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Validity of Ehrenfest's theorem for generalized fields of dyons

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Abstract: The validity of Ehrenfest's theorem with its classical correspondence has been justified for the manifestly covariant equations of dyons. We have also developed accordingly the Lagrangian formulation for electromagnetic fields in a minimum coupled source giving rise to conserved current of dyons. Applying the Gupta subsidiary condition we have extended the validity of the Ehrenfest's theorem for $U(1) \times U(1)$ Abelian gauge theory of dyons. It is shown that the expectation value of the quantum equation of motion reproduces the classical equation of motion, which is the generalized form of Ehrenfest's theorem in quantum field theory.

Key words: Dyons, Ehrenfest's theorem, Lagrangian, quantum field theory, electromagnetic fields

1. Introduction

In classical mechanics the dynamical state of each particle is defined by its position and momentum, at a given instant. In quantum mechanics the dynamical state of the system is represented by its wave function [1]. The expectation values of displacement and momentum obey time evolution equations, which are analogous to those of classical mechanics. This result is called Ehrenfest's theorem [2]. The value of the quantum equation of motion taken with the states in the physical subspace reproduces the classical equations of motion. In a paper Zwanziger calculated the physical states in quantum electrodynamics in terms of observable fields [3]. Thiemann and Winkler [4] also establish the Ehrenfest's Property of compact gauge states, which are labeled by a point, a connection, and an electric field, in the classical phase space. Recently, Parthsarthy [5] has generalized the validity of Ehrenfest's theorem in Abelian and non-Abelian quantum field theories. The first consistent manifestly covariant quantization of electromagnetic fields was formulated by Gupta [6] and Bleuler [7], describing the photon in the Fermi gauge and the Lorentz condition not being an operator identity but a restriction that is imposed on the physical states. Since the Lorentz condition is not consistent with the canonical commutation relations, the former is regarded as a supplement condition that holds only for certain physical states. According to the Gupta–Bleuler formalism, this subspace is a nonnegative subspace so that elements of physical subspace, called physical states, can be probabilistically interpretable. Correspondingly, the photon propagator does not satisfy the Lorentz condition [8]. Cahill [9] also suggested that Fermi's form of the subsidiary condition is the correct one because it does not require the use of an indefinite metric and because it is equivalent to the requirement that physical states be invariant under a certain class of local gauge transformations. The scattering matrix of the relativistic quantum electrodynamics, which is usually obtained with the Gupta–Bleuler method, was deduced with a gauge-independent treatment, in which no use is made of any subsidiary condition.

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The commutation rules between the components of the incoming electromagnetic potentials are replaced by the weaker commutation rules between the components of the incoming electromagnetic-field strengths [10]. The covariant quantization of the electromagnetic field is one of the most peculiar problems of quantum field theory because of the masslessness of the photon. In spite of its vectorial nature, only the two transverse components of the photon are observable, and the third freedom yields the Coulomb interaction between charged particles. It is well known, in the theory of Melo and coworkers [11], the state vectors do not necessarily have positive norms, and the space spanned by them is an indefinite metric Hilbert space. Suzuki [12] used the Gupta subsidiary condition and selected the physical subspace in a unique and simple manner. The question of existence of monopoles [13]-[16] and dyons [17]-[24] has become a challenging new frontier and the object of more interest in recent years in high energy physics. Keeping in view the recent potential importance of monopoles and dyons along with the fact that despite the potential importance of monopoles, the formalism necessary to describe them has been clumsy and not manifestly covariant, Negi and coworkers have already developed a self-consistent quantum field theory of generalized electromagnetic fields associated with dyons (particles carrying electric and magnetic charges) [25–27]. In spite of the enormous potential importance of monopoles (dyons) and the fact that these particles have been extensively studied, no reliable theory has been presented that is as conceptually transparent and predictably tactable as the usual electrodynamics and the formalism necessary to describe them has been clumsy and not manifestly covariant. On the other hand, the concept of electromagnetic (EM) duality has been receiving much attention [28]-[38] in gauge theories, field theories, supersymmetry, and super strings. In a recent paper a unified theory of gravi-electromagnetism is also developed on the generalized Schwinger–Zwanziger [39] formulation on dyons in quaternions in a simple and consistent manner. In this paper, we have made an attempt to show the validity of Ehrenfest's theorem for charged particles and dual charged particles in the case of separate electric and magnetic charge and dyons. Starting from the basic definition of Ehrenfest's theorem, we have discussed the validity of Ehrenfest's theorem in the case of a Dirac particle moving in the electromagnetic field carrying electric charge and it is shown that the Ehrenfest's theorem is valid for a Dirac particle moving in an electromagnetic field. The validity of Ehrenfest's theorem of magnetic charge (i.e. monopole) and the generalization of Ehrenfest's theorem for magnetic monopoles has been obtained. Regarding the Hamiltonian of Dirac fields in the presence of dyons (particles carrying simultaneously the electric and magnetic charge) we have discussed the validity of Ehrenfest's theorem in a static case for electric and magnetic charge. It is also shown that the equation of motion of dyons may be visualized as the generalization of Ehrenfest's theorem for dyons moving in generalized electromagnetic fields. We have also developed accordingly the Lagrangian formulation for the electromagnetic fields in a minimum coupled source justifying the conserved Dirac current for dyons. Applying the Gupta subsidiary condition, we have also reproduced the classical equation of motion and the validity of Ehrenfest's theorem to Abelian quantum field theory has been checked and verified. It is shown that the expectation value of the quantum equation of motion reproduces the classical equation of motion, which is the generalized form of Ehrenfest's theorem in quantum field theory.

2. Basics of Ehrenfest's theorem

Let us define the time derivative of the expectation value of a quantum mechanical operator in terms of the commutator of that operator with the Hamiltonian of the system [40]. Using the Heisenberg equation of motion as

$$\frac{d}{dt}\left\langle \hat{A}\right\rangle = \frac{1}{i}\left\langle \left[\hat{A},\,\hat{\mathcal{H}}\right]\right\rangle + \left\langle \frac{\partial \hat{A}}{\partial t}\right\rangle . \tag{1}$$

Here for brevity we have considered the natural units $(c = \hbar = 1)$. In equation (1) $\hat{A}$ is the quantum mechanical operator and $\langle A \rangle$ is its expectation value. For the case of a massive particle moving in a potential, then the Hamiltonian is defined as

$$\mathcal{H}(x,\, p,\, t) = \frac{p^2}{2m} + V(x,\, t)\,; \tag{2}$$

where x is just the location of the particle. Suppose the instantaneous change in momentum p, using Ehrenfest's theorem, we have

$$\begin{aligned} \frac{d}{dt}\langle p \rangle =& \frac{1}{i}\langle [p,\, \mathcal{H}] \rangle + \left\langle \frac{\partial p}{\partial t} \right\rangle ; \\ =& \frac{1}{i}\langle [p,\, V(x,\, t)] \rangle . \end{aligned} \tag{3}$$

The operator p commutes with itself and has no time dependence. By expanding the right-hand side of equation (3) and replacing p by $-i\nabla$, then

$$\frac{d}{dt}\langle p \rangle = \int \psi^\star V(x,\, t)\, \nabla \psi dx^3 - \int \psi^\star \nabla (V(x,\, t)\, \psi) dx^3. \tag{4}$$

Applying the product rule on the second term in equation (4), then we have

$$\begin{aligned} \frac{d}{dt}\langle p \rangle &= \int \psi^\star V(x,\, t)\, \nabla \psi dx^3 - \int \psi^\star \left(\nabla V(x,\, t)\right) \psi dx^3 - \int \psi^\star V(x\, t)\, \nabla \psi dx^3; \\ &= \int \psi^\star \left(\nabla V(x,\, t)\right) \psi dx^3; \\ &= \langle -\nabla V(x\, ,t) \rangle = \langle F \rangle . \end{aligned} \tag{5}$$

This result manifests as Newton's second law in the case of having so many particles that the net motion is given exactly by the expectation value of a single particle. Similarly, the instantaneous change in the position (x) expectation value is defined as

$$\begin{aligned} \frac{d}{dt}\langle x \rangle =& \frac{1}{i}\langle [x,\, \mathcal{H}] \rangle + \left\langle \frac{\partial x}{\partial t} \right\rangle ; \\ =& \frac{1}{i}\left\langle \left[x,\, \frac{p^2}{2m} + V(x\, ,t)\right] \right\rangle ; \\ =& \frac{1}{i}\left\langle \left[x,\, \frac{p^2}{2m}\right] + V(x,\, t) \right\rangle ; \\ =& \frac{1}{m}\langle p \rangle . \end{aligned} \tag{6}$$

This result is again in accord with the classical equation. As such, equations (5) and (6) are known as Ehrenfest's theorem.

3. Validity of Ehrenfest's theorem for charged particles

Dirac Hamiltonian for the electromagnetic field (Electric Case) is described as [41]

$$\overrightarrow{\mathcal{H}} = \overrightarrow{\alpha} \cdot \overrightarrow{\pi} + \beta m_0 + e\phi, \tag{7}$$

where α, β are usual arbitrary constants given by Dirac in relativistic quantum mechanics and the momentum of charged particle gets modified to

$$\overrightarrow{\pi} = \overrightarrow{p} - eA. \tag{8}$$

In equation (7), where m_0 is the rest mass of the particle carrying electric charge, e is the electric charge and in equation (8) A is the vector part of the electric four-potential $\{A_\mu\} = \left\{\overrightarrow{A},\, i\phi\right.$ where ϕ is the scalar part of $\{A_\mu\}$. In the case of a weak magnetic field, the vector part of the electric four-potential vanishes $\{i.e.\ A = 0\}$ and so equation (8) reduces to $\overrightarrow{\pi} = \overrightarrow{p}$. The Heisenberg equation of motion (1) for a dynamical variable $\hat{F}$ now reduces to

$$\frac{d\left\langle \hat{F}\right\rangle}{dt} = \frac{1}{i\hbar}\left[\left\langle \hat{F}, \hat{\mathcal{H}}\right\rangle\right]. \tag{9}$$

Assuming dynamical variable F corresponding to $\hat{x}$ and using equation (7) for $\mathcal{H}$ we get

$$\begin{aligned}\langle \overrightarrow{v}\rangle = \frac{d\langle \overrightarrow{x}\rangle}{dt} &= \frac{1}{i}\left[\langle \overrightarrow{x},\, \mathcal{H}\rangle\right]\\ &= \langle \overrightarrow{\alpha}\rangle = \frac{\langle \overrightarrow{p}\rangle}{m},\end{aligned} \tag{10}$$

which can be expressed as $\frac{d}{dt}\langle \overrightarrow{x}\rangle = \frac{\langle \overrightarrow{p}\rangle}{m}$ given by equation (6) and thus verifies Ehrenfest's theorem. In equation (10) $\langle \overrightarrow{v}\rangle = \frac{d\langle \overrightarrow{x}\rangle}{dt} = \langle \overrightarrow{\alpha}\rangle$ is known as a velocity operator. Now identifying the dynamical variable $\overrightarrow{F}$ as the momentum operator, we get

$$\begin{aligned}\frac{d\langle \overrightarrow{\pi}\rangle}{dt} &= \frac{\partial\langle \overrightarrow{\pi}\rangle}{\partial t} + \frac{1}{i}\left[\mathcal{H}, \overrightarrow{\pi}\right]\\ &= \frac{\overrightarrow{\alpha}}{i}\left[\langle \overrightarrow{\pi},\, \overrightarrow{\pi}\rangle\right] - \langle (e\nabla\phi)\rangle - e\frac{\partial\left\langle \hat{A}\right\rangle}{\partial t}.\end{aligned} \tag{11}$$

where $[\pi,\, \pi]$ cannot be zero but its commutation relations reduce to

$$\begin{aligned}[\pi_i,\, \pi_j] &= ie\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)(\forall i,\, j = 1,\, 2)\\ &= ie\epsilon_{ijk}H_k\,(\forall ijk = 1, 2, 3)\,.\end{aligned} \tag{12}$$

Hence equation (11) reduces to

$$\begin{aligned}\frac{d\langle \overrightarrow{\pi}\rangle}{dt} &= e\left[\left\langle\left(-\nabla\phi - \frac{\partial\hat{A}}{\partial t}\right) + \left(\overrightarrow{\alpha} \times \overrightarrow{H}\right)\right\rangle\right];\\ &= \left\langle e\left[\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{H}\right]\right\rangle = \left\langle \overrightarrow{F_e}\right\rangle.\end{aligned} \tag{13}$$

Equation (13) is the equation for Lorentz force acting on a charged particle moving the electromagnetic field. In the case of a Coloumb gauge i.e. ($\overrightarrow{H}=0 \Rightarrow \nabla\times A \Rightarrow A=0 \Rightarrow \pi=\overrightarrow{p}$), equation (13) reduces to

$$\frac{d\left\langle \overrightarrow{\pi}\right\rangle }{dt}=\left\langle e\overrightarrow{E}\right\rangle =\left\langle -e\nabla\overrightarrow{\phi}\right\rangle , \tag{14}$$

which can further be expressed as

$$\frac{d}{dt}\left\langle \hat{p}\right\rangle = \quad \left\langle -\nabla\left(e\phi\right)\right\rangle \quad =\left\langle \overrightarrow{F_{e}}\right\rangle , \tag{15}$$

which implies Ehrenfest's theorem. As such Ehrenfest's theorem provides its validity in the case of a Dirac particle moving in an electromagnetic field carrying electric charge and equation (13) is the generalized form of Ehrenfest's theorem for a Dirac particle moving in an electromagnetic field.

4. Validity of Ehrenfest's theorem for dual charged particles

The Dirac Hamiltonian for a dual charge (i.e. magnetic monopole) moving in an electromagnetic field is

$$\overrightarrow{\mathbf{H}}=\overrightarrow{\alpha}\cdot\overrightarrow{\boldsymbol{\pi}}+\beta m_{0}+g\psi, \tag{16}$$

where ψ is the dual (magnetic) scalar potential and g is the dual (magnetic) charge while

$$\overrightarrow{\boldsymbol{\pi}}=\overrightarrow{p}-gB. \tag{17}$$

Here $\overrightarrow{B}$ is the vector part of the magnetic (dual) four-potential, i.e. $\{B_{\mu}\}=\left\{ \overrightarrow{B},\,i\overrightarrow{\psi}\right\}$. Therefore, dynamical quantity $\hat{\mathbf{F}}$ in equation (9) corresponds to $\langle x\rangle$ and can further be described as

$$\begin{aligned}\left\langle \overrightarrow{\boldsymbol{v}}\right\rangle =\frac{d\left\langle \overrightarrow{\boldsymbol{x}}\right\rangle }{dt} & =\frac{1}{i}\left[\left\langle \overrightarrow{\boldsymbol{x}},\,\mathbf{H}\right\rangle \right]\\
 & =\left\langle \overrightarrow{\boldsymbol{v}}\right\rangle =\frac{\left\langle \overrightarrow{p}\right\rangle }{m}.\end{aligned} \tag{18}$$

which verifies Ehrenfest's theorem $\frac{d\left\langle \overrightarrow{\boldsymbol{x}}\right\rangle }{dt}=\frac{\left\langle \overrightarrow{p}\right\rangle }{m}$ for the dynamics of dual charge (magnetic monopole). Likewise identifying the dynamical quantity $\hat{\mathbf{F}}$ corresponding to the momentum operator we get

$$\begin{aligned}\frac{d\left\langle \overrightarrow{\boldsymbol{\pi}}\right\rangle }{dt} & =\frac{\partial\left\langle \overrightarrow{\boldsymbol{\pi}}\right\rangle }{\partial t}+\frac{1}{i}\left[\left\langle \mathbf{H},\overrightarrow{\boldsymbol{\pi}}\right\rangle \right]\\
 & =\frac{\overrightarrow{\alpha}}{i}\left[\left\langle \overrightarrow{\boldsymbol{\pi}},\,\overrightarrow{\boldsymbol{\pi}}\right\rangle \right]-\left\langle \left(g\nabla\psi\right)\right\rangle -g\frac{\partial\left\langle B\right\rangle }{\partial t},\end{aligned} \tag{19}$$

where like the previous case $[\boldsymbol{\pi},\,\boldsymbol{\pi}]$ cannot be zero but its commutation relations reduce to

$$\begin{aligned}\left[\boldsymbol{\pi}_{i},\,\boldsymbol{\pi}_{j}\right] & = & -ig\left(\frac{\partial B_{j}}{\partial x_{i}}-\frac{\partial B_{i}}{\partial x_{j}}\right)(\forall i,\,j=1,\,2)\\
 & = & -ig\epsilon_{ijk}E_{k}\left(\forall ijk=1,2,3\right).\end{aligned} \tag{20}$$

Hence equation (19) reduces to

$$\begin{aligned} \frac{d\left\langle \overrightarrow{\boldsymbol{\pi}} \right\rangle}{dt} &= g\left[\left\langle\left(-\nabla\psi - \frac{\partial \hat{A}}{\partial t}\right)\right\rangle - \left\langle\left(\overrightarrow{\alpha} \times \overrightarrow{E}\right)\right\rangle\right] \\ &= g\left[\left\langle \overrightarrow{H} - \overrightarrow{\boldsymbol{v}} \times \overrightarrow{E} \right\rangle\right] = \left\langle \overrightarrow{\mathbf{F}_g} \right\rangle, \end{aligned} \tag{21}$$

which is a force acting on a pure magnetic monopole moving in an electromagnetic field. In the absence of an electric field (in order to keep in mind the electromagnetic duality), i.e. ($\overrightarrow{E} \longrightarrow 0 \Rightarrow \nabla \times B \Rightarrow B = 0 \Rightarrow \boldsymbol{\pi} = \overrightarrow{p}$), equation (21) may then be generalized to

$$\frac{d\left\langle \overrightarrow{\boldsymbol{\pi}} \right\rangle}{dt} = \left\langle g\overrightarrow{H} \right\rangle = \left\langle -\nabla\left(g\psi\right)\right\rangle = \left\langle \overrightarrow{\mathbf{F}_g} \right\rangle, \tag{22}$$

which shows the validity of Ehrenfest's theorem of magnetic charge (i.e. monopole) and equation (21) is the generalization of Ehrenfest's theorem for magnetic monopole.

5. Validity of Ehrenfest's theorem for dyons

Let us write the Hamiltonian of Dirac particles in generalized electromagnetic fields in the presence of particles simultaneously carrying the electric and magnetic charges, namely dyons, as

$$\begin{aligned} \overrightarrow{\mathrm{H}} &= \overrightarrow{\alpha} \cdot \overrightarrow{\pi} + \beta m_0 + \sum_j a_j \varphi_j \qquad (\forall j = 1, 2) \\ &= \overrightarrow{\alpha} \cdot \overrightarrow{\pi} + \beta m_0 + a_1\varphi_1 + a_2\varphi_2, \end{aligned} \tag{23}$$

where

$$\begin{aligned} \overrightarrow{\pi} &= \overrightarrow{p} - \sum_j a_j V_j; \\ a_1 &= e;\ \varphi_1 = \phi; \\ a_2 &= g;\ \varphi_2 = \psi; \\ V_1 &= \overrightarrow{A};\ V_2 = \overrightarrow{B}. \end{aligned} \tag{24}$$

Hence equation (23) reduces to

$$\overrightarrow{\mathrm{H}} = \overrightarrow{\alpha} \cdot \overrightarrow{\pi} + \beta m_0 + e\phi + g\psi, \tag{25}$$

where π is the modified momentum operator of dyons expressed as

$$\overrightarrow{\pi} = p - eA - gB. \tag{26}$$

Hence we may obtain the velocity operator for dyons as

$$\begin{aligned} \left\langle \overrightarrow{\mathrm{v}} \right\rangle = \frac{d\left\langle \overrightarrow{\mathrm{x}} \right\rangle}{dt} &\Rightarrow \frac{1}{i}\left[\overrightarrow{\mathrm{x}}, \mathrm{H}\right] \\ &\Rightarrow \left\langle \overrightarrow{\mathrm{v}} \right\rangle \Rightarrow \frac{1}{m}\left\langle \overrightarrow{p} \right\rangle, \end{aligned} \tag{27}$$

which is the same as equations (10) and (18) and may be regarded as the velocity operator of dyons. Therefore, the momentum operator of dyons is described as

$$\begin{aligned}\frac{d\left\langle\overrightarrow{\pi}\right\rangle}{dt} &= \frac{\partial\left\langle\overrightarrow{\pi}\right\rangle}{\partial t}+\frac{1}{i}\left[\left\langle \mathrm{H},\,\overrightarrow{\pi}\right\rangle\right]\\ &= \frac{1}{i}\left[\left\langle\overrightarrow{\alpha}\cdot\overrightarrow{\pi},\,\overrightarrow{\pi}\right\rangle\right]+\frac{1}{i}\left[\left\langle\beta\mathbf{m_0},\,\overrightarrow{\pi}\right\rangle\right]+\frac{1}{i}\left[\left\langle e\phi,\,\overrightarrow{\pi}\right\rangle\right]+\frac{1}{i}\left[\left\langle g\psi,\,\overrightarrow{\pi}\right\rangle\right]-e\frac{\partial\left\langle A\right\rangle}{\partial t}-g\frac{\partial\left\langle B\right\rangle}{\partial t},\end{aligned} \tag{28}$$

which is reduced to

$$\frac{d\left\langle\overrightarrow{\pi}\right\rangle}{dt}=\left\langle e\left[\overrightarrow{E}+\overrightarrow{v}\times\overrightarrow{H}\right]+g\left[\overrightarrow{H}-\overrightarrow{v}\times\overrightarrow{E}\right]\right\rangle; \tag{29}$$

where

$$\begin{aligned}\overrightarrow{E} &= -\overrightarrow{\nabla}\phi-\frac{\partial\overrightarrow{A}}{\partial t}-\overrightarrow{\nabla}\times\overrightarrow{B};\\ \overrightarrow{H} &= -\overrightarrow{\nabla}\psi-\frac{\partial\overrightarrow{B}}{\partial t}+\overrightarrow{\nabla}\times\overrightarrow{A}.\end{aligned} \tag{30}$$

Let us decompose $\overrightarrow{E}$ and $\overrightarrow{H}$ in terms of longitudinal and transverse components, i.e.

$$\begin{aligned}\overrightarrow{E} &= \overrightarrow{E_L}+\overrightarrow{E_T},\\ \overrightarrow{H} &= \overrightarrow{H_L}+\overrightarrow{H_T},\end{aligned} \tag{31}$$

where

$$\begin{aligned}\overrightarrow{E_L} &= -\overrightarrow{\nabla}\phi;\\ \overrightarrow{E_T} &= -\frac{\partial\overrightarrow{A}}{\partial t}-\overrightarrow{\nabla}\times\overrightarrow{B};\\ \overrightarrow{H_L} &= -\overrightarrow{\nabla}\psi;\\ \overrightarrow{H_T} &= -\frac{\partial\overrightarrow{B}}{\partial t}+\overrightarrow{\nabla}\times\overrightarrow{A}.\end{aligned} \tag{32}$$

If the particles are stationary, then the transverse part of the electric and magnetic field are vanishing, i.e.

$$\begin{aligned}E_T=0 \quad &\Rightarrow \tfrac{\partial\overrightarrow{A}}{\partial t}=0 \quad ; \overrightarrow{\nabla}\times\overrightarrow{B}=0;\\ H_T=0 \quad &\Rightarrow \tfrac{\partial\overrightarrow{B}}{\partial t}=0 \quad ; \overrightarrow{\nabla}\times\overrightarrow{A}=0.\end{aligned} \tag{33}$$

Equation (33) immediately shows that in the absence of a transverse electromagnetic field or for static dyon $\overrightarrow{A}=\overrightarrow{B}=0$ and $\overrightarrow{E}=-\overrightarrow{\nabla}\phi$ and $\overrightarrow{H}=-\overrightarrow{\nabla}\psi$. Therefore, equation (29) reduces to

$$\frac{d\left\langle\overrightarrow{p}\right\rangle}{dt}=\left\langle\nabla\left(-e\phi\right)+\nabla\left(-g\psi\right)\right\rangle, \tag{34}$$

which is the combination of classical values of $\frac{dp}{dt}$ for electric and magnetic charges in static cases and may be described as the combination of the validity of Ehrenfest's theorem for electric and magnetic charges. Thus the equation of motion of dyons (29) may be visualized as the generalization of Ehrenfest's theorem for dyons moving in generalized electromagnetic fields.

6. Abelian gauge theory and validity of Ehrenfest's theorem

Starting from the Lagrangian density of a generalized electromagnetic field for a minimally coupled source of electric and magnetic charges as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}M_{\mu\nu}M^{\mu\nu} + eA_{\mu}j^{\mu(e)} + gB_{\mu}j^{\mu(g)}, \tag{35}$$

where $j^{\mu(e)}$ is the four-current due to the presence of electric charge and $j^{\mu(g)}$ is the four-current due to the existence of magnetic charges, Lagrangian density (35) yields the following field equations

$$\partial^{\nu}F_{\mu\nu} = ej^{(e)}_{\mu}, \tag{36}$$

and

$$\partial^{\nu}M_{\mu\nu} = gj^{(g)}_{\mu}, \tag{37}$$

where we have used the following subsidiary conditions

$$\begin{aligned} \partial_{\mu}A_{\mu} =&0; \\ \partial_{\mu}B_{\mu} =&0. \end{aligned} \tag{38}$$

Equations (36) and (37) are the classical equations of motion giving to the Generalized Dirac Maxwell's (GDM) equation of dyons. Following Parthasarathy's approach [5] let us introduce the gauge fixing two auxiliary Hermitian scalar field $\alpha(x)$ and $\beta(x)$ in the case of electric and magnetic couplings and accordingly let us consider the following Lagrangian density

$$\begin{aligned} \mathcal{L} =& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}M_{\mu\nu}M^{\mu\nu} + \alpha(x)\partial^{\mu}A_{\mu} + \frac{a}{2}\alpha^{2}(x) \\ &+\beta(x)\partial^{\mu}B_{\mu} + \frac{b}{2}\beta^{2}(x) + eA_{\mu}j^{\mu(e)} + gB_{\mu}j^{\mu(g)}, \end{aligned} \tag{39}$$

where a, b are arbitrary parameters. Here, like Parthasarathy [5], we assume that the gauge field A_{μ} and B_{μ} are operators. Therefore, the Lagrangian density (39) reproduces the following two quantum equations of motion respectively associated with electric and magnetic charges

$$\begin{aligned} \partial_{\mu}F^{\mu\nu} - \partial^{\nu}\alpha(x) =& - ej^{\nu(e)}; \\ \partial^{\mu}A_{\mu} + a\alpha(x) =&0; \end{aligned} \tag{40}$$

and

$$\begin{aligned} \partial_{\mu}M^{\mu\nu} - \partial^{\nu}\beta(x) =& - gj^{\nu(g)}; \\ \partial^{\mu}B_{\mu} + b\beta(x) =&0. \end{aligned} \tag{41}$$

It should be noted that the fields in equation (40) and (41) are operators and thus act on the various functions (state) in indefinite metric quantum mechanical Hilbert space. Therefore, here the method of quantization can be described as the "operator method of quantization". On the other hand, in the classical case, the physical meaningful degrees of freedom contribute only to the observables. Therefore, one can impose the

Gupta subsidiary condition on the two photons respectively associated with electric and magnetic charges of dyons, i.e.

$$\begin{aligned}\alpha^{+}(x) &= 0;\\ \beta^{+}(x) &= 0.\end{aligned} \tag{42}$$

Here the superscript $(+)$ describes the positive frequency part of α and β, and so the quantum mechanical Hilbert space (V_{HS}) is identified with equation (42). Thus the expectation values of the Parthasarathy approach [5] are generalized for equations (40) and (41) as

$$\begin{aligned}\left\langle \phi \left| ej^{\nu(e)} - \partial^{\mu}\alpha(x) + \partial_{\mu}F_{\mu\nu} \right| \phi \right\rangle &= 0;\ \ |\,\phi> \in V_{HS},\\ \left\langle \phi \left| \partial^{\mu}A_{\mu} + a\alpha(x) \right| \phi \right\rangle &= 0;\end{aligned} \tag{43}$$

and

$$\begin{aligned}\left\langle \phi \left| gj^{\nu(g)} - \partial^{\mu}\beta(x) + \partial_{\mu}M_{\mu\nu} \right| \phi \right\rangle &= 0;\ \ |\,\phi> \in V_{HS},\\ \left\langle \phi \left| \partial^{\mu}B_{\mu} + b\beta(x) \right| \phi \right\rangle &= 0.\end{aligned} \tag{44}$$

Now let us use $\alpha^{-} = (\alpha^{+})^{+}$ and $\beta^{-} = (\beta^{+})^{+}$; we get

$$\begin{aligned}\left\langle \phi \left| ej^{\nu(e)} + \partial_{\mu}F_{\mu\nu} \right| \phi \right\rangle &= 0;\ \ |\,\phi> \in V_{HS};\\ \left\langle \phi \left| \partial^{\mu}A_{\mu} \right| \phi \right\rangle &= 0;\end{aligned} \tag{45}$$

and

$$\begin{aligned}\left\langle \phi \left| gj^{\nu(g)} + \partial_{\mu}M_{\mu\nu} \right| \phi \right\rangle &= 0;\ \ |\,\phi> \in V_{HS};\\ \left\langle \phi \left| \partial^{\mu}B_{\mu} \right| \phi \right\rangle &= 0.\end{aligned} \tag{46}$$

Comparing equations (45) and (46) with equations (36) and (37), the expectation values of quantum equation of motion reproduce the classical equation of motion, which is nothing but the generalization of Ehrenfest's theorem to Abelian quantum field theory for generalized fields of dyons as Ehrenfest's theorem establishes a formal connection between the time dependence of quantum mechanical expectation values of observables and the corresponding classical equations of motion. As such, the validity of Ehrenfest's theorem has been justified for Abelian $U(1) \times U(1)$ gauge theory of dyons in terms of two photons.

7. Discussion

Equation (1) represents the time derivative of the expectation value of a quantum mechanical operator in terms of the commutator of the operator with the Hamiltonian of the system. We have discussed the basics of Ehrenfest's theorem in quantum mechanics. Accordingly, equation (2) describes the Hamiltonian. Equation (3) gives rise to the instantaneous change in momentum, which has been expanded in equation (4). Equation (5) gives the representation of Ehrenfest's theorem which manifests Newton's second law for so many particle systems. Similarly, equation (6) describes the other form of Ehrenfest's theorem and it is in accord with the classical equation. In section (3) we have discussed the validity of Ehrenfest's theorem for charged particles. The Dirac Hamiltonian for the electromagnetic field (electric case) is described by equation (7), while equation (8) represents the modified form of momentum of the electric charge. Defining the Heisenberg equation of motion for a dynamical variable by equation (9), we have obtained the relativistic velocity operator by equation (10), which is in accordance with Ehrenfest's theorem at classical level. Identifying equation (11) as the time derivative of the momentum operator for an electric charge particle, we have obtained the commutation relations among momentum operators in equation (12). It is shown that the time derivative of the momentum operator provides equation (13) for Lorentz force acting on a charged particle moving in the electromagnetic field. As such, Ehrenfest's theorem has been verified by equation (15) and it is shown that the validity of Ehrenfest's theorem is consistently applicable for a Dirac particle moving in an electromagnetic field carrying electric charge. In section (4) we have discussed the validity of Ehrenfest's theorem for dual charged particles. Equation (16) describes the Dirac Hamiltonian for a dual charge (i.e. magnetic monopole) moving in an electromagnetic field. We have obtained equation (17) is the modified momentum of dual charge moving in an electromagnetic field. As such we have obtained equation (18) for the velocity operator and verified the validity of Ehrenfest's theorem for the dynamics of dual charge (i.e. magnetic monopole). Describing equation (19) for the time derivative of the momentum operator for a dual charge, we have obtained the commutations relations (20) among momentum operator for monopole. It is shown that the rate of change of momentum of a dual charged particle exerts the Lorentz force equations (21) and (22) acting on a pure magnetic monopole moving in an electromagnetic field. It verifies the validity of Ehrenfest's theorem for a magnetic monopole. In section (5), we have discussed the validity of Ehrenfest's theorem for particles carrying simultaneously existence of electric and magnetic charges (namely dyons). As such, we have written the Hamiltonian of Dirac particle in electromagnetic fields for dyons by equation (23) whereas the momentum operator for the dyons is expressed by equation (24). Accordingly, we have established the momentum operator of dyons by equation (26). Hence we have obtained the velocity operator for a dyonic field by equation (27). The rate of change of the momentum operator of a dyon is described by equation (28), which provides the Lorentz force equation (29) of a dyon carrying the generalized electric and magnetic fields given by equation (30). Decomposing the electric and magnetic field in terms of longitudinal and transverse components of electric and magnetic field by equation (31), we have obtained the transverse and longitudinal components of generalized electric and magnetic fields of dyons and are expressed in terms of electric and magnetic four-potentials as given by equation (32). Restricting the direction of propagation along a particular axis we have made transverse components of generalized electromagnetic fields vanishing by equation (33) so that one can write the horizontal components in terms of gradient of potential. Hence we have obtained the combined term of Ehrenfest's theorem for electric and magnetic charges. It shows the validity of Ehrenfest's theorem for dyons. Thus equations (29) and (34) are the generalization of Ehrenfest's theorem for generalized fields of dyons. In section (6) we have discussed the validity of Ehrenfest's theorem for two potential Abelian gauge theories of dyons. Starting with the Lagrangian density (35) of a generalized electromagnetic field for a minimally coupled source of electric and magnetic charges in an Abelian gauge, it is shown that the Lagrangian density yields the classical equation of motion expressed by equation (36) and (37) respectively for electric and magnetic constituents of dyons, where the subsidiary conditions are imposed as equation (38). Introducing the two different auxiliary Hermitian scalar fields $\alpha(x)$ and $\beta(x)$ for the electric and magnetic couplings of dyons, we have suitably handled the Lagrangian density by equation (39); this Lagrangian density yields the quantum equation of motion given by equations (40) and (41) respectively associated with electric and magnetic charge of dyons. Here the fields in equation (40) and (41) are described as operators acting on the various functions (state) in indefinite metric quantum mechanical Hilbert space. Therefore, here the method of quantization can be described as the "operator method of quantization" contrary to the classical case where the physical meaningful degrees of freedom contribute only to the observables. Imposing the Gupta subsidiary condition (42) on the two photons given in terms of $\alpha(x)$ and $\beta(x)$ respectively associated with electric and magnetic charges of dyons, we have obtained the modified form of field equations (43) and (44) as the expectation values for electric and magnetic charges. Comparing the equations (45) and (46) with equation (36) and (37), it is shown that the expectation values of quantum equation of motion reproduce the classical equations of motion, which is nothing but the generalization of Ehrenfest's theorem to Abelian quantum field theory for generalized fields of dyons since Ehrenfest's theorems establish a formal connection between the time dependence of quantum mechanical expectation values of observables and the corresponding classical equations of motion. As such, the validity of Ehrenfest's theorem has been justified for Abelian $U(1) \times U(1)$ gauge theory of dyons in terms of two photons.

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