nabla_{rad} + \nabla_\mu)^{-1}, \tag{34}$$

where $\nu(= \nu_{mol} + \nu_{rad})$ is the kinematic viscosity, $\nabla_\mu = \partial \ln \mu / \partial \ln P, \nabla_{ad} =$ adiabatic temperature gradient, $\nabla_{rad} =$ radiative temperature gradient and H_p is the pressure scale height. The radiative viscosity is

$$\nu_{rad} = \frac{4aT^4}{15c\kappa\rho^2} = 6.72.10^{-26}\frac{T^4}{\kappa\rho^2} \tag{35}$$

and the molecular viscosity is

$$\nu_{mol} = 2.2.10^{-15}\frac{T^{5/2}}{\rho \ln \lambda}. \tag{36}$$

λ is related to the ratio of Deby length to the mean distance between particles, and is given by

$$\lambda = \frac{1.545}{e^3}(\frac{k^3 T^3 m_e}{\pi\rho})^{1/2},$$

where k is the Boltzman constant, T is temperature, m is the mass of an electron, e is the charge of an electron and ρ is the density.

It has been argued that the formation of μ-gradient (chemical composition gradient) creates an effect which acts against circulation currents and avoid the transport of matter in the radial direction of the developed three dimensional turbulence. At this stage let us define the Brunt-Vaisala frequency. It is the measure of buoyancy force when there is a stable density stratification. When a blob of matter moves from its equilibrium position in radial direction, it oscillates under the effect of buoyancy force. The blob oscillates about its equilibrium position with a frequency of N. If there is a μ- gradient, which is given as $\nabla_\mu = \partial ln\mu / \partial \ln P$, in radial direction we then have

$$N^2 = \frac{g}{H_p}(\nabla_{ad} - \nabla_{rad} + \nabla_\mu) = N_T^2 + N_\mu^2. \tag{37}$$

$N_T^2 = g(\nabla_{ad} - \nabla_{rad})/H_p$ represents the thermal part, and $N_\mu^2 = g\nabla_\mu/H_p$ represents the contribution from the chemical composition gradient.

When the condition given below is satisfied, the chemical composition gradient can inhibit the development of three dimensional turbulence [54],

$$N_\mu^2 > \eta \frac{K}{\nu} \Omega^2.$$

This is infact the turnover frequency of the moving mass cell. The above expression can be rewritten as

$$\left| \frac{\partial \ln \mu}{\partial \ln r} \right| > \frac{K}{\nu} (\frac{\Omega^2 r}{g}) 2 \cdot (\frac{H_p}{r}) (\nabla_{ad} - \nabla_{rad} + \nabla_\mu)^{-1}. \tag{38}$$

For the diffusion to occur this condition must not be satisfied.

As it is clearly seen in equations (38) and (32) there is an Ω dependence. Once we know angular velocity distribution in a star we can then decide if diffusion takes place or not.

Let us say few things about μ-currents. During evolution, at any time, the total velocity of circulation currents is

$$\underline{v} = \underline{v}^\Omega + \underline{v}^\mu. \tag{39}$$

Here, $v^\Omega =$ rotational velocity and $v^\mu =$ circulation velocity of μ-currents.

The rate of change of μ, at any point in the star, is

$$(d\mu/dt)_{nuc} = \partial\mu/\partial t + \underline{v} \cdot \underline{\nabla}\mu. \tag{40}$$

For the unit mass which comoves with the fluid, the mean molecular weight, μ, is constant and $(d\mu/dt)$ is zero.

Mestel [65] investigated the problem of Ω-currents and found out that;

i) (μ-currents oppose Ω currents.

ii) The size of μ-currents might be large enough to choke Ω-currents near the core of the star.

If we assume that v^μ is proportional to $(\Delta\mu/\mu)$, we find from the thermal equilibrium equation (equation 15) that

$$v^\mu \approx \frac{LR^2}{GM^2} \frac{\Delta\mu}{\mu} \tag{41}$$

On the other hand, we have already obtained that

$$v^\Omega \approx \frac{LR^2}{GM^2} \alpha(r). \tag{42}$$

To estimate the magnitude of v^μ we must know the value of $\Delta\mu/\mu$. The change in the chemical composition occurs as a consequence of thermonuclear reactions in the core. μ-currents transport the newly formed matter to the outer regions of the star. During this process a chemical composition gradient of magnitude $\Delta\mu/\mu$ is formed. The time during which Ω- currents carry material from the core is

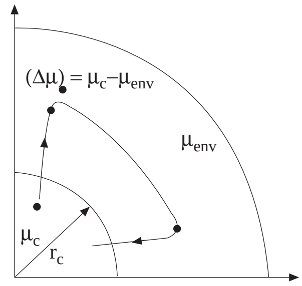

Figure 8. The formation of μ-gradient

$$t_\mu \approx \frac{r_c}{v_c^\Omega} \tag{43}$$

If r_c is small enough, we can write $\bar{\rho}(r) \approx \rho_c$ near the core boundary. Remembering that in equation (42), $\alpha(r)$ is the ratio of centripetal force to gravitational force, and that

$$v^\Omega \propto \alpha(r) = 3\Omega^2/(4\pi G\bar{\rho}(r)),$$

we have

$$\frac{v_c^\Omega}{v_{av}^\Omega} = \frac{\bar{\rho}(r)}{\rho_c}. \tag{44}$$

For a B5- spectral type star, the above ratio has the following value,

$$v_c^\Omega/v_{av}^\Omega \approx 1/30. \tag{45}$$

It is clear in equation (45) that v_c^Ω is smaller than v_{av}^Ω, which means that time is spent almost near the core. Thus, we can take

$$t_\mu \approx t_{cir} \approx \frac{GM^2}{RL\alpha} \tag{46}$$

Next we ask the question: What happens to $\Delta\mu$ in t_μ time? Assume that total hydrogen is converted into helium in the core. In the resulting structure, μ increases. That can be demonsrated by making use of the definition of μ. Let us write it for a completely ionized gas:

$$\frac{1}{\mu} = 2X + (3/4)Y + (1/2)Z \tag{47}$$

with,

$$X + Y + Z = 1.$$

Differentiating the above expression, we get

$$\Delta Y = -\Delta X > 0,$$

where $\Delta Z = 0$, because there is no change in Z, but X decreases, because hydrogen is being converted into helium. From equation (47),

$$\Delta\mu/\mu = \frac{5(-\Delta X)}{6X_0 + Y_0 + Z} > 0. \tag{48}$$

Here, X, Y, Z are the abundances of hydrogen, helium, and heavy elements by weight in one gram of stellar material, respectively, and the zero subscript indicates their initial values. We see in equation (48) that μ increases. Since the conversion time scale for hydrogen is

$$t \approx \frac{(6.10^{18} erg/gm)M}{L}, \tag{49}$$

the amount of fractional hydrogen converted into helium in t_μ time is

$$\Delta X(t_\mu) \approx \frac{t_{cir}}{t} \approx \frac{GM}{6.10^{18} R\alpha}$$

and

$$\Delta\mu/\mu = 1.10^{-19}\frac{GM}{R\alpha},$$

which is an underestimate. And finally,

$$\Delta\mu/\mu \approx 3.10^{-4}(\frac{M/M_\theta}{R/R_\theta})\frac{1}{\alpha}.$$

Remember that near the core $v_{av}^{\Omega} \approx (\bar{\rho}(r)/\rho_c)\ (LR^2/GM^2)\alpha$ (see equations 45 and 19). Using equations (41) and (42) we get, dropping the subscript, in the average velocity v_{av}^{Ω},

$$v^\mu/v^\Omega \approx \frac{\Delta\mu}{\mu}\frac{1}{\alpha}\frac{\rho_c}{\rho(r)} \approx 3.10^{-4}(\frac{M/M_0}{R/R_0})\frac{1}{\alpha^2}\frac{\rho_c}{\bar{\rho}(r)}. \tag{50}$$

Typically for a B5-spectral type star,

$$\rho_c/\bar{\rho}(r) \approx 30, M/M_\circ \approx 7 \text{ and } R/R_\theta \approx 4,$$

hence,

$$\frac{v^\mu}{v^\Omega} \approx \frac{1.575 \times 10^{-2}}{\alpha^2},$$

and since $\alpha < 0.1$ for a B5 type star, we have

$$v^{\mu} > v^{\Omega}. \tag{51}$$

Having v^{μ} larger than v^{Ω} means μ-currents choke Ω-currents. But, if we have a fast rotation $(\alpha = \alpha_{crit})v^{\mu}$ may get smaller than v^{Ω} and we expect mixing. If $\Omega(r)$ evolves in such a way as to increase inward v^{Ω} becomes even larger and we again expect mixing.

5. Conclusion

It has been shown that in an early-type axially rotating star meridional circulation currents were induced, and they led to differential rotation; hence turbulence was developed. As a consequence of this the radial component of the three dimensional turbulence caused mixing via diffusion mechanism if some conditions on efficiency of turbulent motion, chemical composition gradient and the evolution of angular velocity $\Omega(r)$ were satisfied. Besides, for a uniformly rotating star, circulation velocity and relevant time scales have been estimated.

The conclusion is that meridional circulation currents could lead to some chemical mixing in early type stars.

By utilyzing this conclusion which we attained with very crude order of magnitude estimates, one can now do evolutionary calculations of the fast rotating early type stars; to see if one can explain the abnormal chemical abundances in the atmospheres of OBN type stars. The same argument applies to Supernova 1987 A, if proceeded a little bit further in evolving the star in the H-R diagram.Remember that we have similar sort of abundance anamolies in the observations of SN 1987 A.

In fact, Eryurt et al [73] being motivated by such a conclusion have done some calculations on OBN type LMC stars and found that rotational mixing occurs in these stars, and hence succesfully explain the observed abnormal abundances in the atmospheres of these stars.

References

[1] M. Abney M.N. Roy.*Astron. Soc.,* **37** (1877) 278

[2] T. Elvey, O. Struve *Astrophys. J.* **71** (1930) 221

[3] T. Elvey, O. Struve *Astrophys. J.* **72** (1930) 1

[4] A. Slettabak *Astrophys. J.* **110** (1949) 498

[5] E.A. Milne M.N. Roy. *Astron. Soc.* **83** (1923) 118

[6] H. Von Zeipel M.N. Roy. *Astron. Soc.,* **84** (1924) 665

[7] H. Vogt *Astr. Nachr.* No. **5342** (1925)

[8] A.S. Eddington *Observatory* **48** (1925) 73

[9] A.S. Eddington M.N. Roy.*Astron. Soc.,* **90** (1929) 54

[10] A.A. Boyarchuk, I.M. Kopylov *Soviet Astron. J.* **35** (1958) 804

[11] A.A. Boyarchuk, I.M. Kopylov *Public. Crimean Astrophys. Observatory* **31** (1964) 44

[12] A. Uesugi, I. Fukuda *Mem. Fac. Sci. Kyoto Univ. Ser. Phys.,Astrophys., Geophys., Chem.* **33** (1970) 205

[13] P.L. Bernacca, M. Perinotto *Conrib. Oss. Astrofis. Asiago, Univ. Padova* no. **239,** no. 250 (1971)

[14] A. Slettabak, Stellar Rotation, ed. A. Slettabak (Gordon and Breach, New York, 1970) p. 5

[15] H. A. Abt, J. H. Jr. Hunter *Astrophys. J.* **136** (1962) 381

[16] E. C. Olson *Publ. Astron. Soc. Pacific* **80** (1968) 185

[17] J. Jeans, *Astronomy and Cosmogony* (1929)

[18] R. A. Lyttleton, Stability of Rotating Liquid Masses (Cambridge Univ. Press, 1953)

[19] N. Lebovitz *Adv. Astron.* **5** (1967) 465

[20] A. Cassatella, SN 1987 A, ed. J.I. Danziger (ESO, Garching bei Mnchen,1987) p. 25

[21] W. Hillebrandt, P. Hofflich, J.W. Truran, A. Weiss, SN 1987 A, ed. J.I. Danziger (ESO, Garching bei Mnchen, 1987) p. 25

[22] A. Maeder, SN 1987 A, ed. J.I. Danziger (ESO, Garching bei Mnchen, 1987) p. 25

[23] A. Weiss, W. Hillebrandt, J.W. Truran, *Astron. Astrophys.,* **197** (1988) L11

[24] H. Saio, K. Nomoto, M. Kato *Nature* **334** (1988) 508

[25] I. W. Roxburgh, P.A. Strittmatter Z. Astrophys. **63** (1965) 15

[26] J. Faulkner, I.W. Roxburgh, P.A. Strittematter *Astrophys. J.,* **151** (1968) 203

[27] J. W.-K. Mark *Astrophys. J.,* **154** (1968) 627

[28] A. D. Sanderson, R.C. Smith, J. Hazlehurst *Astrophys. J.,* **159** (1970) L69

[29] I.-J. Sackmann, S. P. S. Anand *Astrophys. J.,* **162** (1970) 105.

[30] R. Kippenhahn, H.-C. Thomas, Stellar Rotation, ed. A. Slettabak (IAU col. No. IV, 1970) p. 20

[31] R. Kippenhahn, E. Meyer-Hofmeister, H.-C. Thomas *Astron. Astrophys.,* **5** (1970) 155

[32] J. Crampin, F. Hoyle M.N. Roy. *Astron. Soc.,* **120** (1960) 33

[33] Mestel, L. Stars and Stellar Systems, ed. Aller and Mac Laughlin, vol. VIII (Univ. Of Chicago, 1965) p. 465

[34] R.P. Kraft, Stellar Astronomy, ed. H.K. Chiu, R.L. Warasila, J.H. Remo, vol. 1 (Gordon and Breach Sci. Publ., New York, 1968) p. 317.

[35] K. Fricke, R. Kippenhahn *Ann. Rew. Astron. Astrophys. vol.* **10** (1972) p. 45.

[36] J.-L. Tasoul, Theory of Rotating Stars (Princeton Univ. Press, Princeton, 1978)

[37] I. W. Roxburgh *Astrophys. J.,* **143** (1966) 111.

[38] A.S. Endal, S. Sofia *Astrophys. J.,* **210** (1976) 184

[39] A.S. Endal, S. Sofia *Astrophys. J.,* **220** (1978) 279

[40] A.S. Endal, S. Sofia *Astrophys. J.,* **243** (1981) 625

[41] S.R. Sreenivasan, W.J.F. Wilson *Astrophys. J.,* **254** (1982) 287

[42] E. Schatzman, A. Maeder *Astron. Astrophys.,* **96** (1981) 1

[43] O. Bienayme, A. Maeder, E. Schatzman **Astron. Astrophys., 131** (1984) 316

[44] A. Lebreto, A. Maeder *Astron. Astrophys.,* **175** (1987) 99

[45] A. Maeder *Astron. Astrophys.,* **178** (1987) 159

[46] I. Okamoto, Stellar Rotation, ed. A. Slettabak (Reidel Dordrecht, 1969) p. 73

[47] P. Bodenheimer, J.P. Ostriker *Asrophys. J.,* **161** (1970) 1101

[48] J.P. Ostriker, Stellar Rotation, ed. A. Slettabak (Gordon and Breach, New York, 1970) p. 147

[49] G.S. Bisnovatyi-Kogan, S.I. Bilinnikov, N.D. Kostyuk, A.D. Federova Sov. Astron. **23** (1979) 432

[50] D. Eryurt-Ezer, H. Kırbıyık, N. Özkara *Astrophys. and Space Sci.,* **126** (1986) 29

[51] R. Çelikel, D. Eryurt-Ezer *Astrophys. and Space Sci.,* **153** (1989) 213.

[52] P.A. Strittmatter, J.W. Robertson, D.J. Faulkner *Astron. Astrophys.,* **5** (1972) 426.

[53] R. Kippenhahn, H.-C. Thomas, Fundamental Problems in the Theory of Stellar Rotation, eds. D. Sugimoto, D.Q. Lamb, D.N. Schramm (Reidel, Dordrecht, IAU Symp. No. 93, 1981) p. 237.

[54] J. -P. Zahn, Proceedings of the 13 th Saas Fee Course, eds. B. Hauck, A. Maeder (1983) p.253

[55] M. Tassoul, J.-L. Tassoul *Astrophys. J.,* **279** (1984) 384

[56] E. Schatzman, Proceedings of IAU Symp. No. 105, eds. A. Maeder, A. Renzini (Reidel, Dordrecht, 1984) p. 491.

[57] P.A. Sweet M.N. Roy. *Astron. Soc.,* **110** (1950) 548.

[58] E.J. pik M.N. Roy. *Astron. Soc.,* **111** (1951) 278

[59] N. Baker, R. Kippenhahn *Z. Astrophys.,* **48** (1959) 140

[60] R.C. Smith *Z. Astrophys.,* **63** (1966) 166

[61] R.C. Smith M.N. Roy. *Astron. Soc.,* **148** (1970) 275

[62] L. Mestel *Z. Astrophys.,* **63** (1966) 196

[63] D. Brand, R. C. Smith M.N. Roy. *Astron. Soc.,* **154** (1971) 293

[64] D. Eryurt-Ezer, H. Kırbıyık, N. Kızıloğlu, R. Çelikel *DOĞA, Tu. J. Phys. And Astrophys.,* **16** (1992) 204.

[65] L. Mestel M.N. Roy. *Astron. Soc.,* **113** (1953) 716.

[66] J.P. Zahn, Proceedings of IAU Symp. No. 59, eds. P. Ledoux, A. Noels, R.W. Rogers (Reidel, Dordrecht, 1974) p. 185.

[67] R. Kippenhahn, Proceedings of IAU Symp. no. 66, eds. R.J. Tayler, J.E. Hesser (Reidel, Dordrechr, 1974) p. 21.

[68] H.E. Huppert, E.A. Speigel *Astrophys. J.,* **213** (1977) 157.

[69] J.-L. Tassoul, M. Tassoul *Astrophys. J.,* **261** (1982) 265.

[70] J.-L. Tassoul, Proceedings of IAU Symp. no. 105, eds. A. Maeder, A. Renzini (Reidel, Dordrecht, 1984) p. 475.

[71] L.D. Landau and E.M. Lifshitz, Fluid Mechanics (Pergamon Pres, London, 1959) p. 119.

[72] E. Schatzman *Astron. Astrophys.,* **56** (1977) 211.

[73] D. Eryurt, H. Kırbıyık, N. Kızıloğlu, R. Civelek, A. Weiss *Astron. Astrophys.,* **282** (1994) 485.

[74] E. Meyer-Hofmeister *Astron. Astrophys.,* **16** (1972) 282.

[75] J.C.B. Papaloizou, J.A.J. Whelan M.N. Roy. *Astron. Soc.,* **164** (1973) 1.

[76] P.S. Conti,C. De Loore, Mass Loss and Evolution of O-type Stars (Reidel, Dordrecht, 1979)

[77] S.R. Sreenivasan, W.J.F. Wilson *Astronn. Astrophys.,* **70** (1978) 755.

[78] R.P. Kraft *Astrophys. J.,* **150** (1967) 551.

[79] E. Schatzman *Ann. Astrophys.,* **25** (1962) 18.

[80] L. Hartman, K.B. McGregor *Astrophysc. J.,* **259** (1982) 180.

[81] D. Reimers, Proceedings of the 19 th International Colloquium on Problems in Stellar Hydrodynamics, ed. M. Gabriel (1975).

[82] S. R. Sreenivasan, Highlights of Astronomy, ed. P.A. Wayman vol. **5** (1980) 601.

[83] J. R. Stuffer, L. Hartman,D.R. Soderblom, N. Burnham *Astrophys. J.,* **280** (1984) 202.

[84] J. Bohigas, L. Carrasco, C.A.O. Torres, G.R. Quast *Astron. Astrophys.,* **157** (1986) 278.

[85] L.W. Hartman, R.W. Noyes,*Ann. Rew. Astron. Astrophys.,* vol. **25** (1987) p. 271.

[86] E.A. Kreiken *Z. Astrophys.,* **10** (1935) 199.

[87] S. Chandrasekhar, N.R. Lebovitz *Astrophys. J.,* **135** (1962) 238.