

## The third kind of Darboux transformation and multisoliton solutions for generalized Broer–Kaup equations

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**Abstract:** In this paper, the third kind of Darboux transformation of generalized Broer–Kaup equations is derived from the corresponding spectral problem. By virtue of this Darboux transformation, new  $2N$ -soliton solutions with parameters of the generalized Broer–Kaup equations are obtained. Although  $2N$  is an even number, it is graphically shown that in the cases of  $N = 1$  and  $N = 2$  the obtained  $2N$ -soliton solutions can degenerate into  $M$ -soliton solutions for any positive integer  $M$  less than  $2N$ .

**Key words:** Darboux transformation, multisoliton solution, spectral problem, generalized Broer–Kaup equations

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### 1. Introduction

Since the celebrated Kortweg–de Vries (KdV) equation was exactly solved in 1967 [1], finding soliton solutions of nonlinear partial differential equations (PDEs) has become one of the most exciting and active areas of research investigation. In the past several decades, many effective methods have been proposed for solving nonlinear PDEs, such as the inverse scattering transformation [2], Hirota’s bilinear method [3], Darboux transformation (DT) [4], Painlevé expansion [5], homogeneous balance method [6], function expansion methods [7–12], and others [13–15]. One of the most useful methods to construct soliton solutions is the DT [4], which has been used in a wide range of applications [16–24]. By virtue of DTs, Geng and Tam [16] obtained soliton solutions of a generalized nonlinear Schrödinger equation, Li and Zhang [17] obtained multiple soliton solutions of the classical Boussinesq system, Zhang [18] obtained explicit solutions of a finite-dimensional Hamiltonian system associated with the KdV equation, Chen and Li [19] obtained new soliton solutions of the Boussinesq–Burgers (BK) equation, Zhaqilao and Li [20] obtained some new bidirectional soliton solutions of a  $(2+1)$ -dimensional soliton equation, and Guo et al. [21] obtained two types of breather solutions of a generalized nonlinear Schrödinger–Maxwell–Bloch system. The DT can also be used to construct Laplace sequences in curvature surface and projective space [22]. The DT, which transforms the spectral problem into another spectral problem of the same form, is a gauge transformation [23] of the spectral parameter. More recently, by using a Lie algebra and Tu–Ma scheme [25,26], Zhang et al. [24] constructed a new integrable soliton hierarchy:

$$u_{t_n} = \begin{pmatrix} v \\ w \end{pmatrix}_{t_n} = \begin{pmatrix} 2a_{n+1,x} \\ -2b_{n+1} - 2wa_{n+1} \end{pmatrix}, \quad (1)$$

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with the following recursive relations:

$$\begin{aligned} b_{n+1} &= \frac{1}{2}(b_{nx} - 2a_{nx}) + \frac{1}{2}vb_n + wc_n, \\ a_{n+1} &= -\frac{1}{2}a_{nx} + \frac{1}{2}va_n - c_n, \\ c_{n,x} &= 2wa_n + 2b_n. \end{aligned}$$

The Hamiltonian form of the integrable hierarchy (1) is given by [24]:

$$u_{t_n} = \begin{pmatrix} v \\ w \end{pmatrix}_{t_n} = JL \frac{\delta H_n}{\delta u},$$

where the Hamiltonian operator  $J$  and the recurrence operator  $L$  are employed as

$$J = \begin{pmatrix} 0 & -\partial \\ -\partial & 0 \end{pmatrix}, \quad L = \begin{pmatrix} \frac{1}{2}\partial^{-1}v\partial + \frac{1}{2} & \frac{1}{2}(\partial^{-1}w\partial + w + 2) \\ 2 & \frac{1}{2}(v - \partial) \end{pmatrix}.$$

If we set  $a_0 = b_0 = 0$ ,  $c_0 = n = 2$ ,  $t_2 = t$  [24], the integrable hierarchy (1) reduces to new generalized Broer–Kaup (gBK) equations as follows:

$$v_t = v_{xx} - 2vv_x - 4w_x, \tag{2}$$

$$w_t = -w_{xx} - 2(wv)_x - 2v_x, \tag{3}$$

whose Lax pair matrices include the spectral problem

$$\phi_x = U\phi, \quad U = \begin{pmatrix} -\lambda + \frac{1}{2}v & 1 \\ -2w - 2 & \lambda - \frac{1}{2}v \end{pmatrix} \tag{4}$$

and the auxiliary problem

$$\phi_t = V\phi, \quad V = \begin{pmatrix} 2\lambda^2 + \frac{1}{2}(v_x - v^2) & -2\lambda - v \\ 4\lambda(1 + w) + 2v + 2w_x + 2wv & -2\lambda^2 + \frac{1}{2}(v^2 - v_x) \end{pmatrix}, \tag{5}$$

where  $v = v(x, t)$  and  $w = w(x, t)$  are two potentials, and the constant  $\lambda$  is the spectral parameter.

In [24], the first two kinds of one-fold DTs, bilinear presentation and bilinear Bäcklund transformation of Eqs. (2) and (3), were obtained. Since the third kind of DT is more applicable than the first two kinds of DTs [23], in Section 2 we shall derive the third kind of  $N$ -fold DT of Eqs. (2) and (3) from Eqs. (4) and (5). In Section 3, the derived DT is employed to construct multisoliton solutions of Eqs. (2) and (3).

## 2. Darboux transformation

Constructing the third kind of DT of the gBK Eqs. (2) and (3) is to find the following gauge transformation:

$$\bar{\phi} = T\phi, \tag{6}$$

of the spectral problem of Eqs. (4) and (5). Here  $T$  is a two-order matrix to be determined later, and  $\bar{\phi}$  satisfies other spectral problems with the same form, namely

$$\bar{\phi}_x = \bar{U}\bar{\phi}, \quad \bar{U} = (T_x + TU)T^{-1}, \quad (7)$$

$$\bar{\phi}_t = \bar{V}\bar{\phi}, \quad \bar{V} = (T_x + TV)T^{-1}, \quad (8)$$

where  $\bar{U}$  and  $\bar{V}$  are the resulting matrixes of  $U$  and  $V$  in Eqs. (4) and (5) by replacing the old potentials  $v$  and  $w$  with the new potentials  $\bar{v}$  and  $\bar{w}$ .

To begin with, we suppose that the Darboux matrix is in the form

$$T = T(\lambda) = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \quad (9)$$

where

$$A(\lambda) = \lambda^N + \sum_{k=0}^{N-1} a_k \lambda^k, \quad B(\lambda) = \sum_{k=0}^{N-1} b_k \lambda^k, \quad C(\lambda) = \sum_{k=0}^{N-1} c_k \lambda^k, \quad D(\lambda) = \lambda^N + \sum_{k=0}^{N-1} d_k \lambda^k, \quad (10)$$

and  $\alpha$ ,  $a_k$ ,  $b_k$ ,  $c_k$ , and  $d_k$  ( $0 \leq k \leq N-1$ ) are undetermined functions of  $x$  and  $t$ .

Then we let

$$\phi(\lambda_j) = (\phi_1(x, t, \lambda_j), \phi_2(x, t, \lambda_j))^T, \quad (11)$$

$$\varphi(\lambda_j) = (\varphi_1(x, t, \lambda_j), \varphi_2(x, t, \lambda_j))^T \quad (12)$$

be two basic solutions of Eqs. (4) and (5) when  $\lambda = \lambda_j$ . To determine  $T$  in Eq. (9), we suppose that there exist constants  $\gamma_j$  ( $1 \leq j \leq N$ ) satisfying

$$A(\lambda_j)\phi_1(\lambda_j) + B(\lambda_j)\phi_2(\lambda_j) - \gamma_j[A(\lambda_j)\varphi_1(\lambda_j) + B(\lambda_j)\varphi_2(\lambda_j)] = 0, \quad (13)$$

$$C(\lambda_j)\phi_1(\lambda_j) + D(\lambda_j)\phi_2(\lambda_j) - \gamma_j[C(\lambda_j)\varphi_1(\lambda_j) + D(\lambda_j)\varphi_2(\lambda_j)] = 0, \quad (14)$$

which can be written as a linear algebraic system

$$A(\lambda_j) + \sigma_j B(\lambda_j) = 0, \quad C(\lambda_j) + \sigma_j D(\lambda_j) = 0, \quad (15)$$

or

$$\sum_{k=0}^{N-1} (a_k + \sigma_j b_k) \lambda_j^k = -\lambda_j^N, \quad (16)$$

$$\sum_{k=0}^{N-1} (c_k + \sigma_j d_k) \lambda_j^k = -\sigma_j \lambda_j^N, \quad (17)$$

with

$$\sigma_j = \frac{\varphi_2(\lambda_j) - \gamma_j \phi_2(\lambda_j)}{\varphi_1(\lambda_j) - \gamma_j \phi_1(\lambda_j)}, \quad 1 \leq j \leq 2N, \quad (18)$$

where constants  $\lambda_j$ ,  $\gamma_j$  ( $\lambda_j \neq \lambda_k$  as  $k \neq j$ ) are suitably chosen so that the determinants of coefficients of Eqs. (16) and (17) are nonzero. Therefore,  $a_k$ ,  $b_k$ ,  $c_k$ , and  $d_k$  ( $0 \leq k \leq N-1$ ) are uniquely determined by Eqs. (16) and (17), while  $\alpha$  will be further determined.

It is easy to see from Eqs. (9) and (10) that  $\det T(\lambda)$  is a  $2N$ th-order polynomial in  $\lambda$ :

$$\det T(\lambda_j) = A(\lambda_j)D(\lambda_j) - B(\lambda_j)C(\lambda_j). \quad (19)$$

Using Eq. (15) we have  $\det T(\lambda_j) = 0$ , which implies that  $\lambda_j (1 \leq j \leq 2N)$  are  $2N$  roots of  $\det T(\lambda_j)$ , namely

$$\det T(\lambda_j) = \prod_{j=1}^{2N} (\lambda - \lambda_j). \quad (20)$$

Therefore, we have the following proposition.

**Proposition 1.** *Let  $\alpha$  satisfy the equation in the form of*

$$\partial_x(\ln \alpha) = -\frac{b_{N-1,x}}{1 + 2b_{N-1}}, \quad (21)$$

and then the matrix  $\bar{U}$  determined by Eq. (7) has the same form as  $U$ , that is

$$\bar{U} = \begin{pmatrix} -\lambda + \frac{1}{2}\bar{v} & 1 \\ -2\bar{w} - 2 & \lambda - \frac{1}{2}\bar{v} \end{pmatrix}, \quad (22)$$

where the transformations transform the old potentials  $v$  and  $w$  into new potentials given by

$$\bar{v} = v - \frac{2b_{N-1,x}}{1 + 2b_{N-1}}, \quad (23)$$

$$\bar{w} = (1 + 2b_{N-1})(w + c_{N-1}) + 2b_{N-1}. \quad (24)$$

**Proof** Letting  $T^{-1} = T^*/\det T$  and

$$(T_x + TU)T^* = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix}, \quad (25)$$

then we can see that  $f_{11}(\lambda)$  and  $f_{22}(\lambda)$  are  $(2N + 1)$ th-order polynomials in  $\lambda$  and  $f_{12}(\lambda)$  and  $f_{21}(\lambda)$  are  $2N$ th-order polynomials in  $\lambda$ . On the other hand, when  $\lambda = \lambda_j (1 \leq j \leq 2N)$ , using Eqs. (4), (5), (11), (12), and (18), we can obtain a Riccati equation:

$$\sigma_{j,x} = (-2w - 2) + 2(\lambda_j - \frac{1}{2}v)\sigma_j - \sigma_j^2.$$

Through a direct computation, we can verify that all  $\lambda = \lambda_j (1 \leq j \leq 2N)$  are roots of  $f_{kl}(\lambda) (k, l = 1, 2)$ . Thus, Eq. (25) can be written as

$$(T_x + TU)T^* = (\det T)P(\lambda), \quad (26)$$

where  $P(\lambda)$  has the form

$$P(\lambda) = \begin{pmatrix} P_{11}^{(1)}\lambda + P_{11}^{(0)} & P_{12}^{(0)} \\ P_{21}^{(0)} & P_{22}^{(1)}\lambda + P_{22}^{(0)} \end{pmatrix}, \quad (27)$$

and  $P_{kl}^m(\lambda) (k, l = 1, 2, m = 0, 1)$  are independent of  $\lambda$ .

We further rewrite Eq. (26) as

$$T_x + TU = P(\lambda)T, \quad (28)$$

and then comparing the coefficients of  $\lambda^{N+1}$  and  $\lambda^N$  in Eq. (28) yields

$$P_{11}^{(1)} = -1, \quad P_{11}^{(0)} = \frac{1}{2}v + \partial_x(\ln \alpha), \quad (29)$$

$$P_{12}^{(0)} = \alpha^2(1 + 2b_{N-1}), \quad (30)$$

$$P_{22}^{(1)} = 1, \quad P_{22}^{(0)} = -\left[\frac{1}{2}v + \partial_x(\ln \alpha)\right], \quad (31)$$

$$P_{21}^{(0)} = -\frac{2}{\alpha^2}(w + c_{N-1} + 1). \quad (32)$$

With the help of Eqs. (21), (23), and (24), from Eqs. (29)–(32) we have

$$P_{11}^{(0)} = \frac{1}{2}\bar{v}, \quad P_{12}^{(0)} = 1, \quad P_{22}^{(0)} = -\frac{1}{2}\bar{v}, \quad P_{21}^{(0)} = -2\bar{w} - 2. \quad (33)$$

Then using Eqs. (22), (27), and (33), we can obtain  $\bar{U} = P(\lambda)$ . The proof is completed.  $\square$

**Proposition 2.** *Suppose that the time dependence of  $\alpha$  satisfies the following equation with respect to the variable  $t$ :*

$$\partial_t(\ln \alpha) = \partial_{xx}(\ln \alpha) - 2[\partial_x(\ln \alpha)]^2 - 2v\partial_x(\ln \alpha) - 4(w + 1)b_{N-1} - 2(1 + 2b_{N-1})c_{N-1}. \quad (34)$$

The matrix  $\bar{V}$  determined by Eq. (8) then has the same form as  $V$  in Eq. (5) except for changing  $v$  and  $w$  into  $\bar{v}$  and  $\bar{w}$ , and the old potentials  $v$  and  $w$  are mapped into new potentials according to the same DT in Eqs. (6), (23), and (24).

**Proof** In a similar way to the proof of Proposition 1, we let  $T^{-1} = T^*/\det T$  and

$$(T_t + TV)T^* = \begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda) \\ g_{21}(\lambda) & g_{22}(\lambda) \end{pmatrix}, \quad (35)$$

And then it is easy to see that  $g_{11}(\lambda)$  and  $g_{22}(\lambda)$  are  $(2N+2)$ th-order polynomials in  $\lambda$  and  $g_{12}(\lambda)$  and  $g_{21}(\lambda)$  are  $(2N+1)$ th-order polynomials in  $\lambda$ .

On the other hand, setting  $\lambda_j(1 \leq j \leq 2N)$  and using Eqs. (4), (5), (11), (12), and (18), we can obtain another Riccati equation:

$$\sigma_{j,t} = [4\lambda_j(1 + w) + 2v + 2w_x + 2wv] + 2[-2\lambda_j^2 + \frac{1}{2}(v^2 - v_x)]\sigma_j + (2\lambda_j + v)\sigma_j^2. \quad (36)$$

It is easy to verify that all  $\lambda_j(1 \leq j \leq 2N)$  are roots of  $g_{kl}(\lambda)(k, l = 1, 2)$ . Hence, Eq. (35) can be written as

$$(T_t + TV)T^* = (\det T)Q(\lambda), \quad (37)$$

with

$$Q(\lambda) = \begin{pmatrix} Q_{11}^{(2)}\lambda^2 + Q_{11}^{(1)}\lambda + Q_{11}^{(0)} & Q_{12}^{(1)}\lambda + Q_{12}^{(0)} \\ Q_{21}^{(1)}\lambda + Q_{21}^{(0)} & Q_{22}^{(2)}\lambda^2 + Q_{22}^{(1)}\lambda + Q_{22}^{(0)} \end{pmatrix}, \quad (38)$$

where  $Q_{kl}^{(m)}(\lambda)(k, l = 1, 2, m = 0, 1, 2)$  are independent of  $\lambda$ . Thus, Eq. (37) can be written as

$$T_t + TV = Q(\lambda)T. \quad (39)$$

We compare the coefficients of  $\lambda^{N+2}$ ,  $\lambda^{N+1}$ , and  $\lambda^N$  in Eq. (39) and then obtain

$$Q_{11}^{(2)} = 2, \quad Q_{11}^{(1)} = 0, \quad (40)$$

$$Q_{12}^{(1)} = -2\alpha^2(1 + 2b_{N-1}), \quad (41)$$

$$Q_{12}^{(0)} = \alpha^2(-v - 2a_{N-1} - 4b_{N-2} + 4b_{N-1} + 2d_{N-1} + 4b_{N-1}d_{N-1}), \quad (42)$$

$$Q_{11}^{(0)} = \partial_t(\ln \alpha) + \frac{1}{2}(v_x - v^2) + 4(w + 1)b_{N-1} + 2(1 + 2b_{N-1})c_{N-1}, \quad (43)$$

$$Q_{22}^{(2)} = -2, \quad Q_{22}^{(1)} = 0, \quad (44)$$

$$Q_{21}^{(1)} = \frac{4}{\alpha^2}(c_{N-1} + w + 1), \quad (45)$$

$$Q_{21}^{(0)} = \frac{1}{\alpha^2}[4c_{N-2} + 4(1 + w)d_{N-1} + 2v(1 + w) + 2w_x - 4c_{N-1}(1 + a_{N-1}) - 4(1 + w)a_{N-1}], \quad (46)$$

$$Q_{22}^{(0)} = -\partial_t(\ln \alpha) - \frac{1}{2}(v_x - v^2) - 4(w + 1)b_{N-1} - 2(1 + 2b_{N-1})c_{N-1}, \quad (47)$$

where  $a_{-1} = b_{-1} = c_{-1} = d_{-1} = 0$ .

At the same time, we compare the coefficient of  $\lambda^{N-1}$  in Eq. (28) and then obtain the following formulae:

$$a_{N-1,x} = (2w + 2)b_{N-1} + (1 + 2b_{N-1})c_{N-1}, \quad (48)$$

$$b_{N-1,x} = vb_{N-1} - a_{N-1} - 2b_{N-2} + d_{N-1} + 2b_{N-1}(d_{N-1} + 1), \quad (49)$$

$$c_{N-1,x} = -vc_{N-1} + 2c_{N-2} + (2w + 2)(d_{N-1} - a_{N-1}) - 2(1 + a_{N-1})c_{N-1}, \quad (50)$$

$$d_{N-1,x} = -c_{N-1} - (2w + 2)b_{N-1} - 2b_{N-1}c_{N-1}. \quad (51)$$

Substituting Eqs. (21), (23), (24), and (34) into Eqs. (41)–(43) and (45)–(47) yields

$$Q_{12}^{(1)} = -2, \quad Q_{12}^{(0)} = -\bar{v}, \quad Q_{11}^{(0)} = \frac{1}{2}(\bar{v}_x - \bar{v}^2), \quad (52)$$

$$Q_{21}^{(1)} = 4(\bar{w} + 1), \quad Q_{21}^{(0)} = 2\bar{v} + 2\bar{w}_x + 2\bar{w}\bar{v}, \quad Q_{22}^{(0)} = \frac{1}{2}(\bar{v}^2 - \bar{v}_x). \quad (53)$$

Therefore, we can conclude that  $\bar{V} = Q(\lambda)$  by using Eqs. (8), (52), and (53). The proof is completed.  $\square$

It is shown in Propositions 1 and 2 that transformations (6), (23), and (24) transform the Lax pairs (4) and (5) into other Lax pairs of the same form, i.e. Eqs. (7) and (8). This is to say that Eqs. (2) and (3) can be derived from either Lax pairs (4) and (5) or Lax pairs (7) and (8). The transformation  $(\phi, v, w) \rightarrow (\bar{\phi}, \bar{v}, \bar{w})$  is then called the third kind of DT of Eqs. (2) and (3). In other words, we reach the following theorem.

**Theorem 1.** *The old solutions  $(v, w)$  of the gBK equations (2) and (3) are mapped into their new solutions  $(\bar{v}, \bar{w})$  under the DT in Eqs. (6), (23), and (24), where  $b_{N-1}$  and  $c_{N-1}$  are determined by the linear algebraic equations (16) and (17).*

### 3. Multisoliton solutions

In this section, we apply the third kind of  $N$ -fold DT (6), (23), and (24) to construct multisoliton solutions of the gBK equations (2) and (3).

Firstly, we set seed solutions  $(v, w)$  being constants and we substitute them into the Lax pairs (4) and (5); two basic solutions are then obtained as follows:

$$\varphi(\lambda_j) = \begin{pmatrix} \cosh \xi_j \\ c_j \sinh \xi_j + k_j \cosh \xi_j \end{pmatrix}, \quad \phi(\lambda_j) = \begin{pmatrix} \sinh \xi_j \\ c_j \cosh \xi_j + k_j \sinh \xi_j \end{pmatrix}, \quad (54)$$

where

$$\xi_j = c_j(x + b_j t), \quad c_j = \sqrt{(\lambda_j - \frac{1}{2}v)^2 - 2w - 2}, \quad b_j = -2\lambda_j - v, \quad k_j = \lambda_j - \frac{1}{2}v, \quad (1 \leq j \leq 2N). \quad (55)$$

In view of Eq. (18), we have

$$\sigma_j = c_j \frac{\tanh \xi_j - \gamma_j}{1 - \gamma_j \tanh \xi_j} + k_j, \quad (1 \leq j \leq 2N). \quad (56)$$

Secondly, substituting Eqs. (55) and (56) into Eqs. (16) and (17) and then applying Cramer's rule to the resulting equations, we have the following theorem.

**Theorem 2.** *The third kind of DT  $(\phi, v, w) \rightarrow (\bar{\phi}, \bar{v}, \bar{w})$  in Eqs. (6), (23), and (24) can generate the following  $2N$ -soliton solutions of the gBK equations (2) and (3):*

$$\bar{v}[N] = v - \frac{2b_{N-1,x}}{1 + 2b_{N-1}}, \quad (57)$$

$$\bar{w}[N] = (1 + 2b_{N-1})(w + c_{N-1}) + 2b_{N-1}, \quad (58)$$

where

$$b_{N-1} = \frac{\Delta_{b_{N-1}}}{\Delta}, \quad c_{N-1} = \frac{\Delta_{c_{N-1}}}{\Delta}, \quad (59)$$

$$\Delta = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & \sigma_1 \lambda_1 & \cdots & \lambda_1^k & \sigma_1 \lambda_1^k & \cdots & \lambda_1^{N-1} & \sigma_1 \lambda_1^{N-1} \\ 1 & \sigma_2 & \lambda_2 & \sigma_2 \lambda_2 & \cdots & \lambda_2^k & \sigma_2 \lambda_2^k & \cdots & \lambda_2^{N-1} & \sigma_2 \lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N} & \lambda_{2N} & \sigma_{2N} \lambda_{2N} & \cdots & \lambda_{2N}^k & \sigma_{2N} \lambda_{2N}^k & \cdots & \lambda_{2N}^{N-1} & \sigma_{2N} \lambda_{2N}^{N-1} \end{vmatrix}, \quad (60)$$

$$\Delta_{b_{N-1}} = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & \sigma_1 \lambda_1 & \cdots & \lambda_1^k & \sigma_1 \lambda_1^k & \cdots & \lambda_1^{N-1} & -\lambda_1^N \\ 1 & \sigma_2 & \lambda_2 & \sigma_2 \lambda_2 & \cdots & \lambda_2^k & \sigma_2 \lambda_2^k & \cdots & \lambda_2^{N-1} & -\lambda_2^N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N} & \lambda_{2N} & \sigma_{2N} \lambda_{2N} & \cdots & \lambda_{2N}^k & \sigma_{2N} \lambda_{2N}^k & \cdots & \lambda_{2N}^{N-1} & -\lambda_{2N}^N \end{vmatrix}, \quad (61)$$

$$\Delta_{c_{N-1}} = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & \sigma_1 \lambda_1 & \cdots & \lambda_1^k & \sigma_1 \lambda_1^k & \cdots & -\sigma_1 \lambda_1^N & \sigma_1 \lambda_1^{N-1} \\ 1 & \sigma_2 & \lambda_2 & \sigma_2 \lambda_2 & \cdots & \lambda_2^k & \sigma_2 \lambda_2^k & \cdots & -\sigma_2 \lambda_2^N & \sigma_2 \lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N} & \lambda_{2N} & \sigma_{2N} \lambda_{2N} & \cdots & \lambda_{2N}^k & \sigma_{2N} \lambda_{2N}^k & \cdots & -\sigma_{2N} \lambda_{2N}^N & \sigma_{2N} \lambda_{2N}^{N-1} \end{vmatrix}, \quad (62)$$

and  $\sigma_1, \sigma_2, \dots, \sigma_{2N}$  are determined by Eqs. (55) and (56).

In particular, giving  $N = 1$  to Eqs. (55) and (56), we have

$$b_0 = \frac{\lambda_1 - \lambda_2}{\sigma_2 - \sigma_1}, c_0 = \frac{\sigma_1 \sigma_2 (\lambda_2 - \lambda_1)}{\sigma_2 - \sigma_1}, \quad (63)$$

and hence we obtain two-soliton solutions of the gBK equations (2) and (3):

$$\bar{v}[1] = v + \frac{4(\lambda_1 - \lambda_2)(\lambda_2 \sigma_2 - \lambda_1 \sigma_1)}{(\sigma_2 - \sigma_1)(2\lambda_1 - 2\lambda_2 + \sigma_2 - \sigma_1)} - \frac{2(\lambda_1 - \lambda_2)(\sigma_1 + \sigma_2)}{(2\lambda_1 - 2\lambda_2 + \sigma_2 - \sigma_1)} - \frac{2v(\lambda_1 - \lambda_2)}{(2\lambda_1 - 2\lambda_2 + \sigma_2 - \sigma_1)}, \quad (64)$$

$$\bar{w}[1] = w + \frac{(\lambda_1 - \lambda_2)(2 + 2w - \sigma_1 \sigma_2)}{(\sigma_2 - \sigma_1)} - \frac{2\sigma_1 \sigma_2 (\lambda_1 - \lambda_2)^2}{(\sigma_2 - \sigma_1)^2}, \quad (65)$$

where  $\sigma_1$  and  $\sigma_2$  are determined by Eqs. (55) and (56).

When  $N = 2$ , Eqs. (57) and (58) give the following four-soliton solutions of the gBK equations (2) and (3):

$$\bar{v}[2] = v - \frac{2b_{1,x}}{1 + 2b_1}, \quad (66)$$

$$\bar{w}[2] = (1 + 2b_1)(w + c_1) + 2b_1, \quad (67)$$

where

$$b_1 = \frac{\Delta_{b_1}}{\Delta}, \quad c_1 = \frac{\Delta_{c_1}}{\Delta}, \quad (68)$$

$$\Delta = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & \sigma_1 \lambda_1 \\ 1 & \sigma_2 & \lambda_2 & \sigma_2 \lambda_2 \\ 1 & \sigma_3 & \lambda_3 & \sigma_3 \lambda_3 \\ 1 & \sigma_4 & \lambda_4 & \sigma_4 \lambda_4 \end{vmatrix}, \quad \Delta_{b_1} = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & -\lambda_1^2 \\ 1 & \sigma_2 & \lambda_2 & -\lambda_2^2 \\ 1 & \sigma_3 & \lambda_3 & -\lambda_3^2 \\ 1 & \sigma_4 & \lambda_4 & -\lambda_4^2 \end{vmatrix}, \quad \Delta_{c_1} = \begin{vmatrix} 1 & \sigma_1 & -\sigma_1 \lambda_1^2 & \lambda_1 \sigma_1 \\ 1 & \sigma_2 & -\sigma_2 \lambda_2^2 & \lambda_2 \sigma_2 \\ 1 & \sigma_3 & -\sigma_3 \lambda_3^2 & \lambda_3 \sigma_3 \\ 1 & \sigma_4 & -\sigma_4 \lambda_4^2 & \lambda_4 \sigma_4 \end{vmatrix}, \quad (69)$$

and  $\sigma_1, \sigma_2$  and  $\sigma_3$  are determined by Eqs. (55) and (56).

It is remarkable that  $2N$ -solutions (57) and (58) are such types of soliton solutions that can degenerate into even-soliton solutions and odd-soliton solutions as long as the embedded parameters are suitably selected. In Figures 1 and 2, two seed solutions  $v = 0$  and  $w = -0.05$  are selected to show the spatial structures of two-soliton solutions (64) and (65) with parameters  $\lambda_1 = 2.5, \lambda_2 = -2.5, \gamma_1 = 0.1, \gamma_2 = 0.2$ . However, if we set



$|\gamma_1| = 1$  or  $|\gamma_2| = 1$ , the two-soliton solutions (64) and (65) can degenerate into single-soliton solutions. For example, when  $\lambda_2 = -1$  and other parameters are selected as those in Figures 1 and 2, solutions (64) and (65) degenerate into one-soliton solutions as shown in Figures 3 and 4.

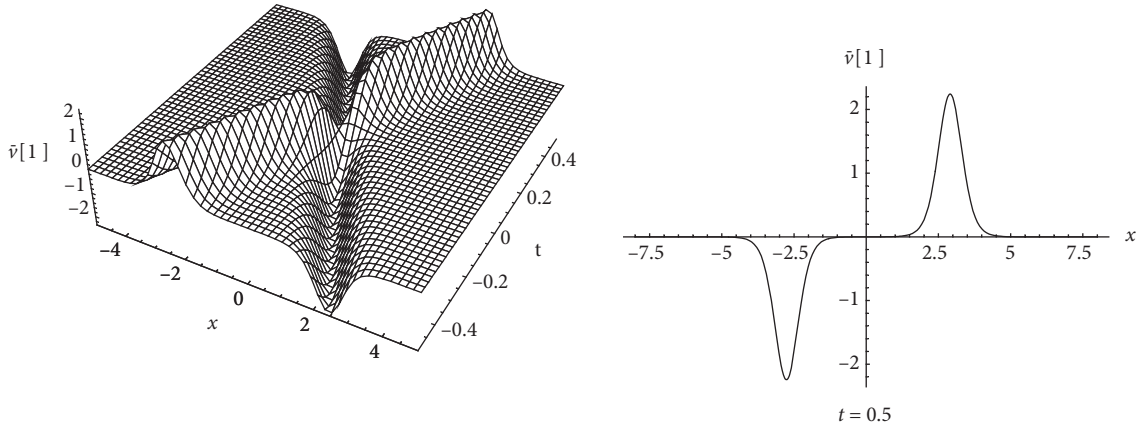


Figure 1. Two-soliton solution determined by solution (64).

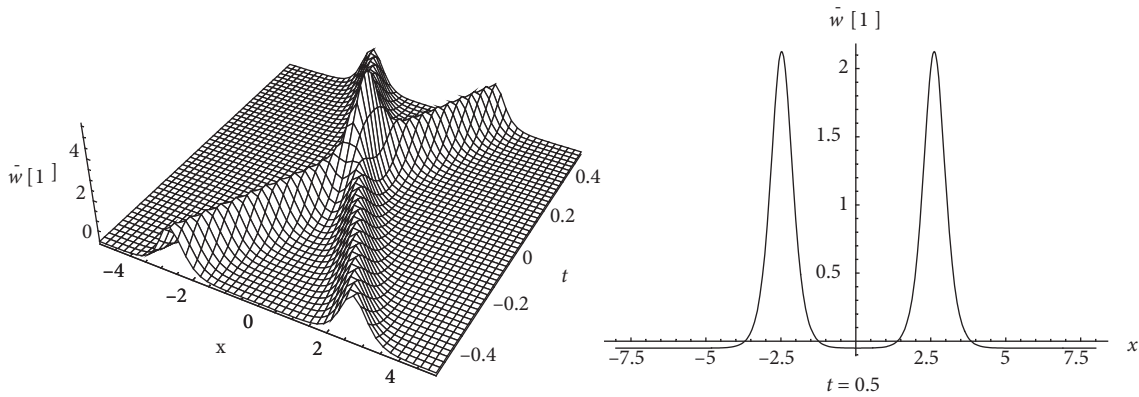


Figure 2. Two-soliton solution determined by solution (65).

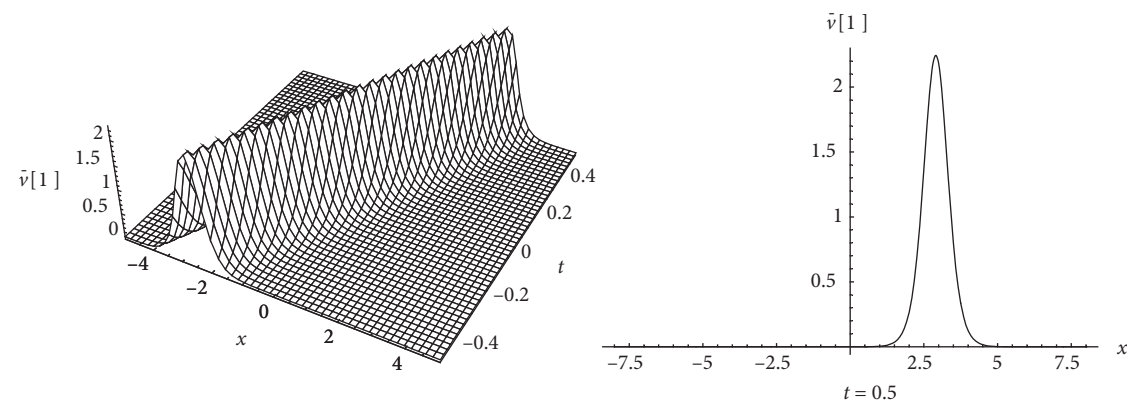
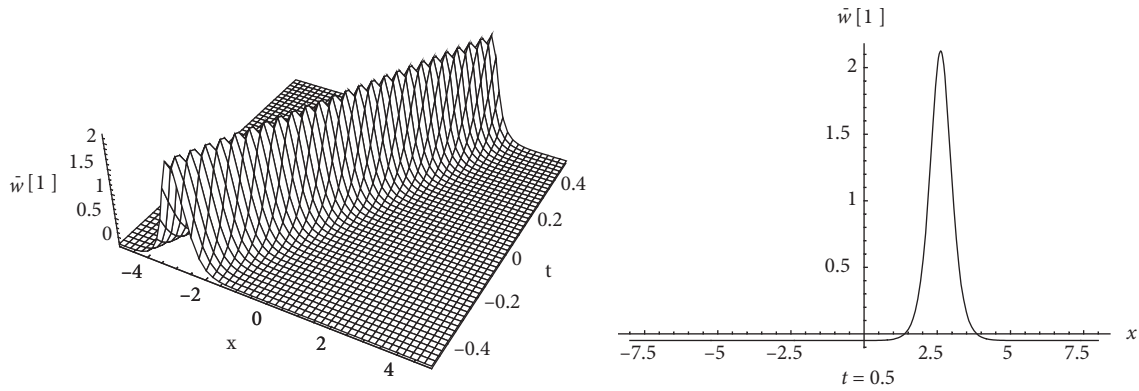
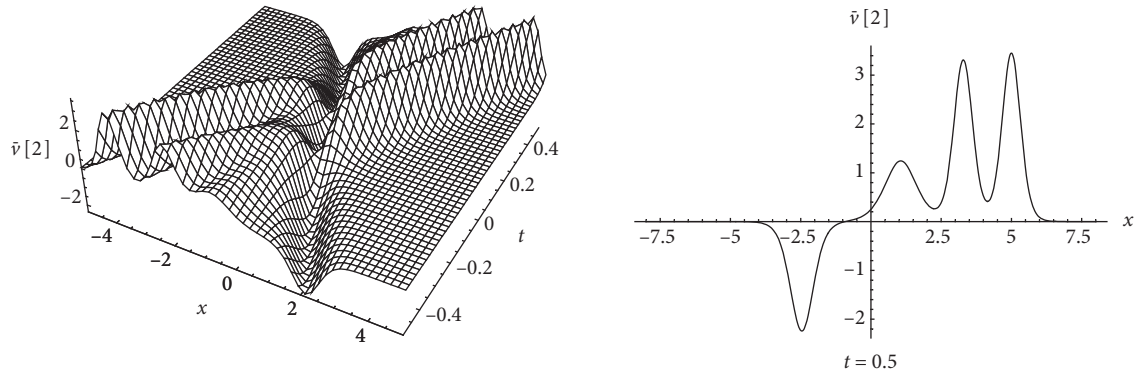


Figure 3. One-soliton solution degenerated from solution (64).

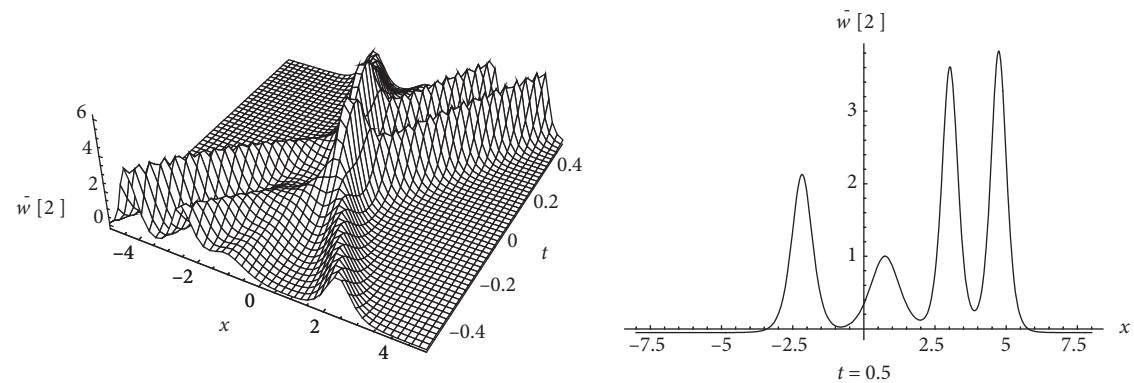


**Figure 4.** One-soliton solution degenerated from solution (65).

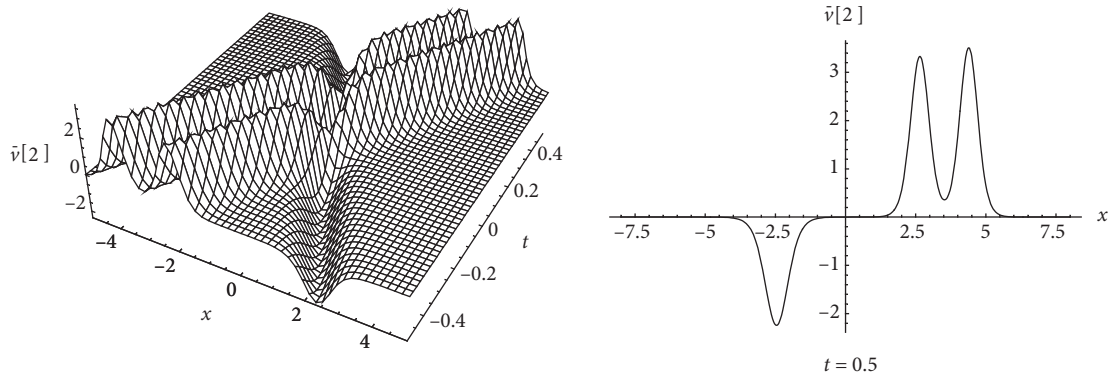
Similarly, we show in Figures 5–12 that the four-soliton solutions (66) and (67) can also degenerate into three-soliton solutions, two-soliton solutions, and one-soliton solutions. In Figures 5–12, the seed solutions are set as  $v = 0$  and  $w = -0.05$ , and the same parameters are chosen as  $\lambda_1 = 2, \lambda_2 = -2.5, \lambda_3 = 3, \lambda_4 = 3.2$  while different parameters are chosen as  $\gamma_1 = 0, \gamma_2 = 0.2, \gamma_3 = 1.5, \gamma_4 = 0.5$  in Figures 5 and 6;  $\gamma_1 = 1, \gamma_2 = 0.2, \gamma_3 = 1.5, \gamma_4 = 0.5$  in Figures 7 and 8;  $\gamma_1 = 1, \gamma_2 = -1, \gamma_3 = 1.5, \gamma_4 = 0.5$  in Figures 9 and 10; and  $\gamma_1 = -1, \gamma_2 = -1, \gamma_3 = 1.5, \gamma_4 = 1$  in Figures 11 and 12.



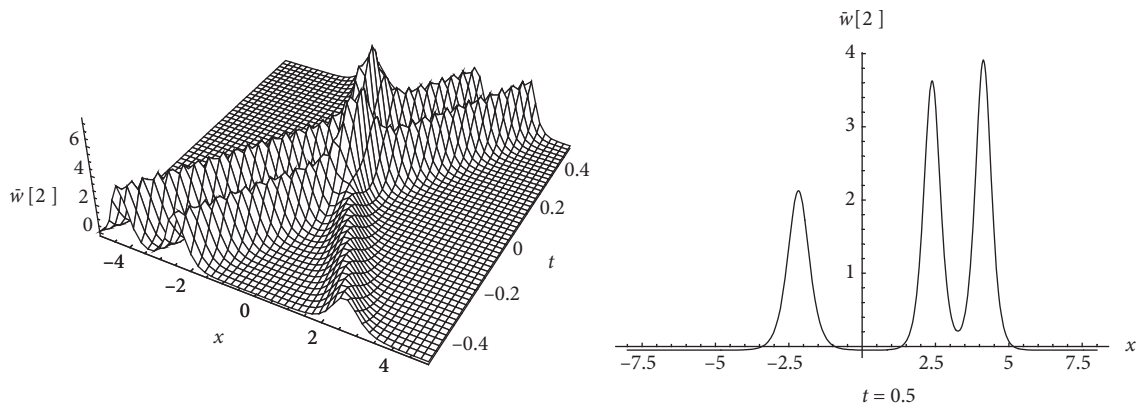
**Figure 5.** Four-soliton solution determined by solution (66)



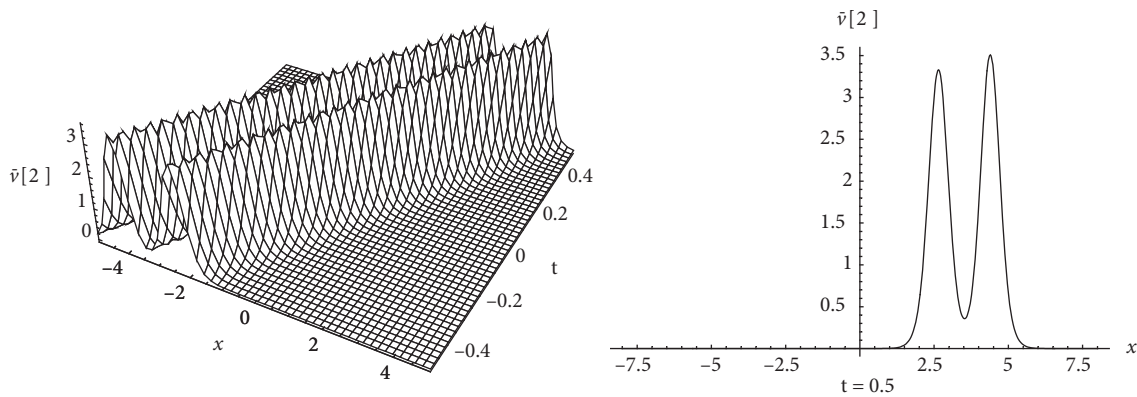
**Figure 6.** Four-soliton solution determined by solution (67).



**Figure 7.** Three-soliton solution degenerated from solution (66).



**Figure 8.** Three-soliton solution degenerated from solution (67).



**Figure 9.** Two-soliton solution degenerated from solution (66).

#### 4. Conclusion

In summary, we have obtained the third kind of  $N$ -fold DT (6), (23), and (24) of the gBK equations (2) and (3). By virtue of this DT,  $2N$ -soliton solutions (57) and (58) with parameters of the gBK equations (2) and (3) are obtained. We would like to note that some relevant studies like those in [27–29] on the known original BK

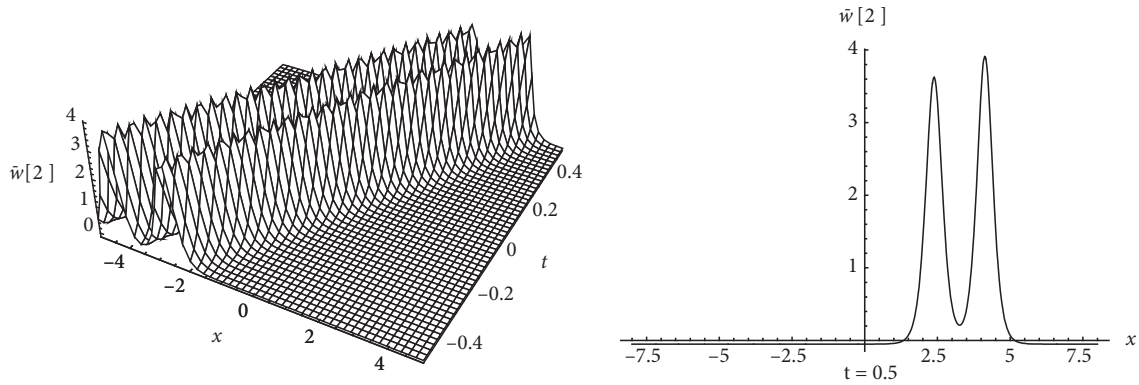


Figure 10. Two-soliton solution degenerated from solution (67).

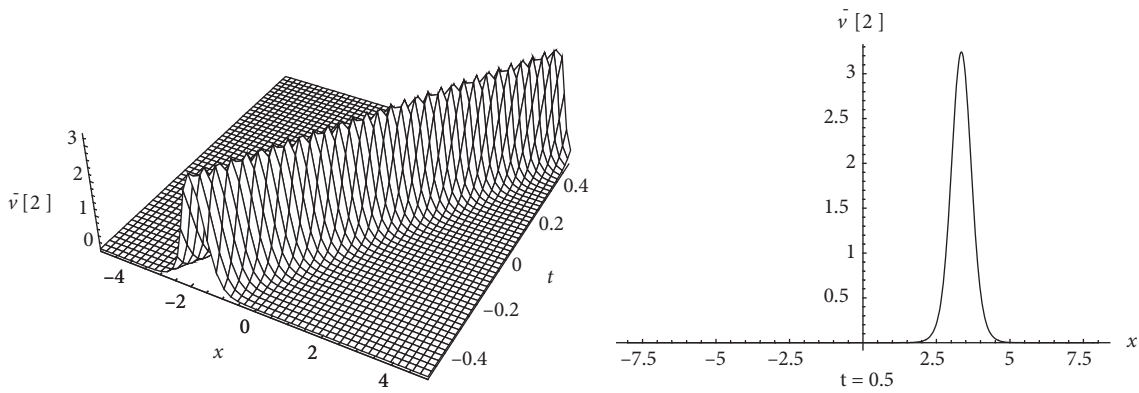


Figure 11. One-soliton solution degenerated from solution (66).

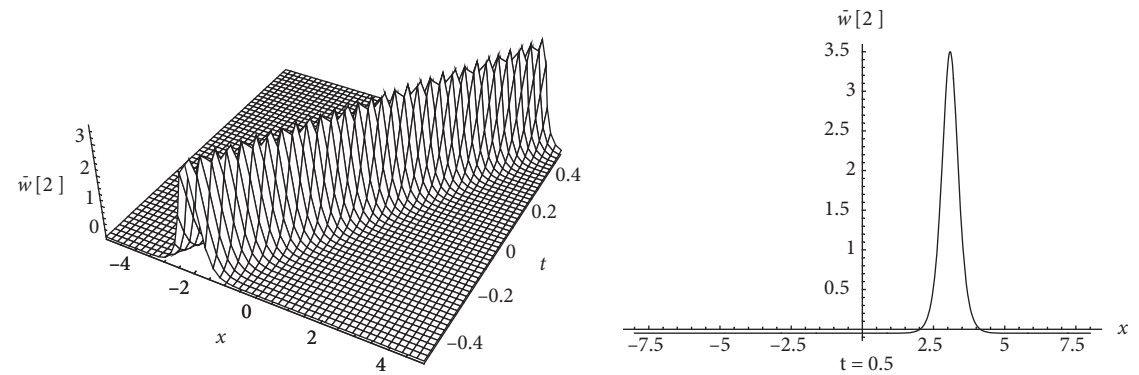


Figure 12. One-soliton solution degenerated from solution (67).

equation and DT have been given. However, solutions (57) and (58) are different from the known solutions [27–29] of the original BK equation because the gBK equations (2) and (3) are new. To the best of our knowledge, the DT (6), (23), and (24) and  $2N$ -soliton solutions (57) and (58) have not been reported in the literature. It is graphically shown that the  $2N$ -soliton solutions (57) and (58) obtained through the third kind of DT can degenerate into even-soliton solutions and odd-soliton solutions. How to construct the third kind of DTs and multisoliton solutions of some other new nonlinear PDEs is worthy of study. This is our task in the future.

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