

Adiabatic loading of rotating bosons into 1D optical lattice

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Abstract: In this paper, the entropy-temperature curves are investigated for rotating boson gases in a 1D optical lattice with transverse harmonic confinement. Regimes at which the atomic sample can be significantly heated or cooled by adiabatically changing the lattice depth and the rotation rate are demonstrated. The suggested approach, which is the semiclassical approximation, includes the correction due to the finite size effect. The obtained results show that the entropy-temperature curves have a monotonically increasing nature for this system, in agreement with the standard thermodynamic arguments. For fixed temperature, entropy always increases with the increasing of the rotation rate or optical potential depth.

Key words: Entropy of boson systems, static properties of condensates, semiclassical theories and applications

1. Introduction

Ultracold rotating bosonic or fermionic atoms in optical lattices are promising tools to simulate problems from condensed matter physics. Experiments have been able to observe a quantum phase transition from a superfluid to Mott-insulating state for bosons [1, 2, 3, 4, 5], the cross-over between quantum tunneling, and thermal activation of phase slips [6, 7]. Straightforward interpretation of these results in certain cases has been complicated by difficulties in measuring temperature related to strong interactions and the lattice potential [8], as well as rotation rate [9, 10, 11].

Usually, preparing a sample of quantum degenerate rotating bosons in an optical lattice consists of first forming an ultracold Bose-Einstein condensation (BEC) in a weak magnetic trap, to which a 1D, 2D, or 3D lattice potential is adiabatically applied [12, 13, 14, 15]. It has been pointed out that loading sufficiently cold, noninteracting atoms into an optical lattice can lead to adiabatic cooling [16, 17, 18], but the available cooling in a real system will clearly depend on and be limited by interactions. It can also depend on the trapping potential (typically harmonic), which provides additional energy in the problem, as well as on the finite size of the sample. Indeed, due to changes of the system's energy spectrum occurring in lattice loading, the initial and final temperatures are not trivially related. It is also of interest to understand how the transition temperature T_0 changes in the lattice to assess the effects that the optical potential and rotation have on the condensation.

Previous studies have considered the entropy-temperature curves of the rotating condensate [12] or optically trapped boson separately [13, 14, 15]. Surprisingly less attention has been paid to the entropy-temperature curves for a system of rotating condensate in an optical lattice. Therefore, it is important to consider the entropy-temperature curves under realistic experimental conditions. These curves can be used for

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analyzing the effect of the optical potential depth and rotation rate on the temperature of the system, and moreover in analyzing the process of loading the rotating condensate into the optical lattice [19, 20, 21].

In this paper, the entropy–temperature relation of rotating bosons in a 1D optical lattice is considered. A good semiclassical approximate [22, 23, 24, 25, 26] is provided. The advantage of the semiclassical approach lies in its simplicity, in comparison to the quantum-mechanical calculations (Bose Hubbard model) [27, 28], and its generality allows the treatment of the finite temperature regime [29, 30]. Our approach can be summarized as follow: a conventional method of quantum mechanics is used to calculate the localized spectrum of this system [31, 32, 33, 34], in which the classical analogy approach first used by Fetter [35] is employed. The parametrized formula for the localized spectrum is used in calculating the accurate density of states (DOS) for the system under consideration. Working with the grand canonical ensemble, the parametrized DOS is used in calculating the thermodynamic potential. The latter is used in calculating analytical expressions for the entropy for various cases. The obtained results include the finite size effect, which allows for comparison of temperatures for adiabatic changes in the rotating lattice.

For the specific trap parameters of the experiment of Spielman et al. [3], the oscillation frequencies are $\omega_{\perp}/2\pi = 500$ Hz, $\omega_z/2\pi = 33000$ Hz, and the number of particles is $N = 1.2 \times 10^4$. The calculated results showed that the entropy of the rotating localized condensate in the optical lattice is completely different from that of a rotating or the optical lattice trap separately.

The paper is organized as follows: Section 2 includes the model and methods for calculating the energy spectrum and DOS of the system. The thermodynamic potential and entropy are presented in Section 3. The conclusion is summarized in the last section.

2. Energy spectrum and density of states of the rotating bosons in a 1D optical lattice

Let us consider a system of noninteracting boson particles confined in an external harmonic potential

$$V(\mathbf{r}) = \frac{1}{2}m(\omega_{\perp}^2 r^2 + \omega_z^2 z^2), \quad (1)$$

where $r^2 = x^2 + y^2$; ω_{\perp} and ω_z are the radial and axial trapping frequencies, respectively; and m is the atomic mass. This cigar-shaped condensate also experiences a one-dimensional optical lattice potential,

$$V_{lat} = sE_R[\sin^2(\frac{\pi z}{d})]. \quad (2)$$

In Eq. (2) the optical potential depth is expressed in terms of the dimensionless factor s and the recoil energy. The recoil energy is defined as the recoil energy that one atom requires when it absorbs one lattice photon, $E_R = \frac{\pi^2 \hbar^2}{2md^2}$, where d is the lattice spacing. The trapped condensate is also subjected to an externally impressed rotation at an angular velocity $\mathbf{\Omega}(\equiv \Omega \hat{e}_z)$ around the z -axis. The noninteracting single particle Hamiltonian has the form

$$H = H_r + H_z \quad (3)$$

with

$$\begin{aligned} H_r &= \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega_{\perp}^2(x^2 + y^2) - \Omega L_z, \\ H_z &= \frac{p_z^2}{2m} + \frac{1}{2}m\omega_z^2 z^2 + sE_R[\sin^2(\frac{\pi z}{d})], \end{aligned} \quad (4)$$

where $L_z = xp_y - yp_x$ is the angular momentum in the z -direction.

The Hamiltonian of Eq. (3) is a completely separable Hamiltonian that can be written as a sum of two independent parts in the radial, H_r , and axial, H_z , direction, respectively. The radial part is characterized by the energy eigenvalues [35],

$$\epsilon_r = n_+ \hbar(\omega_\perp - \Omega) + n_- \hbar(\omega_\perp + \Omega) + \hbar\omega_\perp, \quad (5)$$

with n_\pm being nonnegative integers.

For the axial Hamiltonian part, previous studies have confirmed that the eigenfunctions are localized in a band-like structure [32, 33] for shallow lattice depth. Each one of the eigenfunctions, with initial energy smaller than E_R , is localized in a different well. Viverit et al. [33] pointed out that the energies in the respective pseudoband are approximately given by a sum of the potential energy due to the harmonic trap potential at the position of the particular lattice-well and a term representing the level of excitation with the lattice-well potential itself. The latter can be approximated by the harmonic oscillator ground state energy and moreover allows one to distinguish between the distinct ‘‘pseudobands’’. However, for a shallow lattice the energy eigenvalues are approximated by

$$\epsilon_z^{(0)} = \hbar\omega_z \gamma_z n_z^2 + \sqrt{s} E_R \quad (6)$$

for the first band, with $n_z = 0, \pm 1, \pm 2$. $\sqrt{s} E_R$ can be approximated by $\frac{1}{2} \hbar\omega_z$ for the first pseudoband and $\gamma_z = \frac{\pi^2 \omega_z}{4\omega_R}$ is a dimensionless parameter. The spectrum of Eq. (6) is characteristic of all states in the first pseudoband while the states in the second pseudoband have energy given by $\hbar\omega_z \gamma_z n_z^2 + \frac{3}{2} \hbar\omega_z$. The localized states are here exact eigenstates of the system for finite values of s .

Finally, the single particle energy eigenvalues for the Hamiltonian of Eq. (3) are given by

$$\epsilon_{n_+, n_-, n_z} = \hbar\omega_\perp (\gamma_+ n_+ + \gamma_- n_-) + \epsilon_0 + \hbar\omega_z \gamma_z n_z^2 + \sqrt{s} E_R, \quad (7)$$

where $\gamma_\pm = (1 \mp \alpha)$, $\alpha = \Omega/\omega_\perp$ is the rotation rate, and $\epsilon_0 = \hbar\omega_\perp$. Eigenvalues in Eq. (7) show that the optical potential provided a new energy scale parameter, which is $E_R = \hbar\omega_R$. However, this scale energy parameter does not fulfill the requirement $\hbar\omega_R \gg k_B T$, and consequently the excitation in the z -direction will be suppressed. These suppressions have a more dramatic effect on the appropriate DOS for this system.

The DOS for this system was calculated in our previous paper [36] using the method that was used by Kisten and Toms [25, 26] and given by:

$$\rho(\epsilon) = \frac{2}{\pi} \left(\frac{\omega_R}{\omega_g} \right)^{1/2} \frac{1}{1 - \alpha^2} \left\{ \frac{4}{3} \frac{\epsilon^{3/2}}{(\hbar\omega_g)^{5/2}} + \frac{\omega_\perp + \sqrt{s}\omega_R}{\omega_g} \frac{\epsilon^{1/2}}{(\hbar\omega_g)^{3/2}} \right\}, \quad (8)$$

where $\omega_g = (\omega_\perp^2 \omega_z)^{1/3}$ is the geometrical mean of the harmonic trap frequencies. The DOS in Eq. (8) reveals that, when atoms are loaded into an optical lattice, the DOS changes from the three-dimensional harmonic oscillator result $\rho(\epsilon) \propto \epsilon^2$ to the low-tunneling result $\rho(\epsilon) \propto \epsilon^{2-d/2}$, with d being the optical lattice dimensions. Rotation effect leads to a shift in the radial harmonic oscillator frequencies, but they still fulfill the condition $\hbar\omega_\perp (1 \pm \alpha_c)$, with α_c being the critical rotation rate. The latter provides the criterion stability of the rotating condensate; it does not necessarily indicate the critical frequency for vortex nucleation. The corresponding thermodynamic rotation rate can be estimated using the following relation[37]:

$$\alpha_c \approx 1 - \frac{Na}{\sqrt{8\pi} d_z},$$

where a is the scattering length and $d_z = \sqrt{\frac{\hbar}{m\omega_z\gamma_z}}$ is the ground state spatial extension for the combined harmonic-optical potential.

In the following, we will see that the adiabatical changing of the DOS of a system can lead to the cooling or heating of a gas. Results of Eq. (8) provide a consistent way for treating the effects of the lattice recoil frequency and the rotation rate on the thermodynamic properties. Additionally, its validity is extended to include the finite size effect. Moreover, the effective thermodynamic limit is defined for any confining potential.

3. Entropy of the system

It has been pointed out that loading sufficiently cold, noninteracting boson atoms into a static optical lattice and ramping the rotational frequency can lead to adiabatic cooling. The requirements of this type of loading can be understood from the entropy considerations [16, 20]. For the rotating condensate in an optical lattice, the normalized entropy per particle is given by [29]

$$\frac{\mathcal{S}}{Nk_B} = \frac{q}{N} + \frac{\beta E}{N} - \beta\mu(\Omega, sE_R), \quad (9)$$

where $\beta = (1/k_B T)$, q is the grand-canonical potential, $q = \ln Z$ with Z being the grand canonical partition function, E and N are respectively the total energy and total number of particles, and μ is the chemical potential.

The general expression for the q -potential of an ideal Bose gas is given by summing over all states in 3D [25, 26, 38, 39]:

$$q = - \sum_{n_x, n_y, n_z} \ln(1 - e^{-\beta(\epsilon_{n_x, n_y, n_z} - \mu)}). \quad (10)$$

In Eq. (10), when the thermal energies far exceed the level spacing ($k_B T \gg \omega_\perp^2 \omega_z (1 - \alpha^2) \gamma_z$), it is possible to approximate the discrete sum by an integral. A crucial feature in obtaining a reliable approximation is to use an accurate DOS. After separating the ground state energy level, we have

$$\begin{aligned} q &= q_0 + \sum_{n_x, n_y, n_z} \frac{z e^{-\beta \epsilon_{n_x, n_y, n_z}}}{1 - z e^{-\beta \epsilon_{n_x, n_y, n_z}}} \\ &= q_0 + \sum_{j=1}^{\infty} \frac{z^j}{j} \int_0^{\infty} \rho(\epsilon) e^{-j\beta\epsilon} d\epsilon, \end{aligned} \quad (11)$$

where $z = e^{(\mu - \epsilon_0)}$ is the fugacity.

Substituting Eq. (8) into Eq. (11), the q -potential of our system becomes

$$q = q_0 + \frac{2}{\sqrt{\pi}(1 - \alpha^2)} \left(\frac{\omega_R}{\omega_g} \right)^{1/2} \left\{ \left(\frac{k_B T}{\hbar \omega_g} \right)^{5/2} g_{7/2}(z) + \left(\frac{\omega_\perp + \sqrt{s\omega_R}}{\omega_g} \right) \left(\frac{k_B T}{\hbar \omega_g} \right)^{3/2} g_{5/2}(z) \right\}, \quad (12)$$

where $g_k(z) = \sum_{j=1}^{\infty} (z^j / j^k)$ is the usual Bose function. Once the fugacity z has been determined, all thermodynamically relevant quantities can be calculated by the partial derivative of the grand potential q .

In terms of the q -potential, Eq. (12), the total number of particles is given by [24, 25, 26]:

$$\begin{aligned} N &= z \left(\frac{\partial q}{\partial z} \right)_T, \\ &= N_0 + \frac{2}{\sqrt{\pi}(1-\alpha^2)} \left(\frac{\omega_R}{\omega_g} \right)^{1/2} \left\{ \left(\frac{k_B T}{\hbar \omega_g} \right)^{5/2} g_{5/2}(z) + \left(\frac{\omega_\perp + \sqrt{s} \omega_R}{\omega_g} \right) \left(\frac{k_B T}{\hbar \omega_g} \right)^{3/2} g_{3/2}(z) \right\}, \end{aligned} \quad (13)$$

where $N_0 = z/(1-z)$ is the number of the condensate particles in the ground state and

$$T_0 \approx \frac{\hbar \omega_g \left[\sqrt{\pi}(1-\alpha^2) \sqrt{\omega_g/\omega_R} \right]^{2/5}}{k_B} \left(\frac{N}{\zeta(5/2)} \right)^{2/5} \quad (14)$$

is the BEC transition temperature for the ideal Bose gas trapped in a rotating harmonic oscillator in 1D optical potential. In terms of the q -potential, the total energy is given by

$$E = k_B T^2 \left(\frac{\partial q}{\partial T} \right)_z. \quad (15)$$

The chemical potential of the rotating condensate in an optical lattice can be parametrized using the approaches of Hadzibabic and colleagues [40, 41]. They identified a relevant interaction energy scale to explore the relationship between the nonsaturation of the ideal Bose gases and the interatomic interactions for the pure harmonically trapped gas. The identified energy scale is given by

$$\mu_0(\omega_g) = \frac{\hbar \omega_g}{2} \left(15 N_0 \frac{a}{a_{har}} \right)^{2/5}, \quad (16)$$

where a is the s-wave scattering length and $a_{har} = \sqrt{\hbar/m\omega_g}$ is the ground state spatial extension for the harmonic oscillator. It is clear that the energy in Eq. (16) is equivalent to the mean-field prediction for the nonrotating chemical potential $\mu(T=0)$ of a harmonically trapped gas with N_0 condensate atoms in the Thomas–Fermi limit.

Generalization of the Hadzibabic results that include the rapid rotation effects (slowly rotating condensate with $\Omega < \Omega_c$, where Ω_c is the critical angular velocity for vortex formation) can be obtained by using Fetter [35] results, i.e.

$$\mu_0(\Omega) = \mu_0(\omega_g) (1 - \Omega^2/\omega_\perp^2)^{2/5} = \mu_0(\omega_g) (1 - \alpha^2)^{2/5}, \quad (17)$$

where $\mu_0(\omega_g)$ is the Hadzibabic identified energy scale given in Eq. (16). Eq. (17) identifies the relevant interaction energy scale for the rotating trapped gas. When the rotating bosons are adiabatically loading in the optical potential, the generalization for the result given in Eq. (17) can be obtained by using the results of Pedri et al. [42] for the localized chemical potential at the central lattice site. Thus, the Hadzibabic scaling energy for the rotating boson gas in an optical lattice is:

$$\begin{aligned} \mu(\Omega, sE_R) &= \mu_0(\omega_g) (1 - \alpha^2)^{2/5} \left(\frac{\pi^2 s}{4} \right)^{1/10} \\ &= \frac{1}{2} k_B T_0 \left\{ \frac{2}{\sqrt{\pi}} \left(\frac{\omega_R}{\omega_g} \right)^{1/2} \right\}^{2/5} \left(\frac{\zeta(5/2)}{N} \right)^{2/5} \left[15 N_0 \frac{a}{a_{har}} \right]^{2/5} \left[\frac{\pi^2 s}{4} \right]^{1/10}, \end{aligned} \quad (18)$$

where T_0 is given in Eq. (14). Now one has to use Dalfovo's interaction scaling parameter η [43]. This parameter fixed the chemical potential of the harmonically trapped gas in units of the transition temperature for the noninteracting gas in the same trap:

$$\eta \equiv \frac{\mu_0}{k_B T_0} = \frac{1}{2} \left(\frac{\zeta(5/2)}{N} \right)^{2/5} [15 N_0 \frac{a}{a_{har}}]^{2/5}. \quad (19)$$

In terms of η Eq. (18) becomes

$$\mu(\Omega, s E_R) = \eta k_B T_0 \left\{ \frac{2}{\sqrt{\pi}} \left(\frac{\omega_R}{\omega_g} \right)^{1/2} \right\}^{2/5} \left[\frac{\pi^2 s}{4} \right]^{1/10}. \quad (20)$$

The parametrized $\mu(\Omega, s E_R)$ in Eq. (20) looks like a generalization for the well known Thomas–Fermi approximation of rotating a condensate in an optical lattice [43].

Gathering Eqs. (9), (15), and (20), the entropy of the system is given by:

$$\frac{\mathcal{S}}{N k_B} = \frac{\mathcal{S}_0}{N k_B} + \frac{7}{2} \frac{\zeta(7/2)}{\zeta(5/2)} \mathcal{T}^{5/2} + \frac{5}{2} R(s, \alpha) \mathcal{T}^{3/2} - \eta \left[\frac{2}{\sqrt{\pi}} \left(\frac{\omega_R}{\omega_g} \right)^{1/2} \right]^{2/5} \left[\frac{\pi^2 s}{4} \right]^{1/10} \mathcal{T}^{-1}, \quad (21)$$

where $\mathcal{T} = T/T_0$ is the normalized temperature, \mathcal{S}_0 is the entropy of the condensate at the ground state, and

$$R(s, \alpha) = \left(\frac{\omega_{\perp} + \sqrt{s} \omega_R}{\omega_g} \right) \left[\frac{2 \zeta(5/2) \sqrt{\omega_R/\omega_g}}{\sqrt{\pi} (1 - \alpha^2)} \frac{1}{N} \right]^{2/5}. \quad (22)$$

The parameter $R(s, \alpha)$ is proportional to $(1 - \alpha^2) N^{-2/5}$, as well as being proportional to s and ω_R .

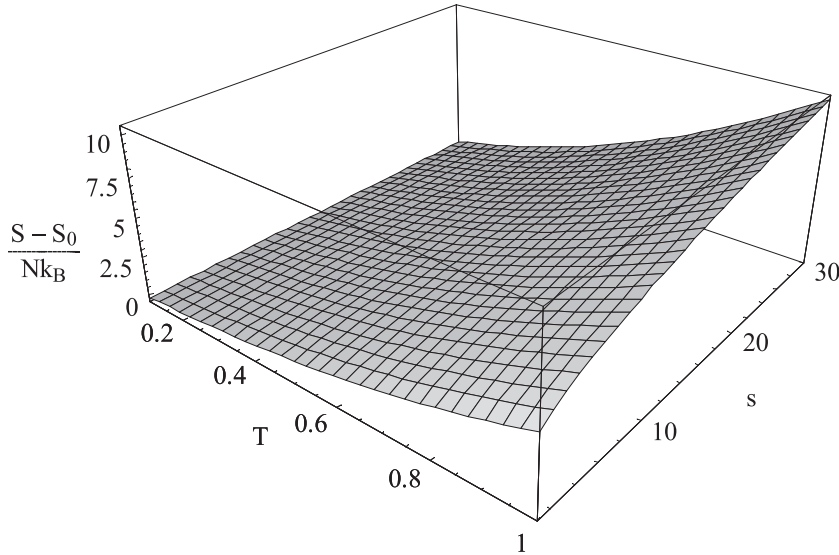


Figure 1. Entropy–temperature curves as a function of the potential depth for rotation rate $\alpha = 0.8$ and $\left(\frac{\omega_R}{\omega_g} \right) = 8.0$.

The result of Eq. (21) shows that it can be used to investigate the process of an adiabatically loading system of rotating N bosons into a 1D lattice. The requirement for adiabaticity in this system is that the entropy

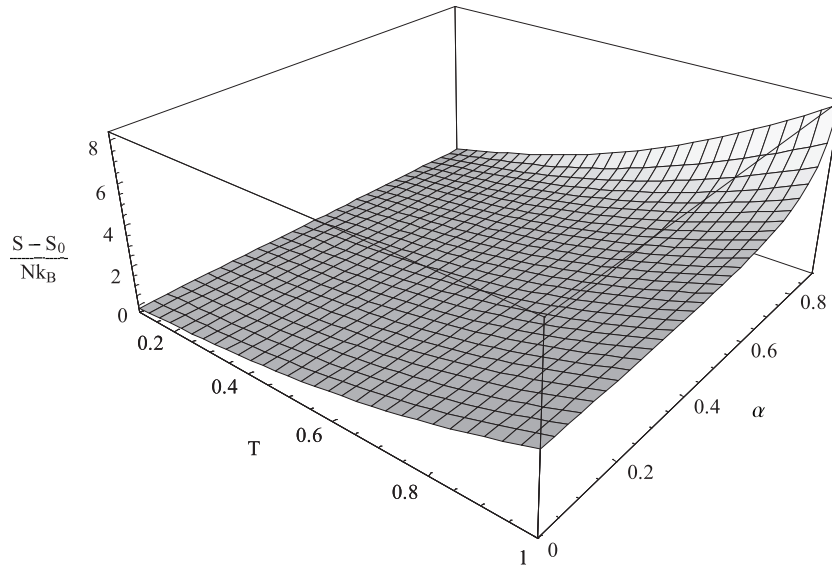


Figure 2. Entropy–temperature curves as a function of the rotation rate α for normalized optical potential $s = 15$ and $\left(\frac{\omega_R}{\omega_g}\right) = 20$.

remains constant throughout this process, and the most useful information can be obtained from knowing how the entropy depends on the system temperature, rotation rate, and the optical potential depth.

In Figures 1 and 2, the entropy–temperature curves as a function of optical potential depth and rotation rate are respectively illustrated graphically. The trap parameters of Spielman’s experiment [3] are used. These figures show that as the temperature increases the entropy has a monotonically increasing nature everywhere. This behavior is comparable with the standard thermodynamic arguments. Moreover, entropy–temperature

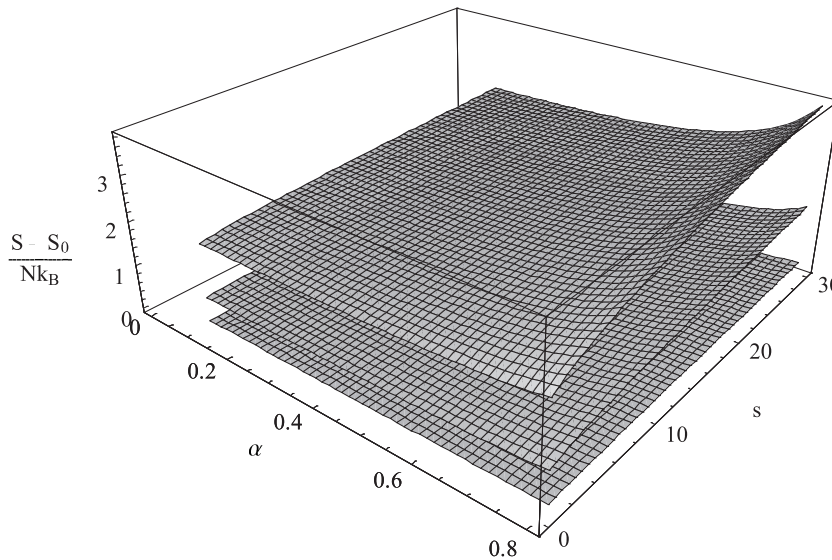


Figure 3. Entropy versus rotation rate α and the potential depth s for various temperatures $T = 0.2, 0.5$, and 0.8 from down to up.

curves always increase quite rapidly with the increase of the rotation rate, i.e. all curves in Figure 2 bend upwards with the increasing of the rotation rate.

In order to assess the degree to which adiabatic loading is affected by the rotation rate, in Figure 3 the entropy versus the lattice depth and rotation rate is considered for three different normalized temperature values, $\mathcal{T} = 0.2, 0.5$, and 0.8 . For small rotation rate values, the entropy remains almost constant. For rapid rotation rates it is clear that the entropy has an increasing nature everywhere. Thus, the adiabatic loading of the rotating condensate into the lattice will greatly depend on the rotation rate.

4. Conclusion

In this paper, the temperature dependence of the entropy of a rotating BEC cloud in a 1D optical lattice was investigated. The obtained results showed that the temperature dependence of the rotating condensate in an optical lattice is greatly affected in this combined potential compared to the pure harmonic or rotating condensate. As an important result, it was found that in order to attain a high ground-state occupation of rotating bosonic gases in an optical lattice, one has to cool to much lower temperatures than necessary for a pure BEC in the corresponding magnetic trap. This concluding remark can be obtained from the entropy-temperature curves. These curves reveal that the entropy has a monotonically increasing nature everywhere, and then no cooling can be achieved during the adiabatic loading of the rotating condensate into the lattice.

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