Turkish Journal of Physics

http://journals.tubitak.gov.tr/physics/

Research Article

Turk J Phys
(2015) 39: 37 – 42
© TÜBİTAK
doi:10.3906/fiz-1403-2

Schrödinger equation with modified Smorodinsky–Winternitz potential

Akpan Ndem IKOT[1,*], **Hassan HASSANABADI**[2], **Elham MAGHSOODI**[2], **Saber ZARRINKAMAR**[3]

[1]Theoretical Physics Group, Department of Physics, University of Port Harcourt, Port Harcourt, Nigeria
[2]Department of Basic Sciences, Shahrood Branch, Islamic Azad University, Shahrood, Iran
[3]Department of Basic Sciences, Garmsar Branch, Islamic Azad University, Garmsar, Iran

Received: 05.03.2014 • **Accepted:** 21.05.2014 • **Published Online:** 23.02.2015 • **Printed:** 20.03.2015

Abstract: We present solutions of the Schrödinger equation for the modified Smorodinsky–Winternitz potential in an exact analytical manner. The considered potential is of noncentral nature and includes the Pöschl–Teller potential in the radial as the radial term and includes both azimuthal and polar angle-dependent terms. The problem, after separation of variables, is solved via the Nikiforov–Uvarov method and the solutions are reported.

Key words: Schrödinger equation, Nikiforov–Uvarov method, SW potential

PACS: 03.65Ge, 03.65.Fd, 02.30Gp

1. Introduction

The solutions of quantum wave equations including Schrödinger, Klein–Gordon, Dirac, Duffin–Kemmer–Petiau (DKP), and spinless-Salpeter equations have been an attractive research subject for both physicists and applied mathematicians and different potentials have been analyzed within the so-called potential model [1–14]. In the theoretical studies, various analytical methods have been used including supersymmetric quantum mechanics (SUSYQM) [15], the Nikiforov–Uvarov (NU) technique [16–19], factorization [20], the path integral approach [21], etc. [22 and references therein]. In addition, the list of considered potentials, due to their possible applications in various physical fields, is quite lengthy. Among the considered potentials, the central ones do not coincide with the experimental results in the case of deformed nuclei or ring-shaped molecules. This has motivated the consideration of angle-dependent terms in the Hamiltonian. The most frequently used angle-dependent potentials are the coulomb ring-shaped [23–27], Hellman [28], Hartmann [29], double ring-shaped coulomb [30], inverse-square angle-dependent [31], Pöschl–Teller double ring-shaped coulomb [32], and double ring-shaped oscillator [33]. The purpose of the present article is to solve the Schrödinger equation with a modified form of a noncentral potential proposed by Smorodinky and Winternitz (SW) [34]. The modified SW (MSW) takes the form

$$V(r,\theta,\varphi)=V(r)+\frac{1}{r^2}\left(\frac{B_1}{\cos^2\theta}+\frac{B_2}{\sin^2\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{B_3}{\sin^2(\alpha\varphi)}+\frac{B_4}{\cos^2(\alpha\varphi)}\right) \tag{1}$$

where $V(r)$ is the SW central potential as will be seen later and $B_i(i=1...4)$ are physical parameters.

*Correspondence: ndemikotphysics@gmail.com

2. The basic equations

The Schrödinger equation with a potential of Eq. (1) takes the form ($\hbar = 2m = 1$),

$$\left(-\nabla^2 + V(r,\theta,\phi)\right)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi) \tag{2}$$

where the Laplacian operator is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{3}$$

Introducing

$$\psi(r,\theta,\phi) = \frac{U(r)}{r}\frac{H(\theta)}{\sqrt{\sin\theta}}\Phi(\varphi) \tag{4}$$

and considering the approximation $\frac{1}{r^2} \simeq \frac{\alpha^2}{\sinh^2(\alpha r)}$, we obtain

$$\frac{d^2U(r)}{dr^2} + \left(E - V(r) + \frac{\alpha^2(l^2+\frac{1}{2})}{\sinh^2(\alpha r)}\right)U(r) = 0 \tag{5}$$

$$\frac{d^2H(\theta)}{d\theta^2} + \left(-\frac{B_1}{\cos^2\theta} - \frac{B_2+m^2}{\sin^2\theta} + \frac{1}{4}\cot^2\theta - l^2\right)H(\theta) = 0, \tag{6}$$

$$\frac{d^2\Phi}{d\varphi^2} + \left(m^2 - [\frac{B_3}{\sin^2(\alpha\varphi)} + \frac{B_4}{\cos^2(\alpha\varphi)}]\right)\Phi(\varphi) = 0 \tag{7}$$

where $l = 0,1,2...,$ and $m = 0,\pm1,\pm2...$

It can be observed in Eq. (5) that if we expand the term $(\sinh\alpha r)^2$ to first order, we have $\sinh^2\alpha r \approx \alpha^2 r^2$. Substituting this approximation into Eq. (1) for $\alpha = 1$, we obtain the noncentral potential reported previously [31]. We intend to study the solutions of Eqs. (5) and (6) with parametric generalization of the NU method [16,19]. In the forthcoming section, we review the NU method.

3. Parametric Nikiforov–Uvarov method

The parametric form of the NU method takes the form [16–19]

$$\frac{d^2\psi}{ds^2} + \frac{\alpha_1 - \alpha_2 s}{s(1-\alpha_3 s)}\frac{d\psi}{ds} + \frac{1}{s^2(1-\alpha_3 s)^2}\left\{-\xi_1 s^2 + \xi_2 s - \xi_3\right\}\psi(s) = 0 \tag{8}$$

The energy equation and eigenfunctions respectively are obtained from

$$\alpha_2 n + (2n+1)\alpha_5 + (2n+1)(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) + n(n-1)\alpha_3 + \alpha_7 + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0, \tag{9}$$

$$\psi(s) = s^{\alpha_{12}}(1-\alpha_3 s)^{\alpha_{13}} P_n^{(\alpha_{10},\alpha_{11})}(1-2\alpha_3 s) \tag{10}$$

where

$$\begin{aligned}
&\alpha_4 = \tfrac{1}{2}(1-\alpha_1), \alpha_5 = \tfrac{1}{2}(\alpha_2 - 2\alpha_3), \alpha_6 = \alpha_5^2 + \xi_1, \\
&\alpha_7 = 2\alpha_4\alpha_5 - \xi_2, \alpha_8 = \alpha_4^2 + \xi_3, \alpha_9 = \alpha_3\alpha_7 + \alpha_3^2\alpha_8 + \alpha_6 \\
&\alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8}, \alpha_{11} = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) \\
&\alpha_{12} = \alpha_4 + \sqrt{\alpha_8}, \alpha_{13} = \alpha_5 - (\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})
\end{aligned} \tag{11}$$

and P_n is the orthogonal Jacobi-polynomial.

4. Solutions

4.1. Solution of the radial equation

In this section, we will solve the radial part of the Schrödinger equation with SW potential defined as [16,28]

$$V(r) = V_o \tanh^2(\alpha r) \tag{12}$$

The SW potential is equivalent to the Pöschl–Teller. This can be immediately seen from $\tanh^2(\alpha r) = 1 - \frac{1}{\cosh^2(\alpha r)}$ [21,34]. Inserting Eq. (12) into Eq. (5), we obtain

$$\frac{d^2U(r)}{dr^2} + \left(E - V_0 \tanh^2(\alpha r) + \frac{\alpha^2(l^2 + \frac{1}{2})}{\sinh^2(\alpha r)}\right) U(r) = 0 \tag{13}$$

By applying the transformation $y = \cosh^2(\alpha r)$, Eq. (13) takes the form

$$\frac{d^2U(y)}{dy^2} + \frac{\left(\frac{1}{2} - y\right)}{y(1-y)}\frac{dU(y)}{dy} + \frac{1}{y^2(1-y)^2}\left\{\frac{(E_{nlm} - V_0)}{4\alpha^2}y^2 + \frac{2V_0 - E_{nlm} + \alpha^2(l^2 + \frac{1}{2})}{4\alpha^2}y - \frac{V_0}{4\alpha^2}\right\} U(y) = 0 \tag{14}$$

By comparing Eq. (14) with Eq. (8), we find

$$\alpha_1 = \frac{1}{2}, \alpha_2 = 1, \alpha_3 = 1, \xi_1 = -\frac{(E_{nlm} - V_0)}{4\alpha^2}, \xi_2 = \frac{2V_0 - E_{nlm} + \alpha^2(l^2 + \frac{1}{2})}{4\alpha^2}, \xi_3 = \frac{V_0}{4\alpha^2} \tag{15}$$

Now, using Eq. (11), we obtain

$$\begin{aligned}
&\alpha_4 = \tfrac{1}{4}, \alpha_5 = -\tfrac{1}{2}, \alpha_6 = \tfrac{1}{4} - \tfrac{(E_{nlm} - V_0)}{4\alpha^2}, \alpha_7 = -\tfrac{1}{4} - \tfrac{2V_0 - E_{nlm} + \alpha^2(l^2+\frac{1}{2})}{4\alpha^2} \alpha_8 = \tfrac{1}{16} + \tfrac{V_0}{4\alpha^2},\\
&\alpha_9 = \tfrac{1}{16} - \tfrac{(l^2+\frac{1}{2})}{4}, \alpha_{10} = 1 + 2\sqrt{\tfrac{1}{16} + \tfrac{V_0}{4\alpha^2}}, \alpha_{11} = 2 + 2\left(\sqrt{\tfrac{1}{16} - \tfrac{(l^2+\frac{1}{2})}{4}} + \sqrt{\tfrac{1}{16} + \tfrac{V_0}{4\alpha^2}}\right),\\
&\alpha_{12} = \tfrac{1}{4} + \sqrt{\tfrac{1}{16} + \tfrac{V_0}{4\alpha^2}}, \alpha_{13} = -\tfrac{1}{2} - \left(\sqrt{\tfrac{1}{16} - \tfrac{(l^2+\frac{1}{2})}{4}} + \sqrt{\tfrac{1}{16} + \tfrac{V_0}{4\alpha^2}}\right)
\end{aligned} \tag{16}$$

Using Eq. (9), we obtain the radial energy eigenvalues for the SW potential as

$$\begin{aligned}
E_{nlm} \;\; = \;\; & -4\alpha^2 n^2 - 4\alpha^2 n - 2\alpha^2 - 4\alpha^2(2n+1)\left(\sqrt{\tfrac{1}{16} - \tfrac{(l^2+\frac{1}{2})}{4}} + \sqrt{\tfrac{1}{16} + \tfrac{V_0}{4\alpha^2}}\right)\\
&+(l^2 + \tfrac{1}{2}) + \tfrac{\alpha^2}{2} - 8\alpha^2\sqrt{(\tfrac{1}{16} + \tfrac{V_0}{4\alpha^2})(\tfrac{1}{16} - \tfrac{(l^2+\frac{1}{2})}{4})},
\end{aligned} \tag{17}$$

and the corresponding radial wave function becomes

$$\begin{aligned}
U_{nlm}(r) \;\; = \;\; & N_{nl}\left(\cosh^2(\alpha r)\right)^{\frac{1}{4} + \sqrt{\frac{1}{16} + \frac{V_0}{4\alpha^2}}}\left(1 - \cosh^2(\alpha r)\right)^{\frac{1}{4} + \sqrt{\frac{1}{16} - \frac{(l^2+\frac{1}{2})}{4}}}\\
&\times P_n^{(2\sqrt{\frac{1}{16} + \frac{V_0}{4\alpha^2}}, 2\sqrt{\frac{1}{16} - \frac{(l^2+\frac{1}{2})}{4}})}(1 - 2\cos^2 \alpha r).
\end{aligned} \tag{18}$$

where N_{nl} is the normalization constant.

4.2. Solution of the polar equation

Rearranging Eq. (6) in view of the transformation $z = \cos^2\theta$, we obtain

$$\frac{d^2H_l(z)}{dz^2} + \frac{\left(\frac{1}{2} - z\right)}{z(1-z)}\frac{dH_l(z)}{dz} + \frac{1}{z^2(1-z)^2}\left\{(\frac{1}{16} + \frac{l^2}{4})z^2 + \frac{1}{4}\left(-l^2 + B_1 - B_2 - m^2\right)z - \frac{B_1}{4}\right\}H_l(z) = 0 \quad (19)$$

Now comparing Eq. (19) with Eq. (8) and using Eq. (11), we find

$$\begin{aligned}
&\alpha_1 = \tfrac{1}{2}, \alpha_2 = 1, \alpha_3 = 1, \xi_1 = -(\tfrac{1}{16} + \tfrac{l^2}{4}),\ \xi_2 = \tfrac{1}{4}\left(-l^2 + B_1 - B_2 - m^2\right),\ \xi_3 = \tfrac{B_1}{4} \\
&\alpha_4 = \tfrac{1}{4}, \alpha_5 = -\tfrac{1}{2}, \alpha_6 = \tfrac{1}{4} - (\tfrac{1}{16} + \tfrac{l^2}{4}), \alpha_7 = -\tfrac{1}{4} - \tfrac{1}{4}\left(-l^2 + B_1 - B_2 - m^2\right), \\
&\alpha_8 = \tfrac{1}{16} + \tfrac{B_1}{4}, \alpha_9 = \tfrac{1}{16} + (-\tfrac{1}{16} + \tfrac{B_2+m^2}{4}),\ \alpha_{10} = 1 + 2\sqrt{\tfrac{1}{16} + \tfrac{B_1}{4}}, \\
&\alpha_{11} = 2 + 2\left(\sqrt{\tfrac{1}{16} + (-\tfrac{1}{16} + \tfrac{B_2+m^2}{4})} + \sqrt{\tfrac{1}{16} + \tfrac{B_1}{4}}\right), \alpha_{12} = \tfrac{1}{4} + \sqrt{\tfrac{1}{16} + \tfrac{B_1}{4}} \\
&\alpha_{13} = -\tfrac{1}{2} - \left(\sqrt{\tfrac{1}{16} + (-\tfrac{1}{16} + \tfrac{B_2+m^2}{4})} + \sqrt{\tfrac{1}{16} + \tfrac{B_1}{4}}\right)
\end{aligned} \quad (20)$$

Substituting Eq. (20) into the energy equation (9), we find

$$\begin{aligned}
&n^2 + \tfrac{(2n+1)}{2} + (2n+1)(\sqrt{\tfrac{1}{16} + (-\tfrac{1}{16} + \tfrac{B_2+m^2}{4})} + \sqrt{\tfrac{1}{16} + \tfrac{B_1}{4}}) - \tfrac{1}{4} - \tfrac{1}{4}\left(-l^2 + B_1 - B_2 - m^2\right) \\
&+2(\tfrac{1}{16} + \tfrac{B_1}{4}) + 2\sqrt{(\tfrac{1}{16} + \tfrac{B_1}{4})[\tfrac{1}{16} + (-\tfrac{1}{16} + \tfrac{B_2+m^2}{4})]} = 0,
\end{aligned} \quad (21)$$

or or, more neatly,

$$l = \left\{\begin{array}{l} -4n^2 - 4n - \frac{3}{2} - 4(2n+1)(\sqrt{\frac{1}{16} + (-\frac{1}{16} + \frac{B_2+m^2}{4})} + \sqrt{\frac{1}{16} + \frac{B_1}{4}}) \\ -B_2 - m^2 - B_1 - 8\sqrt{(\frac{1}{16} + \frac{B_1}{4})[\frac{1}{16} + (-\frac{1}{16} + \frac{B_2+m^2}{4})]} \end{array}\right\}^{\frac{1}{2}} \quad (22)$$

The corresponding polar wave function is obtained from Eqs. (10) and (20) as

$$\begin{aligned}
H_l(\theta) &= N_m\left(\cos^2\theta\right)^{\frac{1}{4} + \sqrt{\frac{1}{16} + \frac{B_1}{4}}}\left(\sin^2\theta\right)^{\frac{1}{4} + \sqrt{\frac{1}{16} + (-\frac{1}{16} + \frac{B_2+m^2}{4})}} \\
&\times P_n^{\left(2\sqrt{\frac{1}{16} + \frac{B_1}{4}}, 2\sqrt{\frac{1}{16} + (-\frac{1}{16} + \frac{B_2+m^2}{4})}\right)}(1 - 2\cos^2\theta)
\end{aligned} \quad (23)$$

4.3. Solution of the azimuthal equation

Now we consider the azimuthal equation. By applying the transformation $s = \cos^2(\alpha\phi)$ to Eq. (7), we have

$$\frac{d^2\Phi_{nl}(\phi)}{ds^2} + \frac{\frac{1}{2} - s}{s(1-s)}\frac{d\Phi_{nl}(\phi)}{ds} + \frac{1}{s^2(1-s)^2}\left\{-\frac{m^2}{4\alpha^2}s^2 + \frac{\left(B_4 - B_3 + m^2\right)}{4\alpha^2}s - \frac{B_4}{4\alpha^2}\right\}\Phi_{nl}(\phi) = 0 \quad (24)$$

The required set of parameters are

$$\begin{aligned}
&\alpha_1=\tfrac{1}{2},\alpha_2=1,\alpha_3=1,\xi_1=\tfrac{m^2}{4\alpha^2},\xi_2=\tfrac{\left(B_4-B_3+m^2\right)}{4\alpha^2},\xi_3=\tfrac{B_4}{4\alpha^2},\alpha_4=\tfrac{1}{4},\alpha_5=-\tfrac{1}{2},\\
&\alpha_6=\tfrac{1}{4}+\tfrac{m^2}{4\alpha^2},\alpha_7=-\tfrac{1}{4}-\tfrac{\left(B_4-B_3+m^2\right)}{4\alpha^2},\alpha_8=\tfrac{1}{16}+\tfrac{B_4}{4\alpha^2},\alpha_9=\tfrac{1}{16}+\tfrac{B_3}{4\alpha^2},\\
&\alpha_{10}=1+2\sqrt{\tfrac{1}{16}+\tfrac{B_4}{4\alpha^2}},\alpha_{11}=2+2\left(\sqrt{\tfrac{1}{16}+\tfrac{B_3}{4\alpha^2}}+\sqrt{\tfrac{1}{16}+\tfrac{B_4}{4\alpha^2}}\right),\alpha_{12}=\tfrac{1}{4}+\sqrt{\tfrac{1}{16}+\tfrac{B_4}{4\alpha^2}},\\
&\alpha_{13}=-\tfrac{1}{2}-\left(\sqrt{\tfrac{1}{16}+\tfrac{B_3}{4\alpha^2}}+\sqrt{\tfrac{1}{16}+\tfrac{B_4}{4\alpha^2}}\right)
\end{aligned} \tag{25}$$

Using Eq. (9), we get

$$m=\left\{\begin{array}{l} 4\alpha^2n^2+4\alpha^2n+2\alpha^2+4\alpha^2(2n+1)(\sqrt{\tfrac{1}{16}+\tfrac{B_3}{4\alpha^2}}+\sqrt{\tfrac{1}{16}+\tfrac{B_4}{4\alpha^2}})\\ +B_3-\tfrac{\alpha^2}{2}+B_4+8\alpha^2\sqrt{(\tfrac{1}{16}+\tfrac{B_4}{4\alpha^2})(\tfrac{1}{16}+\tfrac{B_3}{4\alpha^2})}\end{array}\right\}^{\frac{1}{2}} \tag{26}$$

Eq. (26) completely determines the value of m. Now using the parameters of Eq. (25), we obtain the wave function of the azimuthal part as

$$\begin{aligned}
\Phi_m(\phi) \;&=\; \left(\cos^2(\alpha\phi)\right)^{\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{B_4}{4\alpha^2}}}\left(\sin^2(\alpha\phi)\right)^{\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{B_3}{4\alpha^2}}}\\
&\times P_n^{\left(2\sqrt{\frac{1}{16}+\frac{B_4}{4\alpha^2}},2\sqrt{\frac{1}{16}+\frac{B_3}{4\alpha^2}}\right)}\left(1-2\cos^2(\alpha\phi)\right)
\end{aligned} \tag{27}$$

Finally, using Eq. (4), we can determine the total wave function as

$$\begin{aligned}
&\psi_{nlm}(r,\theta,\varphi)=N_{nlm}[\left(\cosh^2(\alpha r)\right)^{\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{V_0}{4\alpha^2}}}\left(1-\cosh^2(\alpha r)\right)^{\frac{1}{4}+\sqrt{\frac{1}{16}-\frac{(l^2+\frac{1}{2})}{4}}}\\
&\times P_n^{(2\sqrt{\frac{1}{16}+\frac{V_0}{4\alpha^2}},2\sqrt{\frac{1}{16}-\frac{(l^2+\frac{1}{2})}{4}})}(1-2\cos^2\alpha r)]\left[\begin{array}{l}\left(\cos^2\theta\right)^{\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{B_1}{4}}}\left(\sin^2\theta\right)^{\frac{1}{4}+\sqrt{\frac{1}{16}+(-\frac{1}{16}+\frac{B_2+m^2}{4})}}\\ \times P_n^{\left(2\sqrt{\frac{1}{16}+\frac{B_1}{4}},2\sqrt{\frac{1}{16}+(-\frac{1}{16}+\frac{B_2+m^2}{4})}\right)}(1-2\cos^2\theta)\end{array}\right]\\
&\times\left(\cos^2(\alpha\varphi)\right)^{\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{B_4}{4\alpha^2}}}\left(\sin^2(\alpha\varphi)\right)^{\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{B_3}{4\alpha^2}}}\times P_n^{\left(2\sqrt{\frac{1}{16}+\frac{B_4}{4\alpha^2}},2\sqrt{\frac{1}{16}+\frac{B_3}{4\alpha^2}}\right)}\left(1-2\cos^2(\alpha\varphi)\right).
\end{aligned} \tag{28}$$

5. Conclusion

In this paper, we solved the Schrödinger equation for noncentral potential MSW potential using the NU method. We obtained the solutions of the radial, polar, and azimuthal parts of the Schrodinger equation explicitly. This result is the generalization of the work of Yesiltas [31], Maghsoodi et al. [32], and Chen et al. [21]. The potential, because of including both polar and azimuthal angle-dependent terms, can be a good candidate to consider the deformation effects and nonspherical molecules. In addition, as the higher states are reported, we can immediately investigate the transition patterns and energy splitting.

References

[1] Hassanabadi, H.; Maghsoodi, E.; Ikot, A. N.;Zarrinkamar, S. *Appl. Math. & Comp.* **2013**, *219*, 9388–9398.

[2] Ikot, A. N.; Maghsoodi, E.; Zarrinkamar, S.; Hassanabadi, H. *Few-Body Syst.* Doi.10.1007/s00601-013-0701-6.

[3] Ikot, A. N.; Hassanabadi, H.; Yazarloo, B. H.; Zarrinkamar, S. *Int. J. Mod. Phys. E*, **2013**, *22*, 1350048.

[4] Ikot, A. N.; Maghsoodi, E.; Antia, A. D.; Zarrinkamar, S.; Hassanabadi, H. *Can. J. Phys.* **2013**, *91*, 560–575.

[5] Salpeter, E. E.; Bethe, H. A. *Phys. Rev.* **1985**, *84*, 1232–1242.

[6] Wick, G. C. *Phys. Rev.* **1954**, *96*, 1124–1234.

[7] Chang, L.; Roberts, C. D. *Phys. Rev. Lett.* **2009**, *103*, 081601.

[8] Hassanabadi, H.; Zarrinkamar, S.; Yazarloo, B. H. *Chin. J. Phys.* **2012**, *50*, 783–795.

[9] Lucha, W.; Schoberl, F. F. *Int. J. Mod. Phys. A* **2002**, *17*, 2233.

[10] Ikot, A. N.; Maghsoodi, E.; Zarrinkamar, S.; Salehi, N.; Hassanabadi, H. *Int. J. Mod. Phys.E* **2013**, *22*, 1350052.

[11] Greiner, W. *Relativistic Quantum Mechanics*; Springer: Berlin, Germany, 2000.

[12] Landau, L. D.; Lifshitz, E. M. *Quantum Mechanics, Non-Relativistic Theory*; Pergamon: New York, NY, USA, 1977.

[13] Dong, S. H.; Qiang, W. C.; Sun, G. H.; Bezerra, V. B. *J. Phys. A* **2007**, *40*, 10535.

[14] Qiang, W. C.; Dong, S. H. *Phys. Lett. A* **2007**, *368*, 13.

[15] Cooper, F.; Khare, A.; Sukhatme, U. *Phys. Rep.* **1995**, *251*, 267–385.

[16] Nikforov, A. F.; Uvarov, V. B. *Special Functions of Mathematical Physics*; Birkhauser: Basel, Switzerland, 1988.

[17] Tezcan, C.; Sever, R. *Int. J. Theor. Phys.* **2009**, *48*, 337–352.

[18] Miranda, M. G.; Sun G. H.; Dong, S. H. *Int. J. Mod. Phys. E* **2010**, *19*, 123.

[19] Zhang, M. C.; Sun G. H.; Dong, S. H. *Phys. Lett. A* **2010**, *374*, 704.

[20] Dong, S. H. *Factorization Method in Quantum Mechanics*; Springer: Dordrecht, Holland, 2007.

[21] Bentag, B.; Allal, M. S. *Chin. J. Phys.* **2013**, *51*, 26–40.

[22] Hassanabadi, H.; Maghsoodi, E.; Ikot, A. N.; Zarrinkmar, S. *Euro. Phys. J. Plus* **2013**, *128*, 79.

[23] Lu, F. L.; Zhuoang, G. C.; Chen, C. Y. *J. At. Mol. Phys.* **2006**, *23*, 493–503.

[24] Hassanabadi, H.; Kamali, M. *Chin. Phys. B* **2013**, *22*, 100304.

[25] Eshghi, M. *Chin. Phys. Letts.* **2012**, *29*, 110504.

[26] Kibler, M.; Winternitz, P. J. *J. Phys. A* **1987**, *20*, 4097–4117.

[27] Quesne, C. *J. Phys. A* **1988**, *21*, 3093–3100.

[28] Ikhdair, S. M.; Sever, R. arxiv/quant-ph/0603205.

[29] Hartmann, H. *Theor. Chim. Acta* **1972**, *24*, 201–209.

[30] Chen, Y. C.; Lu, F. L.; Sun, D. S.; Dong, S. H. *Chin. Phys. B* **2013**, *22*, 100302.

[31] Yesiltas, O. *Chin. Phys. Lett.* **2008**, *25*, 1172.

[32] Maghsoodi, E.; Hassanabadi, H.; Zarrinkamar, S. *Chin. Phys. B* **2013**, *22*, 030302.

[33] Chen, C. Y.; You, Y.; Wang, X. H.; Dong, S. H. *Phys. Lett. A*. **2013**, *377*, 1521.

[34] Fris, J.; Mandrosov, V.; Smorodinsky, Y. A.; Uhlir, M.; Winternitz, P. *Phys. Lett.* **1965**, *16*, 354–365.