




Octonion form of duality-invariant field equations for dyons

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Abstract: The hypothetical particles dyons, which carry both electric and magnetic charges simultaneously, are widely discussed in application to electromagnetic theory and magnetohydrodynamics. Particularly, the duality-invariant field equations were suggested with suitable definitions of the dyon's electromagnetic characteristics. In this study, we propose an alternative formulation of the duality-invariant field equations for dyons based on octonion algebra. Octonions have been used to express the equations for potentials, field strengths, and sources in a more compact and consistent manner. Additionally, the octonionic form of the energy conservation law for dyons has been derived.

Key words: Octonions, Maxwell equations, magnetic monopole, dyon, conservation of energy

1. Introduction

The hypothetical particles dyons, which carry both electric and magnetic charges simultaneously, were first proposed by Julian Schwinger as a phenomenological alternative to quarks [1, 2]. Many physical theories of electromagnetism and linear gravity have been developed for dyons in the literature; see [2]–[10]. Recently after Olesen's study [11], Coceal et al. constructed the equations of magnetic plasma, usual plasma, and the pair of both magnetic and electric plasma with dyons, which are deeply connected to the idea of electric-magnetic duality [4].

Mathematics, physics, and many physical applications are performed by using vector, matrix, and tensor algebra. In addition to these algebraic structures, the solutions and representations of physical problems are discussed with quaternions, octonions, sedenions, octons, and sedeons, which are multicomponent number systems. These algebras offer important and alternative notations of physical systems for representing them in a more compact, effective, and elegant way. Particularly, there are many studies on electromagnetism, linear gravity, and quantum mechanics that were performed in terms of hypercomplex values; see [12]–[48].

The purpose of this study is to identify the duality-invariant field equations and energy conservation law for dyons in octonionic algebra representation. The paper has the following structure. In Section 2, we briefly recall the information about octonion algebra and its complex type. A short review of equations and results for dyonic electrodynamics is presented in Section 3. The generalized representation of the duality-invariant form of equations for dyons in terms of octonion algebra is given in Section 4. In the final section, the energy conservation law for dyons is derived via octonions.

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2. Mathematical preliminaries

Octonions are eight-component hypercomplex numbers also called Cayley numbers. They are a unique alternative and division algebra with both noncommutative and nonassociative structures [49, 50]. According to different systems, they form some styles such as classical, complexified, dual, hyperbolic or split types.

The real octonion \mathbf{X} is defined as

$$\mathbf{X} = \sum_{n=0}^7 x_n \mathbf{e}_n = x_0 \mathbf{e}_0 + x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3 + x_4 \mathbf{e}_4 + x_5 \mathbf{e}_5 + x_6 \mathbf{e}_6 + x_7 \mathbf{e}_7, \quad (1)$$

where the x_n terms are real number coefficients of the octonion and the \mathbf{e}_n s are its basis elements. For two real octonions \mathbf{X} and \mathbf{Y} , the sum and difference are given as

$$\begin{aligned} \mathbf{X} \pm \mathbf{Y} &= \sum_{n=0}^7 (x_n \pm y_n) \mathbf{e}_n = (x_0 \mathbf{e}_0 + x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3 + x_4 \mathbf{e}_4 + x_5 \mathbf{e}_5 + x_6 \mathbf{e}_6 + x_7 \mathbf{e}_7) \\ &\quad \pm (y_0 \mathbf{e}_0 + y_1 \mathbf{e}_1 + y_2 \mathbf{e}_2 + y_3 \mathbf{e}_3 + y_4 \mathbf{e}_4 + y_5 \mathbf{e}_5 + y_6 \mathbf{e}_6 + y_7 \mathbf{e}_7). \end{aligned} \quad (2)$$

The circular octonion \mathbf{X} consists of two parts:

$$S_{\mathbf{X}} = x_0 \mathbf{e}_0, \quad V_{\mathbf{X}} = \vec{\mathbf{X}} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3 + x_4 \mathbf{e}_4 + x_5 \mathbf{e}_5 + x_6 \mathbf{e}_6 + x_7 \mathbf{e}_7 \quad (3)$$

where $S_{\mathbf{X}}$ and $V_{\mathbf{X}}$ show the scalar and vectorial parts, respectively. Therefore, the real octonion \mathbf{X} can be written in a shorter form as

$$\mathbf{X} = S_{\mathbf{X}} + V_{\mathbf{X}} = x_0 \mathbf{e}_0 + \vec{\mathbf{X}}. \quad (4)$$

In this study, the Cayley–Dickson multiplication rules, as one form of construction processes in the literature, are used in all steps as shown in the Table [16]–[20].

Table 1. Multiplication rules of an octonion’s units in Cayley–Dickson form.

| | \mathbf{e}_1 | \mathbf{e}_2 | \mathbf{e}_3 | \mathbf{e}_4 | \mathbf{e}_5 | \mathbf{e}_6 | \mathbf{e}_7 |
|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \mathbf{e}_1 | -1 | \mathbf{e}_3 | $-\mathbf{e}_2$ | \mathbf{e}_5 | $-\mathbf{e}_4$ | $-\mathbf{e}_7$ | \mathbf{e}_6 |
| \mathbf{e}_2 | $-\mathbf{e}_3$ | -1 | \mathbf{e}_1 | \mathbf{e}_6 | \mathbf{e}_7 | $-\mathbf{e}_4$ | $-\mathbf{e}_5$ |
| \mathbf{e}_3 | \mathbf{e}_2 | $-\mathbf{e}_1$ | -1 | \mathbf{e}_7 | $-\mathbf{e}_6$ | \mathbf{e}_5 | $-\mathbf{e}_4$ |
| \mathbf{e}_4 | $-\mathbf{e}_5$ | $-\mathbf{e}_6$ | $-\mathbf{e}_7$ | -1 | \mathbf{e}_1 | \mathbf{e}_2 | \mathbf{e}_3 |
| \mathbf{e}_5 | \mathbf{e}_4 | $-\mathbf{e}_7$ | \mathbf{e}_6 | $-\mathbf{e}_1$ | -1 | $-\mathbf{e}_3$ | \mathbf{e}_2 |
| \mathbf{e}_6 | \mathbf{e}_7 | \mathbf{e}_4 | $-\mathbf{e}_5$ | $-\mathbf{e}_2$ | \mathbf{e}_3 | -1 | $-\mathbf{e}_1$ |
| \mathbf{e}_7 | $-\mathbf{e}_6$ | \mathbf{e}_5 | \mathbf{e}_4 | $-\mathbf{e}_3$ | $-\mathbf{e}_2$ | \mathbf{e}_1 | -1 |

By means of these rules, the product of two classical octonions is given as follows:

$$\mathbf{X}\mathbf{Y} = x_0 y_0 + x_0 \vec{\mathbf{Y}} + \vec{\mathbf{X}} y_0 - \vec{\mathbf{X}} \cdot \vec{\mathbf{Y}} + \vec{\mathbf{X}} \times \vec{\mathbf{Y}}. \quad (5)$$

The octonion conjugate process can be shown by changing the signs outside of the scalar part of the octonion \mathbf{X} :

$$\bar{\mathbf{X}} = x_0 \mathbf{e}_0 - \sum_{n=1}^7 x_n \mathbf{e}_n = x_0 - \vec{\mathbf{X}}. \quad (6)$$

For two real octonions, the octonionic conjugate process can be summarized as follows:

$$(\overline{\overline{\mathbf{X}}}) = \mathbf{X}, \quad (\overline{\mathbf{X} + \mathbf{Y}}) = \overline{\mathbf{X}} + \overline{\mathbf{Y}}, \quad (\overline{\mathbf{XY}}) = (\overline{\mathbf{Y}})(\overline{\mathbf{X}}). \quad (7)$$

In the case of having vectorial parts in equation (5), the scalar and vectorial products can be defined in terms of two real octonions as follows:

$$\vec{\mathbf{X}} \cdot \vec{\mathbf{Y}} = -\frac{1}{2} [\mathbf{XY} + (\overline{\mathbf{XY}})], \quad (8)$$

$$\vec{\mathbf{X}} \times \vec{\mathbf{Y}} = \frac{1}{2} [\mathbf{XY} - (\overline{\mathbf{XY}})]. \quad (9)$$

The norm process, which satisfies the commutative property, is given by multiplying the octonion and its conjugate in the following form:

$$N(\mathbf{X}) = \mathbf{X}\overline{\mathbf{X}} = \overline{\mathbf{X}}\mathbf{X} = \sum_{n=0}^7 x_n^2. \quad (10)$$

If the octonion is in the nonzero form, the multiplicative inverse \mathbf{X}^{-1} can be expressed by the conjugate and norm as follows:

$$\mathbf{X}^{-1} = \overline{\mathbf{X}}/N(\mathbf{X}). \quad (11)$$

A complex octonion \mathbb{C} is a combination of two real octonions \mathbf{X} and \mathbf{Y} with a new unit i :

$$\mathbb{C} = \mathbf{X} + i\mathbf{Y} = \sum_{n=0}^7 (x_n + iy_n)\mathbf{e}_n = \sum_{n=0}^7 c_n \mathbf{e}_n. \quad (12)$$

It is easily seen that the c_n terms are complex numbers and i represents the complex unit, which has the property $i^2 = -1$. As complex octonions have algebraic characteristics similar to those of the real ones, they differ from the others in terms of their components due to the additional complex unit i . Note that the additional complex conjugate occurs in a simultaneous way. Octonion conjugate $\overline{\mathbb{C}}$ and complex conjugate \mathbb{C}^* are written as follows [17, 18]:

$$\begin{aligned} \overline{\mathbb{C}} &= (x_0 + iy_0)\mathbf{e}_0 - (x_1 + iy_1)\mathbf{e}_1 - (x_2 + iy_2)\mathbf{e}_2 - (x_3 + iy_3)\mathbf{e}_3 \\ &\quad - (x_4 + iy_4)\mathbf{e}_4 - (x_5 + iy_5)\mathbf{e}_5 - (x_6 + iy_6)\mathbf{e}_6 - (x_7 + iy_7)\mathbf{e}_7, \end{aligned} \quad (13)$$

$$\begin{aligned} \mathbb{C}^* &= (x_0 - iy_0)\mathbf{e}_0 + (x_1 - iy_1)\mathbf{e}_1 + (x_2 - iy_2)\mathbf{e}_2 + (x_3 - iy_3)\mathbf{e}_3 \\ &\quad + (x_4 - iy_4)\mathbf{e}_4 + (x_5 - iy_5)\mathbf{e}_5 + (x_6 - iy_6)\mathbf{e}_6 + (x_7 - iy_7)\mathbf{e}_7. \end{aligned} \quad (14)$$

It is clear that the other algebraic properties are similar to the circular octonions. Note that they now have sixteen components due to the complex unit i .

3. Duality-invariant field equations for dyons

Let us consider the general case of the Maxwell equations written for electric and magnetic sources [51]. The existence of magnetic charges can be understood by means of Dirac's classical papers [52, 53]. In addition to

the ρ_e electric charge and \vec{J}_e current densities, one introduces ρ_m magnetic charge and \vec{J}_m current densities [16]. This equation set is written as follows:

$$\begin{aligned}
 \vec{\nabla} \cdot \vec{E} &= 4\pi\rho_e, \\
 \vec{\nabla} \cdot \vec{B} &= 4\pi\rho_m, \\
 \vec{\nabla} \times \vec{E} &= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} - \frac{4\pi}{c}\vec{J}_m, \\
 \vec{\nabla} \times \vec{B} &= \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}_e,
 \end{aligned} \tag{15}$$

where \vec{E} , \vec{B} , and c are the electric field, magnetic induction, and speed of light, respectively. Although the magnetic monopole is not experimentally observed, it is frequently discussed in mathematical physics, high energy physics, and solid state physics. In addition, the presence of the magnetic monopole is closely linked to the presence of the short-U (1) gauge group and specifies the C-invariability [7]. The monopole is better comprehended in non-Abelian gauge theories. Maxwell's equations, which are the fundamental structures for electromagnetism generated by both electric and magnetic fields, are also invariant under the duality transformations as well as Lorentz transformations. The duality transformations are frequently presented for the electromagnetic terms as follows:

$$\vec{E} \rightarrow \vec{E} \cos \theta + \vec{B} \sin \theta; \quad \vec{B} \rightarrow -\vec{E} \sin \theta + \vec{B} \cos \theta. \tag{16}$$

For the special case of $\theta = \frac{\pi}{2}$, they will transform into

$$\vec{E} \rightarrow \vec{B}; \quad \vec{B} \rightarrow -\vec{E}, \tag{17}$$

and by using the transformation matrices, Eqs. (16) and (17) can be briefly transcribed by using matrices as follows thanks to [2], [3], and [7]–[10]:

$$\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}, \tag{18a}$$

$$\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}. \tag{18b}$$

The most significant property of these transformation matrices is that the determinant is equal to +1. This means that the transformations remain unchanged. Many times, the other electromagnetic equalities can be given in the following form:

$$\begin{aligned}
 \rho_e &\longrightarrow \rho_m, & \rho_m &\longrightarrow -\rho_e, \\
 \vec{J}_e &\longrightarrow \vec{J}_m, & \vec{J}_m &\longrightarrow -\vec{J}_e, \\
 \phi_e &\longrightarrow \phi_m, & \phi_m &\longrightarrow -\phi_e, \\
 \vec{A}_e &\longrightarrow \vec{A}_m, & \vec{A}_m &\longrightarrow -\vec{A}_e.
 \end{aligned} \tag{19}$$

Because of the different spatial and temporal symmetries in terms of various physical systems, the Maxwell equations with monopole term denoted in Eq. (15) are invariant under the electromagnetic duality transformations in general forms [38]:

$$\begin{aligned}
\vec{E} &\rightarrow \alpha\vec{E} + \beta\vec{B}, \\
\vec{B} &\rightarrow \alpha\vec{B} - \beta\vec{E}, \\
\vec{J}_e &\rightarrow \alpha\vec{J}_e + \beta\vec{J}_m, \\
\vec{J}_m &\rightarrow \alpha\vec{J}_m - \beta\vec{J}_e, \\
\rho_e &\rightarrow \alpha\rho_e + \beta\rho_m, \\
\rho_m &\rightarrow \alpha\rho_m - \beta\rho_e,
\end{aligned} \tag{20}$$

where the α and β terms denote any parameters. After Coceal et al's work in [4] in terms of the electromagnetic duality form, Mironov and Mironov developed new duality notations for dyons that satisfy the transformation matrix equal to +1 [38].

In the case of dyons, the duality-invariant electric and magnetic fields can be defined by using Coceal's suggestion [4, 6] and reforming Mironov's developments [38] in the following form:

$$\vec{\mathcal{E}} = \frac{1}{\lambda}(\epsilon\vec{E} + g\vec{B}), \tag{21}$$

$$\vec{\mathcal{B}} = \frac{1}{\lambda}(-g\vec{E} + \epsilon\vec{B}). \tag{22}$$

Here, the duality-invariant fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are not the physical fields. They are quantities, which occurred from linear combinations of \vec{E} electric field and \vec{B} magnetic induction [4, 6, 38]. λ is the impact constant depending on electric and magnetic charges in the form of $\lambda = \sqrt{\epsilon^2 + g^2}$. In this sense, the determinant of the new transformation denoted in (21) and (22) is equal to +1 and plays the invariability role just as $\cos^2\theta + \sin^2\theta = +1$. In addition to the Maxwell equations, the electric-magnetic field strengths and Lorenz's gauge conditions defined by potential terms are given in the following expressions, respectively:

$$\vec{E} = -\vec{\nabla}\phi_e - \frac{1}{c}\frac{\partial\vec{A}_e}{\partial t} - \vec{\nabla} \times \vec{A}_m, \tag{23}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}_e - \vec{\nabla}\phi_m - \frac{1}{c}\frac{\partial\vec{A}_m}{\partial t}. \tag{24}$$

$$\vec{\nabla} \cdot \vec{A}_e + \frac{1}{c}\frac{\partial\phi_e}{\partial t} = 0, \tag{25}$$

$$\vec{\nabla} \cdot \vec{A}_m + \frac{1}{c}\frac{\partial\phi_m}{\partial t} = 0. \tag{26}$$

To describe the dyons it is convenient to introduce the renormalized sources:

$$\rho = \frac{\lambda}{e} \rho_e = \frac{\lambda}{g} \rho_m, \quad (27a)$$

$$\vec{J} = \frac{\lambda}{e} \vec{J}_e = \frac{\lambda}{g} \vec{J}_m, \quad (27b)$$

and the renormalized field potentials [38]:

$$\phi = \frac{\lambda}{e} \phi_e = \frac{\lambda}{g} \phi_m, \quad (28a)$$

$$\vec{A} = \frac{\lambda}{e} \vec{A}_e = \frac{\lambda}{g} \vec{A}_m. \quad (28b)$$

The Maxwell-like equations for dyons have been redefined in the following form [38]:

$$\begin{aligned} \vec{\nabla} \cdot \vec{\mathcal{E}} &= 4\pi\rho, \\ \vec{\nabla} \cdot \vec{\mathcal{B}} &= 0, \\ \vec{\nabla} \times \vec{\mathcal{E}} &= -\frac{1}{c} \frac{\partial \vec{\mathcal{B}}}{\partial t}, \\ \vec{\nabla} \times \vec{\mathcal{B}} &= \frac{1}{c} \frac{\partial \vec{\mathcal{E}}}{\partial t} + \frac{4\pi}{c} \vec{J}, \end{aligned} \quad (29)$$

where the ρ and \vec{J} terms indicate the duality-invariant electrical charge and current densities, respectively. Then the dyonic strengths of electric and magnetic terms can be rewritten as:

$$\begin{aligned} \vec{\mathcal{E}} &= -\vec{\nabla}\phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \\ \vec{\mathcal{B}} &= \vec{\nabla} \times \vec{A}. \end{aligned} \quad (30)$$

In addition, the Lorenz-like condition can be rearranged for dyons in duality-invariant form as:

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0. \quad (31)$$

Furthermore, an analogy can also be set in terms of the electromagnetic energy conservation between the classical electromagnetism and dyons. The energy conservation equation can be written as:

$$\vec{\nabla} \cdot (\vec{\mathcal{E}} \times \vec{\mathcal{B}}) = -\frac{1}{c} \vec{\mathcal{B}} \cdot \frac{\partial \vec{\mathcal{B}}}{\partial t} - \frac{1}{c} \vec{\mathcal{E}} \cdot \frac{\partial \vec{\mathcal{E}}}{\partial t} - \frac{4\pi}{c} \vec{\mathcal{E}} \cdot \vec{J}, \quad (32)$$

and, using the vector identity, the Poynting vector $\vec{\mathcal{S}}$ or the electromagnetic energy flux is defined as $\vec{\mathcal{S}} = \vec{\mathcal{E}} \times \vec{\mathcal{B}}$. Let \mathcal{U} be an electromagnetic energy density, and the time-dependent changing of this term $\frac{\partial \mathcal{U}}{\partial t}$ can be written as follows:

$$\frac{\partial \mathcal{U}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\mathcal{E}^2 + \mathcal{B}^2) = \frac{1}{2} \frac{\partial}{\partial t} (\vec{\mathcal{E}} \cdot \vec{\mathcal{E}} + \vec{\mathcal{B}} \cdot \vec{\mathcal{B}}). \quad (33)$$

By using electromagnetic energy flux and density, the Poynting theorem can be briefly written for dyons in the following expression:

$$\vec{\nabla} \cdot \vec{\mathcal{S}} + \frac{1}{c} \frac{\partial \mathcal{U}}{\partial t} = -\frac{4\pi}{c} \vec{\mathcal{E}} \cdot \vec{\mathcal{J}}. \quad (34)$$

Thus, the well-known energy conservation equation for classical electromagnetism via both electric and magnetic sources is obtained once again as follows:

$$\vec{\nabla} \cdot \vec{\mathcal{S}} + \frac{1}{c} \frac{\partial \mathcal{U}}{\partial t} = -\frac{4\pi}{c} (\vec{\mathcal{J}}_e \cdot \vec{\mathcal{E}} + \vec{\mathcal{J}}_m \cdot \vec{\mathcal{B}}). \quad (35)$$

It is easily seen that equations (34) and (35) are similar to each other with the above duality operations. In order to have more detailed expressions, one can use the equalities of dyonic systems in the framework of Mironov's paper [38].

4. Duality-invariant field equations in octonion form

In order to obtain the octonionic equations for dyons, the complex octonionic differential operator and its conjugate should be defined in Cayley–Dickson notation as

$$\square = \frac{i}{c} \frac{\partial}{\partial t} \mathbf{e}_0 + \frac{\partial}{\partial x} \mathbf{e}_5 + \frac{\partial}{\partial y} \mathbf{e}_6 + \frac{\partial}{\partial z} \mathbf{e}_7 \quad (36)$$

and

$$\square^* = -\frac{i}{c} \frac{\partial}{\partial t} \mathbf{e}_0 + \frac{\partial}{\partial x} \mathbf{e}_5 + \frac{\partial}{\partial y} \mathbf{e}_6 + \frac{\partial}{\partial z} \mathbf{e}_7. \quad (37)$$

The multiplication of these operators is shown as follows:

$$\square = \square \square^* = \square^* \square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \quad (38)$$

and this satisfies the commutativity where the ∇^2 or Δ symbol is the Laplace operator in Cartesian coordinates. The expressions (21) and (22) for dyons can be first suggested in the complex octonionic form as

$$\mathcal{E} = \frac{1}{\lambda} [\mathbf{e}\mathbf{E} + g(\mathbf{e}_4)\mathbf{B}], \quad (39)$$

$$\mathcal{B} = \frac{1}{\lambda} [-g(-\mathbf{e}_4)\mathbf{E} + \mathbf{e}\mathbf{B}], \quad (40)$$

where \mathcal{E} and \mathcal{B} represent the octonionic duality-invariant field terms. Using these definitions, the complex field octonion can be defined as

$$\mathbb{F} = \mathcal{E} + i\mathcal{B} = \mathcal{E}_x \mathbf{e}_1 + \mathcal{E}_y \mathbf{e}_2 + \mathcal{E}_z \mathbf{e}_3 + i\mathcal{B}_x \mathbf{e}_5 + i\mathcal{B}_y \mathbf{e}_6 + i\mathcal{B}_z \mathbf{e}_7 \quad (41)$$

or

$$\mathbb{F} = \frac{1}{\lambda} [\mathbf{e}\mathbf{E} + g(\mathbf{e}_4)\mathbf{B}] + \frac{i}{\lambda} [-g(-\mathbf{e}_4)\mathbf{E} + \mathbf{e}\mathbf{B}]. \quad (42)$$

This expression can be rewritten in a more clear form as

$$\begin{aligned} \mathbb{F} &= \frac{1}{\lambda} [e(E_x \mathbf{e}_1 + E_y \mathbf{e}_2 + E_z \mathbf{e}_3) + g(\mathbf{e}_4)(B_x \mathbf{e}_5 + B_y \mathbf{e}_6 + B_z \mathbf{e}_7)] \\ &+ \frac{i}{\lambda} [-g(-\mathbf{e}_4)(E_x \mathbf{e}_1 + E_y \mathbf{e}_2 + E_z \mathbf{e}_3) + e(B_x \mathbf{e}_5 + B_y \mathbf{e}_6 + B_z \mathbf{e}_7)] \end{aligned} \quad (43)$$

or

$$\begin{aligned} \mathbb{F} &= \frac{1}{\lambda} [e(E_x \mathbf{e}_1 + E_y \mathbf{e}_2 + E_z \mathbf{e}_3) + g(B_x \mathbf{e}_1 + B_y \mathbf{e}_2 + B_z \mathbf{e}_3)] \\ &+ \frac{i}{\lambda} [-g(E_x \mathbf{e}_5 + E_y \mathbf{e}_6 + E_z \mathbf{e}_7) + e(B_x \mathbf{e}_5 + B_y \mathbf{e}_6 + B_z \mathbf{e}_7)]. \end{aligned} \quad (44)$$

If the complex octonionic differential operator denoted in equation (36) is applied to the complex octonionic field given in equation (44), the following equality is obtained:

$$\begin{aligned} \square \mathbb{F} &= i\mathbf{e}_0 \frac{1}{\lambda} [g\vec{\nabla} \cdot \vec{E} - e\vec{\nabla} \cdot \vec{B}] + i\mathbf{e}_1 \frac{1}{\lambda} \left[\frac{e}{c} \frac{\partial E_x}{\partial t} + \frac{g}{c} \frac{\partial B_x}{\partial t} + g(\vec{\nabla} \times \vec{E})_x - e(\vec{\nabla} \times \vec{B})_x \right] \\ &+ i\mathbf{e}_2 \frac{1}{\lambda} \left[\frac{e}{c} \frac{\partial E_y}{\partial t} + \frac{g}{c} \frac{\partial B_y}{\partial t} + g(\vec{\nabla} \times \vec{E})_y - e(\vec{\nabla} \times \vec{B})_y \right] \\ &+ i\mathbf{e}_3 \frac{1}{\lambda} \left[\frac{e}{c} \frac{\partial E_z}{\partial t} + \frac{g}{c} \frac{\partial B_z}{\partial t} + g(\vec{\nabla} \times \vec{E})_z - e(\vec{\nabla} \times \vec{B})_z \right] \\ &+ \mathbf{e}_4 \frac{1}{\lambda} [e\vec{\nabla} \cdot \vec{E} + g\vec{\nabla} \cdot \vec{B}] + \mathbf{e}_5 \frac{1}{\lambda} \left[\frac{g}{c} \frac{\partial E_x}{\partial t} - \frac{e}{c} \frac{\partial B_x}{\partial t} - e(\vec{\nabla} \times \vec{E})_x - g(\vec{\nabla} \times \vec{B})_x \right] \\ &+ \mathbf{e}_6 \frac{1}{\lambda} \left[\frac{g}{c} \frac{\partial E_y}{\partial t} - \frac{e}{c} \frac{\partial B_y}{\partial t} - e(\vec{\nabla} \times \vec{E})_y - g(\vec{\nabla} \times \vec{B})_y \right] \\ &+ \mathbf{e}_7 \frac{1}{\lambda} \left[\frac{g}{c} \frac{\partial E_z}{\partial t} - \frac{e}{c} \frac{\partial B_z}{\partial t} - e(\vec{\nabla} \times \vec{E})_z - g(\vec{\nabla} \times \vec{B})_z \right]. \end{aligned} \quad (45)$$

Using divergence, curl, and time derivative expressions of (21) and (22), expression (45) can be represented in a simple form:

$$\begin{aligned} \square \mathbb{F} &= i\mathbf{e}_0 [-\vec{\nabla} \cdot \vec{B}] + i\mathbf{e}_1 \left[\frac{1}{c} \frac{\partial \mathcal{E}_x}{\partial t} - (\vec{\nabla} \times \vec{B})_x \right] + i\mathbf{e}_2 \left[\frac{1}{c} \frac{\partial \mathcal{E}_y}{\partial t} - (\vec{\nabla} \times \vec{B})_y \right] \\ &+ i\mathbf{e}_3 \left[\frac{1}{c} \frac{\partial \mathcal{E}_z}{\partial t} - (\vec{\nabla} \times \vec{B})_z \right] + \mathbf{e}_4 [\vec{\nabla} \cdot \vec{\mathcal{E}}] + \mathbf{e}_5 \left[-\frac{1}{c} \frac{\partial \mathcal{B}_x}{\partial t} - (\vec{\nabla} \times \vec{\mathcal{E}})_x \right] \\ &+ \mathbf{e}_6 \left[-\frac{1}{c} \frac{\partial \mathcal{B}_y}{\partial t} - (\vec{\nabla} \times \vec{\mathcal{E}})_y \right] + \mathbf{e}_7 \left[-\frac{1}{c} \frac{\partial \mathcal{B}_z}{\partial t} - (\vec{\nabla} \times \vec{\mathcal{E}})_z \right]. \end{aligned} \quad (46)$$

On the other hand, the relationship between the complexified octonion source and Maxwell-like equations can be correlated as

$$\mathbb{J} = i \frac{4\pi}{c} [-J_x \mathbf{e}_1 - J_y \mathbf{e}_2 - J_z \mathbf{e}_3] + 4\pi \rho \mathbf{e}_4 = 4\pi \left[-\frac{i}{c} \vec{J} + \rho \mathbf{e}_4 \right]. \quad (47)$$

The more compact and basic form between field and source can be written as

$$\square\mathbb{F} = \mathbb{J}. \quad (48)$$

A potential for the dyonic system can be defined in octonion form as

$$\mathbb{A} = -iA_x e_1 - iA_y e_2 - iA_z e_3 + \phi e_4 = -i\vec{A} + \phi e_4, \quad (49)$$

and then, if the operator described in equation (37) is operated to this potential, the following equality can be obtained:

$$\begin{aligned} \square^*\mathbb{A} = & e_1 \left[-\frac{\partial\phi}{\partial x} - \frac{1}{c} \frac{\partial A_x}{\partial t} \right] + e_2 \left[-\frac{\partial\phi}{\partial y} - \frac{1}{c} \frac{\partial A_y}{\partial t} \right] + e_3 \left[-\frac{\partial\phi}{\partial z} - \frac{1}{c} \frac{\partial A_z}{\partial t} \right] \\ & + i e_4 \left[-\vec{\nabla} \cdot \vec{A} - \frac{1}{c} \frac{\partial\phi}{\partial t} \right] + i e_5 \left[(\vec{\nabla} \times \vec{A})_x \right] + i e_6 \left[(\vec{\nabla} \times \vec{A})_y \right] + i e_7 \left[(\vec{\nabla} \times \vec{A})_z \right]. \end{aligned} \quad (50)$$

Using equations (30) and (31), the following expression can be written

$$\square^*\mathbb{A} = \mathcal{E}_x e_1 + \mathcal{E}_y e_2 + \mathcal{E}_z e_3 + i\mathcal{B}_x e_5 + i\mathcal{B}_y e_6 + i\mathcal{B}_z e_7. \quad (51)$$

Equation (51) is the same as equation (41) and it can be briefly rearranged as

$$\square^*\mathbb{A} = \mathbb{F}. \quad (52)$$

Also, if both sides of equation (52) are multiplied from the left by the operator in equation (36), it will be

$$\square\square^*\mathbb{A} = \square\mathbb{F}. \quad (53)$$

This equation can be shown using equations (38) and (48) as

$$\square\mathbb{A} = \mathbb{J}. \quad (54)$$

Hence, this expression will be referred to as the wave equation in octonion form and can be rewritten as

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right] \left[\phi e_4 - i\vec{A} \right] = 4\pi \left[\rho e_4 - \frac{i}{c} \vec{J} \right], \quad (55)$$

and then this dyonic wave equation can clearly be separated in terms of scalar and vectorial components:

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = 4\pi\rho, \quad (56a)$$

$$\frac{1}{c^2} \frac{\partial^2 A_x}{\partial t^2} - \frac{\partial^2 A_x}{\partial x^2} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} = \frac{4\pi}{c} J_x, \quad (56b)$$

$$\frac{1}{c^2} \frac{\partial^2 A_y}{\partial t^2} - \frac{\partial^2 A_y}{\partial x^2} - \frac{\partial^2 A_y}{\partial y^2} - \frac{\partial^2 A_y}{\partial z^2} = \frac{4\pi}{c} J_y, \quad (56c)$$

$$\frac{1}{c^2} \frac{\partial^2 A_z}{\partial t^2} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2} - \frac{\partial^2 A_z}{\partial z^2} = \frac{4\pi}{c} J_z. \quad (56d)$$

It is understood that the wave equation for dyons gains a different and significant form.

5. Octonionic energy conservation for dyons

In order to obtain the energy conservation law for dyons, the complex conjugate of the octonionic field equation given in equation (41) should be defined as

$$\mathbb{F}^* = \mathcal{E} - i\mathcal{B} = \mathcal{E}_x e_1 + \mathcal{E}_y e_2 + \mathcal{E}_z e_3 - i\mathcal{B}_x e_5 - i\mathcal{B}_y e_6 - i\mathcal{B}_z e_7. \quad (57)$$

Using the property of the scalar product of two octonions as shown in equation (8), $\mathbb{F}^* \cdot (\square\mathbb{F})$ and $\mathbb{F}^* \cdot \mathbb{J}$ terms should be obtained. The scalar product rule for a complex octonion is parallel to a circular one. Hence, the following equality can be written:

$$\begin{aligned} \mathbb{F}^* \cdot (\square\mathbb{F}) &= \mathbb{F}^* \cdot \mathbb{J} \\ -\frac{1}{2} \left[\mathbb{F}^*(\square\mathbb{F}) + \overline{\mathbb{F}^*(\square\mathbb{F})} \right] &= -\frac{1}{2} \left[\mathbb{F}^*\mathbb{J} + \overline{\mathbb{F}^*\mathbb{J}} \right] \end{aligned} \quad (58)$$

As the first step of this operation, the left side of equation (58) can be multiplied in octonion form. The product of $\mathbb{F}^*(\square\mathbb{F})$ and $\overline{\mathbb{F}^*(\square\mathbb{F})}$ terms is obtained in the following line:

$$\mathbb{F}^*(\square\mathbb{F}) = \overline{\mathbb{F}^*(\square\mathbb{F})} = i\mathbf{e}_0 \left[\frac{1}{c} \vec{\mathcal{E}} \cdot \frac{\partial \vec{\mathcal{E}}}{\partial t} - \vec{\mathcal{E}} \cdot (\vec{\nabla} \times \vec{\mathcal{B}}) + \frac{1}{c} \vec{\mathcal{B}} \cdot \frac{\partial \vec{\mathcal{B}}}{\partial t} + \vec{\mathcal{B}} \cdot (\vec{\nabla} \times \vec{\mathcal{E}}) \right]. \quad (59)$$

Then the following expression is also attained for the right side of equation (58) in a similar way by applying the octonion product rule:

$$\mathbb{F}^*\mathbb{J} = \overline{\mathbb{F}^*\mathbb{J}} = i\mathbf{e}_0 \left[-\frac{4\pi}{c} \vec{\mathcal{E}} \cdot \vec{\mathcal{J}} \right]. \quad (60)$$

It can be easily seen from equations (59) and (60) that all components in the same basis element without the \mathbf{e}_0 scalar unit zeroize each other. By using equations (59) and (60), the following equality can be rearranged in a complex scalar basis element as

$$i\mathbf{e}_0 \left[\frac{1}{c} \vec{\mathcal{E}} \cdot \frac{\partial \vec{\mathcal{E}}}{\partial t} - \vec{\mathcal{E}} \cdot (\vec{\nabla} \times \vec{\mathcal{B}}) + \frac{1}{c} \vec{\mathcal{B}} \cdot \frac{\partial \vec{\mathcal{B}}}{\partial t} + \vec{\mathcal{B}} \cdot (\vec{\nabla} \times \vec{\mathcal{E}}) \right] = i\mathbf{e}_0 \left[-\frac{4\pi}{c} \vec{\mathcal{E}} \cdot \vec{\mathcal{J}} \right], \quad (61)$$

and then, using electromagnetic energy flux and density, the Poynting theorem can be briefly written for dyons in the following expression denoted in equation (34) once again:

$$\vec{\nabla} \cdot \vec{\mathcal{S}} + \frac{1}{c} \frac{\partial \mathcal{U}}{\partial t} = -\frac{4\pi}{c} \vec{\mathcal{E}} \cdot \vec{\mathcal{J}}. \quad (62)$$

At the end of this process, it is easily seen that the duality-invariant field strengths (21) and (22) proposed by Mironov are more correct. In this way, the equality $\mathcal{E}^2 + \mathcal{B}^2 = E^2 + B^2$ is obtained between classical electrodynamics and dyonic systems, and then the energy conservation is confirmed for two systems by using the result of the determinant equal to 1 by means of the transformation matrix elements.

6. Conclusion

The electromagnetic equations are used in not only classical electromagnetism but also other subfields of physics such as wave theory, magnetohydrodynamics, geophysics, and astrophysics. The duality-invariant equalities were studied by Coceal et al. and then a different perspective was presented for physical science. After Coceal's paper, Mironov and Mironov presented duality-invariant equalities for electromagnetism by using the sedeon structure, which is a space-time algebra having 16 components and associative properties [38]. The duality-invariant expressions have not been studied by using eight components and an alternative division ring such as octonions yet. In this context, this study fills a gap in the literature.

The dyonic field equations in octonionic form have been written with duality-invariant quantities $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ as a linear combination of the physical \vec{E} and \vec{B} fields using the λ coefficient by including charge terms for dyons. Note that this λ factor shows the dyonic system to be invariable under the duality transformations. We used octonionic combinations of the electric field and magnetic flux density with a factor (e_4). The reason for this contribution is that physically octonionic fields \mathbf{E} and \mathbf{B} cannot be described on the same basis.

We investigated the dyonic equations in complex octonion algebra and the results were compared with the classical equations. The law of energy conservation, known as the Poynting theorem, has been obtained for dyons and written via octonion algebra in a different and alternative manner. Due to the usage of the λ parameter for dyonic strengths, it has shown neither increases nor decreases in terms of energy. In other words, the energy is conserved all the time, even if it transforms from one to another type. As a consequence, it is seen that the equalities between the duality-invariant and classical ones are presented in an alternative and elegant way via octonion algebra, which has the last member of the division ring in a high-dimensional mathematical structure.

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