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Spin (Pseudospin) doublet in view of energy-dependent potential

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Abstract: Behavior of Dirac particles in the presence of scalar, vector, and tensor potentials respectively represented by energy-dependent Morse and Coulomb-like potentials is examined by working out the Dirac equation under the condition of spin (pseudospin) symmetry. The closed form of the energy eigenvalue equation and corresponding wave functions in terms of hypergeometric functions are acquired by making use of the asymptotic iteration method. We investigate the effect of energy-dependent potential on both bound states and energy splitting between members of spin (pseudospin) doublets.

Key words: Dirac equation, energy dependent potential, pseudospin and spin doublets

1. Introduction

Since the reducing Dirac equation with a fermion for a scalar or a vector potential to the Pauli–Schrödinger equation [1], wave equations including energy-dependent potential have been studied in nonrelativistic and relativistic quantum mechanics [2–6]. In order to describe a heavy quark system, wave equations with an energy-dependent potential have been studied [7]. In a recent study, Martinez et al. introduced a technique including energy-dependent potential to solve the Schrödinger equation exactly [8]. Bound states for nonrelativistic particles in view of an energy-dependent potential have also been investigated [9]. Yekken et al. recently analyzed the supersymmetry features of energy-dependent harmonic oscillators and Pöschl–Teller potentials in a one-dimensional space [10]. In order to describe the subbarrier fusion cross-section of diverse systems, the Woods–Saxon potential having energy dependency has been used in a recent paper [11]. Moreover, various authors have recently proposed and/or used energy-dependent potentials to investigate different physical systems and physical concepts [12–16].

Quasidegeneracy was experimentally monitored in nuclei amongst states with $(\mathbf{nlj}-\frac{\mathbf{1}}{\mathbf{2}}=\mathbf{l})$ and $(\mathbf{n}-\mathbf{l}+\mathbf{2j}-\frac{\mathbf{3}}{\mathbf{2}}=\mathbf{l})$ for the single nucleon. Here, $\mathbf{j}$, $\mathbf{nl}$ represent quantum numbers of the total angular, radial, and orbital angular momentums, respectively [17,18]. The notion of pseudospin $\tilde{s}=\mathbf{1/2}$ was defined [17,18] to elucidate this quasidegeneracy in view of nonrelativistic quantum mechanics. $\tilde{s}$ (pseudospin) and $\tilde{l}$ (pseudo-orbital angular momentum) concepts were introduced to define a pseudospin doublet with total angular momentum $\mathbf{j}=\tilde{s}+\tilde{l}$. Identical bands, magnetic moment, deformation, and superdeformation in nuclei were successfully expounded by considering the pseudospin concept [19–25]. In spite of knowing that the pseudospin initially was a nonrelativistic concept, its relativistic characteristic was discovered [26]; Ginocchio

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reported that usual $\mathbf{l}$ of the lower part of the Dirac spinor corresponds to the pseudo-orbital angular momentum. Thenceforth, the pseudospin concept has been also examined in view of relativistic quantum mechanics. The condition $\frac{\mathbf{d\Sigma(r)}}{\mathbf{dr}}=\frac{\mathbf{d(V(r)+S(r))}}{\mathbf{dr}}=\mathbf{0}$, where $\mathrm{V}(\mathbf{r})$ denotes a repulsive vector potential and $\mathbf{S(r)}$ is an attractive scalar potential, corresponds to the exact pseudospin symmetry in the Dirac equation [27,28], while the case $\frac{\mathbf{d\Delta(r)}}{\mathbf{dr}}=\frac{\mathbf{d(S(r)-V(r))}}{\mathbf{dr}}=\mathbf{0}$ corresponds to exact spin symmetry, which is generally applied for mesons [29]. On the other hand, the efficacy of the tensor potential on spin (pseudospin) doublets has been recently studied [30–36]. Those authors [30–36] showed that the tensor potential removes degeneracies between the energy states that form a pseudospin or spin doublet.

The Morse potential describing the interaction between two atoms (or molecules) is given as [37]

$$V_M(r) = D_e\left(e^{-2\delta(r-r_0)} - 2e^{-\delta(r-r_0)}\right), \tag{1}$$

where the dissociation energy, equilibrium distance, and a parameter related to the width of potential well are represented with $\mathbf{D_e}$, δ, and $\mathbf{r_0}$, respectively. The Morse potential solution of the Dirac equation under pseudospin symmetry has been found by using the methods of Nikiforov–Uvarov [38], exact quantization rule [39], and asymptotic iteration method (AIM) [40]. Afterwards, Aydogdu and Sever approximately solved the Dirac equation for the mentioned potential together with Coulomb-like tensor potential under the circumstances of pseudospin and spin symmetry [41].

In the present study, the Dirac equation with energy-dependent scalar, vector, and tensor potentials for condition of spin (pseudospin) symmetry is solved analytically in view of AIM. The efficacy of the energy-dependent potentials on both bound states and spin (pseudospin) doublets is investigated. To do this, we use the scalar and vector energy-dependent Morse (EDM) and energy-dependent Coulomb-like (EDCL) tensor potentials having the following forms:

$$V_{EDM}(r,\, E_{n\kappa}) = D_e\left(1+\gamma_1 E_{n\kappa}^{\sigma_1}\right)\left(e^{-2\delta(r-r_0)} - 2e^{-\delta(r-r_0)}\right), \tag{2}$$

$$U(r,\, E_{n\kappa}) = V_{EDCL}(r,\, E_{n\kappa}) = -\frac{\tau(1+\gamma_2 E_{n\kappa}^{\sigma_2})}{r}, \tag{3}$$

where γ_1, γ_2, σ_1, and σ_2 are real constants. The above forms of energy-dependent potential are chosen in order to obtain an analytic solution of the Dirac equation with the case of spin (pseudospin) symmetry. In addition, it has been shown that the coupling constant that leads to a coherent theory is affected by energy-dependent potential [6,9]. Our principal task is to investigate under which condition Dirac particles compose a bound state in the presence of energy-dependent potential. Secondly, we show how the energy-dependent potential affects energy splitting between members of spin (pseudospin) doublets in view of the tensor potential.

The present paper has the following structure. In the next section, starting with the Dirac equation for energy-dependent scalar, vector, and tensor potentials, we get the second-order differential equation by making use of spin (pseudospin) symmetry. In Section 3, spin and pseudospin symmetric solutions of second-order differential equation are put forward. Section 4 is reserved to interpret and discuss the results. A summary and the conclusion are given in the final section.

2. The Dirac equation with energy-dependent potentials

When a particle with mass M and $spin-1/2$ moves in energy-dependent vector $V(r,\,E_{n\kappa})$, scalar $S(r,\,E_{n\kappa})$, and tensor $U(r,\,E_{n\kappa})$ potentials, it is described by the following differential equation [30]:

$$\left[\alpha\cdot\mathbf{p}+\beta\left(M+S(r,\,E_{n\kappa})\right)+V(r,\,E_{n\kappa})-i\beta\alpha\cdot\hat{r}U(r,\,E_{n\kappa})\right]\Psi_{n\kappa}(r,\,E_{n\kappa})=E_{n\kappa}\Psi_{n\kappa}(r,\,E_{n\kappa}), \tag{4}$$

where $E_{n\kappa}$ and $\mathbf{p}=-i\nabla$ represent the total relativistic energy and momentum operator in three dimensions, respectively. Here we use natural units: $\hbar=c=1$. Furthermore, the Dirac spinor can be taken in the following form:

$$\Psi_{n\kappa}(\vec{r},\,E_{n\kappa})=\begin{pmatrix}\frac{F_{n\kappa}(r,\,E_{n\kappa})}{r}Y_{jm}^{l}(\theta,\,\varphi)\\ i\frac{G_{n\kappa}(r,\,E_{n\kappa})}{r}Y_{jm}(\theta,\,\varphi)\end{pmatrix}, \tag{5}$$

where $Y_{jm}^{l}(\theta,\,\varphi)$ represents the spherical harmonic for spin while $Y_{jm}(\theta,\,\varphi)$ is the spherical harmonic for pseudospin. Here $G_{n\kappa}(r,\,E_{n\kappa})$ and $F_{n\kappa}(r,\,E_{n\kappa})$ are the lower and upper radial wave functions, respectively. Inserting Eq. (5) into Eq. (4) and considering the following relations [42]

$$(\sigma\cdot\mathbf{A})(\sigma\cdot\mathbf{B})=\mathbf{A}\cdot\mathbf{B}+i\sigma\cdot(\mathbf{A}\times\mathbf{B}), \tag{6}$$

$$\sigma\cdot\mathbf{p}=\sigma\cdot\hat{r}\left(\hat{r}\cdot\mathbf{p}+i\frac{\sigma\cdot\mathbf{L}}{r}\right), \tag{7}$$

with the properties [42]

$$\sigma\cdot\mathbf{L}\begin{Bmatrix}Y_{jm}^{\tilde{l}}(\theta,\,\varphi)\\ Y_{jm}^{l}(\theta,\,\varphi)\end{Bmatrix}=\begin{Bmatrix}(\kappa-1)Y_{jm}^{\tilde{l}}(\theta,\,\varphi)\\ -(\kappa+1)Y_{jm}^{l}(\theta,\,\varphi)\end{Bmatrix} \tag{8}$$

$$\frac{\sigma\cdot\vec{r}}{r}\begin{Bmatrix}Y_{jm}^{\tilde{l}}(\theta,\,\varphi)\\ Y_{jm}^{l}(\theta,\,\varphi)\end{Bmatrix}=\begin{Bmatrix}-Y_{jm}^{l}(\theta,\,\varphi)\\ -Y_{jm}^{\tilde{l}}(\theta,\,\varphi)\end{Bmatrix} \tag{9}$$

Eq. (4) is converted into a set of two coupled differential equations in the first-order for the lower and upper radial wave functions:

$$\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r,\,E_{n\kappa})\right)F_{n\kappa}(r,\,E_{n\kappa})=\left[M+E-\Delta(r,\,E_{n\kappa})\right]G_{n\kappa}(r,\,E_{n\kappa}) \tag{10}$$

$$\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r,\,E_{n\kappa})\right)G_{n\kappa}(r,\,E_{n\kappa})=\left[M-E+\Sigma(r,\,E_{n\kappa})\right]F_{n\kappa}(r,\,E_{n\kappa}), \tag{11}$$

where

$$\Delta(r,\,E_{n\kappa})=V(r,\,E_{n\kappa})-S(r,\,E_{n\kappa}), \tag{12}$$

$$\Sigma(r,\,E_{n\kappa})=V(r,\,E_{n\kappa})+S(r,\,E_{n\kappa}). \tag{13}$$

Eliminating $F_{n\kappa}(r,\,E_{n\kappa})$ from Eq. (10) and $G_{n\kappa}(r,\,E_{n\kappa})$ from Eq. (11), one can obtain the following second-order differential equations:

$$\begin{aligned}&\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa+1)}{r^2}+\left(\frac{2\kappa}{r}-U(r,\,E_{n\kappa})-\frac{d}{dr}\right)U(r,\,E_{n\kappa})+\frac{\frac{d\Delta(r,\,E_{n\kappa})}{dr}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r,\,E_{n\kappa})\right)}{M+E-\Delta(r,\,E_{n\kappa})}\right]F_{n\kappa}(r,\,E_{n\kappa})\\ &+\left(E+M-\Delta(r,\,E_{n\kappa})\right)\left(E-M-\Sigma(r,\,E_{n\kappa})\right)F_{n\kappa}(r,\,E_{n\kappa})=0\end{aligned} \tag{14}$$

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}+\left(\frac{2\kappa}{r}-U(r,\,E_{n\kappa})+\frac{d}{dr}\right)U(r,\,E_{n\kappa})-\frac{\frac{d\Sigma(r,\,E_{n\kappa})}{dr}\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r,\,E_{n\kappa})\right)}{M-E+\Sigma(r,\,E_{n\kappa})}\right]G_{n\kappa}(r,\,E_{n\kappa}) \\ +\left(E+M-\Delta(r,\,E_{n\kappa})\right)\left(E-M-\Sigma(r,\,E_{n\kappa})\right)G_{n\kappa}(r,\,E_{n\kappa})=0 \tag{15}$$

where $\kappa(\kappa-1)=(+1)$ and $\kappa(\kappa+1)=l(l+1)$. Our procedure for solving Eqs. (14) and (15) is to find eigenvalue equations and corresponding wave functions for the Dirac particles in view of energy-dependent potential.

3. Solution of the second-order differential equation

The pseudospin and spin symmetric solutions of Eqs. (14) and (15) are presented in this section. The AIM is used in the calculations. Ciftci et al. [43] introduced the AIM in order to solve the following type second-order differential equations:

$$\frac{d^2}{dz^2}f_n(z)=\left(\lambda_0(z)\frac{d}{dz}+s_0(z)\right)f_n(z). \tag{16}$$

Here we give only the fundamental ingredients of the method. The reader can find more details about the method in the literature [43].

Fundamental ingredients of the AIM:

i) $\lambda_0(z)\neq 0$,

ii) $\lambda_0(z)$ and $s_0(z)$ are sufficiently differentiable,

iii) $\lambda_k(z)=\frac{d}{dz}\lambda_{k-1}(z)+s_{k-1}(z)+\lambda_0(z)\lambda_{k-1}(z)$ is the recurrence relation for the $\lambda_k(z)$,

iv) $s_k(z)=\frac{d}{dz}s_{k-1}(z)+s_0(z)\lambda_{k-1}(z)$ is known as the recurrence relation for the $s_k(z)$,

v) $\frac{s_k(z)}{\lambda_k(z)}=\frac{s_{k-1}(z)}{\lambda_{k-1}(z)}=\alpha(z)$ is the asymptotic sight of method,

vi) $\delta_k(z)=\lambda_k(z)s_{k-1}(z)-\lambda_{k-1}(z)s_k(z)=0$, $k=1,\,2,\,3...$ gives the eigenvalues equation,

vii) k denotes the iteration number,

viii) $f_n(z)=C\exp\left(-\int^z\frac{s_n(z_1)}{\lambda_n(z_1)}\right)$ is the wave function generator,

ix) n is the radial quantum number.

3.1. Exact spin symmetric solution

Spin symmetry appears exactly in the Dirac equation if $\frac{d\Delta(r)}{dr}=0$, namely $\Delta(r)$ is a constant. Considering potentials in the forms of

$$V(r,\,E_{n\kappa})+S(r,\,E_{n\kappa})=\Sigma(r,\,E_{n\kappa})=\varepsilon_1e^{-2\delta(r-r_0)}-2\varepsilon_1e^{-\delta(r-r_0)}, \tag{17}$$

$$U(r,\,E_{n\kappa})=-\frac{\varepsilon_2}{r}, \tag{18}$$

where $\varepsilon_1 = D_e(1+\gamma_1 E^{\sigma_1})$, $\varepsilon_2 = \tau(1+\gamma_2 E^{\sigma_2})$ and the exact spin symmetry $\Delta(r) = C_s = \text{constant}$, Eq. (14) becomes

$$\begin{aligned}&\left(\tfrac{d^2}{dr^2} - \tfrac{(\kappa+\varepsilon_2)(\kappa+\varepsilon_2+1)}{r^2} + (E_{n\kappa}+M-C_s)(E_{n\kappa}-M)\right.\\&\left.+(E_{n\kappa}+M-C_s)\left(-\varepsilon_1 e^{-2\delta(r-r_0)} + 2\varepsilon_1 e^{-\delta(r-r_0)}\right)\right) F_{n\kappa}(r,\, E_{n\kappa}) = 0\end{aligned} \tag{19}$$

Because of the centrifugal term, one cannot find the exact solution of Eq. (19). Therefore, we use the Pekeris approximation [44] to overcome this obstacle.

Pekeris approximation:

The centrifugal term can be expanded in a series around $x = 0$:

$$V_{CT}(x,\, E_{n\kappa}) = \frac{(\kappa+\varepsilon_2)(\kappa+\varepsilon_2+1)}{r_0^2\left(1+\frac{x}{r_0}\right)^2} = \frac{(\kappa+\varepsilon_2)(\kappa+\varepsilon_2+1)}{r_0^2}\left(1-2\frac{x}{r_0}+3\frac{x^2}{r_0^2}+O\left(\frac{x^3}{r_0^3}\right)\right). \tag{20}$$

Here we firstly change the coordinate as $r - r_0 = x$. It should be noted that the above expansion is viable only for low vibrational energy states. This means that one can use the approximation near the minimum point $r \approx r_0$. Therefore, anyone can take expansion terms up to 2nd order [45]. Moreover, one can consider the centrifugal term in the following form:

$$\tilde{V}_{CT}(x,\, E_{n\kappa}) = \frac{(\kappa+\varepsilon_2)(\kappa+\varepsilon_2+1)}{r_0^2}\left(\chi_0 + \chi_1 e^{-\delta x} + \chi_2 e^{-2\delta x}\right), \tag{21}$$

where χ_0, χ_1, and χ_2 are constants. Performing series expansion of Eq. (21) up to second order and comparing this expansion with Eq. (20), it is easily obtained that

$$\chi_0 = 1 - \frac{3}{\delta r_0} + \frac{3}{\delta^2 r_0^2}, \tag{22}$$

$$\chi_1 = \frac{4}{\delta r_0} - \frac{6}{\delta^2 r_0^2}, \tag{23}$$

$$\chi_2 = -\frac{1}{\delta r_0} + \frac{3}{\delta^2 r_0^2}. \tag{24}$$

Henceforth, we are going to use the $\tilde{V}_{CT}(x,\, E_{n\kappa})$ instead of $V_{CT}(x,\, E_{n\kappa})$ to get an approximate solution of Eq. (19).

Introducing a new variable $y = e^{-\delta x}$ and then using the factorization $F_{n\kappa}(y,\, E_{n\kappa}) = y^{\Lambda_1} e^{-\Lambda_3 y} f_{n\kappa}(y,\, E_{n\kappa})$, Eq. (19) becomes

$$\frac{d^2}{dy^2} f_{n\kappa}(y,\, E_{n\kappa}) = \left(\frac{2\Lambda_3 y - (2\Lambda_1+1)}{y}\frac{d}{dy} + \frac{(2\Lambda_1+1)\Lambda_3 - \Lambda_2^2}{y}\right) f_{n\kappa}(y,\, E_{n\kappa}), \tag{25}$$

where

$$\Lambda_1^2 = \frac{(\kappa+\varepsilon_2)(\kappa+\varepsilon_2+1)}{r_0^2\delta^2}\chi_0 - \frac{(E_{n\kappa}+M-C_s)(E_{n\kappa}-M)}{\delta^2}, \tag{26}$$

$$\Lambda_2^2 = -\frac{(\kappa+\varepsilon_2)(\kappa+\varepsilon_2+1)}{r_0^2\delta^2}\chi_1 + \frac{(E_{n\kappa}+M-C_s)2\varepsilon_1}{\delta^2}, \tag{27}$$

$$\Lambda_3^2 = \frac{(\kappa+\varepsilon_2)(\kappa+\varepsilon_2+1)}{r_0^2\delta^2}\chi_2 + \frac{(E_{n\kappa}+M-C_s)\varepsilon_1}{\delta^2}. \tag{28}$$

Comparing Eq. (25) with Eq. (16) and using the recurrence relations given in the fundamental ingredients, we obtain that

$$\lambda_0(y) = \frac{2\Lambda_3 y - (2\Lambda_1+1)}{y}, \tag{29}$$

$$s_0(y) = \frac{(2\Lambda_1+1)\Lambda_3 - \Lambda_2^2}{y}, \tag{30}$$

$$\lambda_1(y) = \frac{2+4\Lambda_1^2-\Lambda_2^2 y-3\Lambda_3 y+4\Lambda_3^2 y^2+6\Lambda_1-6\Lambda_1\Lambda_3 y}{y^2}, \tag{31}$$

$$s_1(y) = \frac{2(1+\Lambda_1-\Lambda_3 y)(\Lambda_2^2-\Lambda_3-2\Lambda_1\Lambda_3)}{y^2}, \tag{32}$$

$$\dots$$

Substituting $\lambda_0, \lambda_1, \lambda_2, \dots$ and $s_0, s_1, s_2, \dots$ into the quantization condition of AIM $\lambda_k(z)s_{k-1}(z) - \lambda_{k-1}(z)s_k(z) = 0$ gives

$$\Lambda_{10} = \frac{\Lambda_2^2-\Lambda_3}{2\Lambda_3} \; for\; k=1, \tag{33}$$

$$\Lambda_{11} = \frac{\Lambda_2^2-3\Lambda_3}{2\Lambda_3} \; for\; k=2, \tag{34}$$

$$\Lambda_{12} = \frac{\Lambda_2^2-5\Lambda_3}{2\Lambda_3} \; for\; k=3, \tag{35}$$

$$\dots$$

$$\Lambda_{1n} = \frac{\Lambda_2^2-(2n+1)\Lambda_3}{2\Lambda_3} \; for\; k=n-1. \tag{36}$$

The last equation generates the energy eigenvalue equation as follows:

$$\begin{aligned}&\left(\frac{(C_s-E_{n\kappa}-M)2r_0^2\varepsilon_1+(\kappa+\varepsilon_2)(\kappa+\varepsilon_2+1)\chi_1}{\left((C_s-E_{n\kappa}-M)r_0^2\varepsilon_1-(\kappa+\varepsilon_2)(\kappa+\varepsilon_2+1)\chi_2\right)^{1/2}}\right.\\&\left.+2\left((E_{n\kappa}+M-C_s)(E_{n\kappa}-M)r_0^2-(\kappa+\varepsilon_2)(\kappa+\varepsilon_2+1)\chi_0\right)^{1/2}\right)^2+\delta^2 r_0^2(2n+1)^2=0\end{aligned} \tag{37}$$

The corresponding eigenfunctions are calculated by making use of the wave functions generator given in the fundamental ingredients. Eigenfunctions of relativistic particles in the existence of energy-dependent scalar, vector, and tensor potentials are found with a little calculation as follows:

$$F_{n\kappa}(r,\, E_{n\kappa}) = Ne^{-\delta\Lambda_1(r-r_0)}e_1^{-\Lambda_3 e^{-\delta(r-r_0)}}F_1\left(-n,\, 1+2\Lambda_1,\, 2\Lambda_3 e^{-\delta(r-r_0)}\right). \tag{38}$$

Here the normalization constant is represented by N. It is important to mention here that Λ_1, Λ_2, and Λ_3 include $E_{n\kappa}$. One can easily see that, using Eq. (10), the other component of the Dirac spinor can be found in the framework of spin symmetry.

3.2. Exact pseudospin symmetric solution

Pseudospin symmetry appears exactly in the Dirac equation if $\frac{d\Sigma(r)}{dr} = 0$, namely $\Sigma(r)$ is a constant and it is defined as C_{ps}. Considering potentials in the following form

$$S(r, E_{n\kappa}) - V(r, E_{n\kappa}) = \Delta(r, E_{n\kappa}) = \varepsilon_1 e^{-2\delta(r-r_0)} - 2\varepsilon_1 e^{-\delta(r-r_0)}, \tag{39}$$

$$U(r, E_{n\kappa}) = -\frac{\varepsilon_2}{r}, \tag{40}$$

and using the exact pseudospin symmetry $\Sigma(r) = C_{ps} = \text{constant}$, Eq. (15) becomes

$$\begin{aligned} &\left(\tfrac{d^2}{dr^2} - \tfrac{(\kappa+\varepsilon_2)(\kappa+\varepsilon_2-1)}{r^2} + (E_{n\kappa} - M - C_{ps})(E_{n\kappa} + M)\right. \\ &\left. +(E_{n\kappa} - M - C_{ps})\left(-\varepsilon_1 e^{-2\delta(r-r_0)} + 2\varepsilon_1 e^{-\delta(r-r_0)}\right)\right) G_{n\kappa}(r, E_{n\kappa}) = 0 \end{aligned} \tag{41}$$

Because of the centrifugal term, the exact solution of Eq. (41) cannot be obtained. We use again the Pekeris approximation to remove this obstruction as in the case of the spin symmetric solution. Utilizing the Pekeris approximation with $y = e^{-\delta x}$ and then introducing factorization $G_{n\kappa}(y, E_{n\kappa}) = y^{\tilde{\Lambda}_1} e^{-\tilde{\Lambda}_3 y} g_{n\kappa}(y, E_{n\kappa})$, Eq. (41) becomes

$$\frac{d^2}{dy^2} g_{n\kappa}(y, E_{n\kappa}) = \left(\frac{2\tilde{\Lambda}_3 y - (2\tilde{\Lambda}_1 + 1)}{y}\frac{d}{dy} + \frac{(2\tilde{\Lambda}_1 + 1)\tilde{\Lambda}_3 - \tilde{\Lambda}_2^2}{y}\right) g_{n\kappa}(y, E_{n\kappa}), \tag{42}$$

where

$$\tilde{\Lambda}_1^2 = \frac{(\kappa+\varepsilon_2)(\kappa+\varepsilon_2-1)}{r_0^2\delta^2}\chi_0 - \frac{(E_{n\kappa} - M - C_{ps})(E_{n\kappa} + M)}{\delta^2}, \tag{43}$$

$$\tilde{\Lambda}_2^2 = -\frac{(\kappa+\varepsilon_2)(\kappa+\varepsilon_2-1)}{r_0^2\delta^2}\chi_1 + \frac{(E_{n\kappa} - M - C_{ps})2\varepsilon_1}{\delta^2}, \tag{44}$$

$$\tilde{\Lambda}_3^2 = \frac{(\kappa+\varepsilon_2)(\kappa+\varepsilon_2-1)}{r_0^2\delta^2}\chi_2 + \frac{(E_{n\kappa} - M - C_{ps})\varepsilon_1}{\delta^2}. \tag{45}$$

It is easily realized that Eq. (41) has a similar form to Eq. (25) obtained for the spin symmetric solution. It is also possible to perform mapping among $\tilde{\Lambda}_i$ and Λ_i where $i = 1, 2, 3$ as:

mapping	*outcome*
$\varepsilon_2 \to \varepsilon_2 - 1$	$\Lambda_1^2 \to \tilde{\Lambda}_1^2$
$M \to -M$	$\Lambda_2^2 \to \tilde{\Lambda}_2^2$
$C_s \to C_{ps}$	$\Lambda_3^2 \to \tilde{\Lambda}_3^2$

As a consequence of this, the energy eigenvalue equation and corresponding wave functions of the Dirac particles in view of energy-dependent scalar, vector, and tensor potentials in the framework of pseudospin symmetry can be immediately written from Eqs. (37) and (38) by using the above mapping:

$$\left(\frac{(C_{ps}-E_{n\kappa}+M)2r_0^2\varepsilon_1+(\kappa+\varepsilon_2)(\kappa+\varepsilon_2-1)\chi_1}{\left((C_{ps}-E_{n\kappa}+M)r_0^2\varepsilon_1-(\kappa+\varepsilon_2)(\kappa+\varepsilon_2-1)\chi_2\right)^{1/2}}\right.$$
$$\left.+2\left((E_{n\kappa}-M-C_{ps})(E_{n\kappa}+M)r_0^2-(\kappa+\varepsilon_2)(\kappa+\varepsilon_2-1)\chi_0\right)^{1/2}\right)^2+\delta^2r_0^2(2n+1)^2=0 \tag{46}$$

$$G_{n\kappa}(r,\,E_{n\kappa})=Ne^{-\delta\tilde{\Lambda}_1(r-r_0)}e_1^{-\tilde{\Lambda}_3e^{-\delta(r-r_0)}}F_1\left(-n,\,1+2\tilde{\Lambda}_1,\,2\tilde{\Lambda}_3e^{-\delta(r-r_0)}\right), \tag{47}$$

where N denotes a normalization parameter.

4. Results

4.1. Correctness of the energy eigenvalue equations

The energy eigenvalue equations (37) and (47) turn out to be more tangled and they require careful analysis. On the other hand, our results' accuracy can be checked by comparing them with the results published previously.

Spin Symmetry:

If we ignore the energy dependence of potential, i.e. $\gamma_1=\gamma_2=0$, and arrange the energy eigenvalue equation (37) by replacing $\chi_0\to D_0$, $\chi_1\to D_1$, $\chi_2\to D_2$, $\delta\to a$, $r_0\to r_e$, $\tau\to\lambda$ and using definitions $\tilde{\beta}=\frac{(\kappa+\lambda)(\kappa+\lambda+1)}{r_e^2}$ and $E_{n\kappa}=C_s-M-\tilde{E}_{n\kappa}$, then it becomes

$$\left[\frac{\tilde{\beta}D_1+2\tilde{E}_{n\kappa}D_e}{\sqrt{\tilde{E}_{n\kappa}D_e-\tilde{\beta}D_2}}+2\sqrt{\tilde{E}_{n\kappa}(\tilde{E}_{n\kappa}-C_s+2M)-\tilde{\beta}D_0}\right]^2+(1+2n)^2a^2=0, \tag{48}$$

which is identical to equation (45) in Aydogdu and Sever [41].

Pseudospin Symmetry:

As we consider the vector and scalar Morse and Coulomb-like tensor potential without energy dependence, i.e. $\gamma_1=\gamma_2=0$, and organize Eq. (47) by replacing $\chi_0\to D_0$, $\chi_1\to D_1$, $\chi_2\to D_2$, $\delta\to a$, $r_0\to r_e$, $\tau\to\lambda$ and using definitions $\beta=\frac{(\kappa+\lambda)(\kappa+\lambda-1)}{r_e^2}$ and $E_{n\kappa}=C_{ps}+M-\tilde{E}_{n\kappa}$, then the pseudospin symmetric energy eigenvalues equation (46) becomes

$$\left[\frac{\beta D_1+2\tilde{E}_{n\kappa}D_e}{\sqrt{\tilde{E}_{n\kappa}D_e-\beta D_2}}+2\sqrt{\tilde{E}_{n\kappa}(\tilde{E}_{n\kappa}-C_{ps}-2M)-\beta D_0}\right]^2+(1+2n)^2a^2=0. \tag{49}$$

Eq. (49) turns out to be identical to equation (33) in Aydogdu and Sever [41]. Moreover, it is a trivial matter to check that the above equation is reduced to equation (34) in Aydogdu and Sever [41], equation (41) in Bayrak and Boztosun [40], equation (34) in Qiang et al. [39], and equation (37) in Berkdemir [38] in the absence of tensorial interaction $\tau=0$.

4.2. Effect of energy dependence

As the energy eigenvalue equations (37) and (47) are very complicated, we conduct numerical analysis here in order to discuss the effect of energy-dependent potential on both bound states and spin and pseudospin doublets. In the calculation, we use the same value of parameters used in the literature [38–41] to provide compatibility: $M = 10fm^{-1}$, $D_e = 5fm^{-1}$, $r_0 = 2.40873$, $\delta = 0.988879$.

Spin Symmetry:

In the absence of tensor interaction, i.e. $\tau = 0$, we determine the minimum value of C_s to obtain a bound state for the Dirac particle in view of energy-dependent scalar and vector EDM potential. The power of $E_{n\kappa}$ is chosen as $\sigma_1 = 1, -1$. Although there are many other options for σ_1, we restrict the numerical analysis to these choices to investigate the effect of energy dependent potential on spin doublets without losing generality. The numerical results are given in Table 1. According to Table 1, the minimum value of C_s to obtain a bound state is 9.822110770016 within the energy-independent potential $\gamma_1 = 0$. It takes the same value for $\gamma_1 > 1$. An interesting result is obtained for $\gamma_1 < 0$. In this case, the minimum value of C_s goes to zero.

Table 1. Spin symmetry: the minimum C_s value to obtain a bound state whose energy is given in units of fm^{-1} for $\tau = 0$.

$1g_{9/2}$	$\sigma_1 = 1$		$\sigma_1 = -1$	
γ_1	$C_s(min)$	$E_{n\kappa}$	$C_s(min)$	$E_{n\kappa}$
1.0	9.822110770016	4.34436×10^{-13}	10.000000000001	1.57729×10^{-13}
0.5	9.822110770016	4.81952×10^{-13}	10.000000000001	3.80620×10^{-13}
	9.822110770016	4.81952×10^{-13}	9.822110770016	4.81952×10^{-13}
-0.5	0	2.01197	0	0.49671
-1.0	0	1.00654	0	0.99371

Table 2 shows how the energy dependence of potential affects the spin doublet. As an example, we consider the doublet $(0p_{3/2}, 0p_{1/2})$ and choose $C_s = 10fm^{-1}$. In Aydogdu and Sever [41], energy splitting between $0p_{3/2}$ and $0p_{1/2}$ is calculated as $0.01903916fm^{-1}$ in view of Morse plus Coulomb-like tensor potentials without energy dependence. The results given in Table 2 indicate that the magnitude of energy splitting between members of this doublet is changed according to γ_1. It should be noted in Table 2 that energy splitting increases for $\gamma_1 < 0$ while decreasing for $\gamma_1 \geq 0$.

Table 2. Spin symmetry: the bound state energy eigenvalues in units of fm^{-1} in the absence and presence of tensor potential with the energy-dependent scalar and vector potentials.

$\sigma_1 = \sigma_2 = 1$		$p_{3/2}$		$p_{1/2}$	
γ_1	γ_2	$\tau = 0$	$\tau = 0.75$	$\tau = 0$	$\tau = 0.75$
1	1	0.0186380	0.0104055	0.0186380	0.0293636
1	-1	0.0186380	0.0105497	0.0186380	0.0287002
-1	1	0.0190688	0.0105497	0.0190688	0.0305392
-1	-1	0.0190688	0.0106631	0.0190688	0.0297919
1		0.0186380	0.0104770	0.0186380	0.0290273
-1		0.0190688	0.0105875	0.0190688	0.0301597
	1	0.0188481	0.0104588	0.0188481	0.0299270
	-1	0.0188481	0.0106057	0.0188481	0.0292245

Pseudospin Symmetry:

In the pseudospin symmetric case, the minimum value of C_{ps} was determined as $-10fm^{-1}$ in Berkdemir [38] and $-9.96fm^{-1}$ in Bayrak and Boztosun [40] for the state $2f_{7/2}$. From Table 3, it is easily seen that this value is nearly $-9.95fm^{-1}$ in the presence of potential without energy dependence. However, in view of the energy-dependent potential, the minimum value of C_{ps} goes to zero when one takes $\gamma_1 > 0$. Unlike the spin symmetric case, it is necessary to take a positive value of γ_1 to obtain more bound states.

Table 3. Pseudospin symmetry: the minimum C_{ps} value to obtain a bound state whose energy is given in units of fm^{-1} for $\tau = 0$.

$2f_{7/2}$	$\sigma_1 = 1$		$\sigma_1 = -1$	
γ_1	$C_{ps}(min)$	$E_{n\kappa}$	$C_{ps}(min)$	$E_{n\kappa}$
1.0		-1.98792		-1.00622
0.5		-0.99381		-0.50326
	-9.95076952	-9.73659×10^{-9}	-9.95076952	-9.73659×10^{-9}
-0.5	-9.95076952	-9.49086×10^{-9}	-9.95076952	-6.48101×10^{-16}
-1.0	-9.95076952	-9.25723×10^{-9}	-9.95076952	-9.49704×10^{-16}

We also investigate the effect of energy dependence on a pseudospin doublet. Numerical results with $C_{ps} = -10fm^{-1}$ are presented in Table 4 for a pseudospin doublet $(1d_{5/2}, 0g_{7/2})$. The results show that energy splitting between the pseudospin doublet increases with $\gamma_1 > 0$ while decreasing with $\gamma_1 \leq 0$ in contrast to the spin symmetric case.

Table 4. Pseudospin symmetry: the bound state energy eigenvalues in units of fm^{-1} in the absence and presence of tensor potential with the energy-dependent scalar and vector potentials.

$\sigma_1 = \sigma_2 = 1$		$1d_{5/2}$		$g_{7/2}$	
γ_1	γ_2	$\tau = 0$	$\tau = 0.75$	$\tau = 0$	$\tau = 0.75$
1	1	-0.0250464	-0.0183943	-0.0183943	-0.0302338
1	-1	-0.0250464	-0.0181312	-0.0181312	-0.0304945
-1	1	-0.0237564	-0.0177032	-0.0177032	-0.0283285
-1	-1	-0.0237564	-0.0174689	-0.0174689	-0.0285393
1		-0.0250464	-0.0182619	-0.0182619	-0.0303647
-1		-0.0237564	-0.0175853	-0.0175853	-0.0284344
	1	-0.0243675	-0.0180356	-0.0180356	-0.0292197
	-1	-0.0243675	-0.0177878	-0.0177878	-0.0294527

5. Discussion

In the present study, we approximately obtained pseudospin and spin symmetric solutions of the Dirac equation with scalar and vector EDM potential and EDCL tensor potential. Energy eigenfunctions were acquired in terms of hypergeometric functions. Furthermore, we obtained the energy eigenvalue equation. In calculations, the AIM was used. We showed that our results are consistent with those presented [38–41] previously. The effect of energy-dependent potential on both bound states and pseudospin and spin doublets was also investigated. We found that energy dependence of potential changes the minimum value of C_s (or C_{ps}) to get a bound state. Moreover, it is interesting to note here that this value goes to zero for $\gamma_1 < 0$ in the spin symmetric case and for $\gamma_1 > 0$ in the pseudospin symmetric case. We have, finally, analyzed the energy splitting between spin

(pseudospin) doublets in view of potential with energy dependence. It is seen that the magnitude of the energy splitting between members of the spin (pseudospin) doublet increases (decreases) for $\gamma_1 < 0$ $(\gamma_1 > 0)$ while decreasing (increasing) for $\gamma_1 \geq 0$ $(\gamma_1 \leq 0)$.

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