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Representation of electromagnetic and gravitoelectromagnetic Poynting theorems in higher dimensions

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Abstract: This study investigates whether the electromagnetic and gravitoelectromagnetic energy conservation equations are obtained together by using octonion algebra or not. Maxwell and Maxwell-like equations for linear gravity with magnetic monopole terms are used in the SI unit system. A new complex octonionic field term is suggested for the first time. The complex octonionic source equation is then obtained. Finally, Poynting theorems for both electromagnetism and gravitoelectromagnetism are defined for the first time by using higher dimensional algebra.

Key words: Octonion, electromagnetism, gravitoelectromagnetism, magnetic monopole, electromagnetic energy, Poynting theorem

1. Introduction

As physical problems or applications can be solved by using vector, tensor, and matrix algebras, they may be also represented by quaternions, octonions, and sedenions [1–5]. These algebraic structures are also known as hypernumber systems and are useful tools for the representations and generalizations of quantities in different physical subfields such as electromagnetism (EM), gravitoelectromagnetism (GEM), quantum mechanics, acoustics, group theory, or supersymmetric quantum mechanics. There are many studies about EM and GEM with different algebras in the literature [6–29].

Octonions with 8 components have both noncommutative and nonassociative algebraic properties and they satisfy alternative division rings. According to the different physical systems, octonions have some types such as real, complex, split, and hyperbolic forms. In this paper, energy conservation equations for both EM and GEM are expressed by using complex octonions. Some studies related to complex octonions can be summarized as follows: Tolan et al. reformulated classical electromagnetism [22], Tanışlı and Kansu studied electromagnetism for bi-isotropic media [23], Tanışlı and Jancewicz represented electromagnetic Lorenz conditions with magnetic monopoles [24], and Kansu et al. showed electromagnetic energy conservation without sources [25].

To date, classical electromagnetism and energy conservation equations have been presented using complexified quaternions by Tanışlı [26] and complex octonions by Kansu et al. [25]. Additionally, Tanışlı studied energy conservations with quaternions for acoustics [27]. Due to the analogy between EM and GEM, it is thought that the Poynting theorem can be adapted together in higher dimensions alternatively. Therefore,

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complex octonion algebra with 16 components has valid and useful hypercomplex numbers.

After the introduction of this study, octonion algebra and its properties are given in Section 2. In the next section, Maxwell and Maxwell-like equations for gravity are expressed with magnetic and gravitomagnetic monopole terms. The energy conservation equations for EM and GEM are then written in vectorial forms. In Section 4, the complex octonionic differential operator and field equation are presented. In addition, by obtaining the complex octonionic source equation, the energy conservation equations, energy fluxes, and densities in terms of EM and GEM are attained via octonionic representations. The results, conclusions, and fundamental features of this study are presented in the last section.

2. Octonion algebra

Octonions are included among the hypercomplex numbers. They have 8 real components and generate a normed alternative division algebra over the real numbers. They were discovered by John T. Graves in 1843 and were independently developed by Arthur Cayley in 1845. Therefore, octonions are also known as "Cayley numbers" in the literature. Octonion algebra has both noncommutative and nonassociative algebraic structures [2,3,22–25]. A real octonion, $\boldsymbol{A}$, is shown as follows:

$$\boldsymbol{A}=\sum_{n=0}^{7} a_n\boldsymbol{e}_n = a_0\boldsymbol{e}_0+a_1\boldsymbol{e}_1+a_2\boldsymbol{e}_2+a_3\boldsymbol{e}_3+a_4\boldsymbol{e}_4+a_5\boldsymbol{e}_5+a_6\boldsymbol{e}_6+a_7\boldsymbol{e}_7. \tag{1}$$

Here, the a_n and $\boldsymbol{e}_n$ terms represent the real numbers and basis elements of the real octonion, respectively. For 2 real octonions $\boldsymbol{A}$ and $\boldsymbol{B}$, some algebraic processes can be defined. The summation and subtraction are given as:

$$\begin{aligned}\boldsymbol{A}\pm\boldsymbol{B} &= \sum_{n=0}^{7}(a_n\pm b_n)\,\boldsymbol{e}_n \\ &= (a_0\boldsymbol{e}_0+a_1\boldsymbol{e}_1+a_2\boldsymbol{e}_2+a_3\boldsymbol{e}_3+a_4\boldsymbol{e}_4+a_5\boldsymbol{e}_5+a_6\boldsymbol{e}_6+a_7\boldsymbol{e}_7) \\ &\quad\pm(b_0\boldsymbol{e}_0+b_1\boldsymbol{e}_1+b_2\boldsymbol{e}_2+b_3\boldsymbol{e}_3+b_4\boldsymbol{e}_4+b_5\boldsymbol{e}_5+b_6\boldsymbol{e}_6+b_7\boldsymbol{e}_7)\,.\end{aligned} \tag{2}$$

The real octonion $\boldsymbol{A}$ consists of 2 parts: scalar and vectorial. The scalar and vectorial parts can be given as below, respectively:

$$S_{\boldsymbol{A}} = a_0\boldsymbol{e}_0, \tag{3}$$

$$V_{\boldsymbol{A}} = a_1\boldsymbol{e}_1+a_2\boldsymbol{e}_2+a_3\boldsymbol{e}_3+a_4\boldsymbol{e}_4+a_5\boldsymbol{e}_5+a_6\boldsymbol{e}_6+a_7\boldsymbol{e}_7. \tag{4}$$

Therefore, the real octonion $\boldsymbol{A}$ can be written briefly as:

$$\boldsymbol{A}=S_{\boldsymbol{A}}+V_{\boldsymbol{A}}=a_0\boldsymbol{e}_0+\vec{\boldsymbol{A}}, \tag{5}$$

where $\vec{\boldsymbol{A}}$ denotes the other vectorial notation of the real octonion. Before multiplying 2 real octonions, the product rules between the basis elements should be given. There are many multiplication rules in the literature. In this study, Cayley–Dickson construction rules are used with the following properties:

$$\begin{aligned}-\boldsymbol{e}_4\boldsymbol{e}_i &= \boldsymbol{e}_i\boldsymbol{e}_4=\hat{\boldsymbol{e}}_i,\ \boldsymbol{e}_4\hat{\boldsymbol{e}}_i=-\hat{\boldsymbol{e}}_i\boldsymbol{e}_4=\boldsymbol{e}_i,\ \boldsymbol{e}_4\boldsymbol{e}_4=-\boldsymbol{e}_0, \\ \boldsymbol{e}_i\boldsymbol{e}_j &= -\delta_{ij}\boldsymbol{e}_0+\varepsilon_{ijk}\boldsymbol{e}_k,\ \hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j=-\delta_{ij}\boldsymbol{e}_0-\varepsilon_{ijk}\boldsymbol{e}_k,\ i,j,k\in(1,2,3)\,, \\ -\hat{\boldsymbol{e}}_j\boldsymbol{e}_i &= \boldsymbol{e}_i\hat{\boldsymbol{e}}_j=-\delta_{ij}\boldsymbol{e}_4-\varepsilon_{ijk}\hat{\boldsymbol{e}}_k.\end{aligned} \tag{6}$$

Here, $\hat{\boldsymbol{e}}_k\equiv\boldsymbol{e}_{4+k}$, $k\in(1,2,3)$ and $\boldsymbol{e}_0=1$ [22-25]. These rules can also be summarized as a table.

Table. Cayley–Dickson multiplication rules for octonions.

	$\boldsymbol{e}_0$	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$	$\boldsymbol{e}_4$	$\boldsymbol{e}_5$	$\boldsymbol{e}_6$	$\boldsymbol{e}_7$
$\boldsymbol{e}_0$	1	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$	$\boldsymbol{e}_4$	$\boldsymbol{e}_5$	$\boldsymbol{e}_6$	$\boldsymbol{e}_7$
$\boldsymbol{e}_1$	$\boldsymbol{e}_1$	-1	$\boldsymbol{e}_3$	$-\boldsymbol{e}_2$	$\boldsymbol{e}_5$	$-\boldsymbol{e}_4$	$-\boldsymbol{e}_7$	$\boldsymbol{e}_6$
$\boldsymbol{e}_2$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_3$	-1	$\boldsymbol{e}_1$	$\boldsymbol{e}_6$	$\boldsymbol{e}_7$	$-\boldsymbol{e}_4$	$-\boldsymbol{e}_5$
$\boldsymbol{e}_3$	$\boldsymbol{e}_3$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1$	-1	$\boldsymbol{e}_7$	$-\boldsymbol{e}_6$	$\boldsymbol{e}_5$	$-\boldsymbol{e}_4$
$\boldsymbol{e}_4$	$\boldsymbol{e}_4$	$-\boldsymbol{e}_5$	$-\boldsymbol{e}_6$	$-\boldsymbol{e}_7$	-1	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$
$\boldsymbol{e}_5$	$\boldsymbol{e}_5$	$\boldsymbol{e}_4$	$-\boldsymbol{e}_7$	$\boldsymbol{e}_6$	$-\boldsymbol{e}_1$	-1	$-\boldsymbol{e}_3$	$\boldsymbol{e}_2$
$\boldsymbol{e}_6$	$\boldsymbol{e}_6$	$\boldsymbol{e}_7$	$\boldsymbol{e}_4$	$-\boldsymbol{e}_5$	$-\boldsymbol{e}_2$	$\boldsymbol{e}_3$	-1	$-\boldsymbol{e}_1$
$\boldsymbol{e}_7$	$\boldsymbol{e}_7$	$-\boldsymbol{e}_6$	$\boldsymbol{e}_5$	$\boldsymbol{e}_4$	$-\boldsymbol{e}_3$	$-\boldsymbol{e}_2$	$\boldsymbol{e}_1$	-1

By using these rules, the product of 2 real octonions, $\boldsymbol{A}$ and $\boldsymbol{B}$, is equal to the following expression:

$$\boldsymbol{AB}=a_0b_0 + a_0\vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}b_0 - \vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}. \tag{7}$$

Octonions also have an octonionic conjugate, denoted by changing all signs of the vectorial part.

$$\bar{\boldsymbol{A}} = S_{\boldsymbol{A}} - V_{\boldsymbol{A}} = a_0\boldsymbol{e}_0 - \vec{\boldsymbol{A}} \tag{8}$$

The octonionic conjugate process can be also applied for 1 or 2 real octonions, $\boldsymbol{A}$ and $\boldsymbol{B}$, as follows:

$$\left(\bar{\bar{\boldsymbol{A}}}\right) = \boldsymbol{A},\;\; \left(\overline{\boldsymbol{A}+\boldsymbol{B}}\right) = \bar{\boldsymbol{A}} + \bar{\boldsymbol{B}},\;\; \left(\overline{\boldsymbol{AB}}\right) = \bar{\boldsymbol{B}}\bar{\boldsymbol{A}}. \tag{9}$$

If there are no scalar parts in Eq. (7), the scalar and vectorial products of real octonions $\boldsymbol{A}$ and $\boldsymbol{B}$ are given the following manner, respectively [22–25]:

$$\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}} = -\frac{1}{2}\left[\boldsymbol{AB} + \left(\overline{\boldsymbol{AB}}\right)\right], \tag{10}$$

$$\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}} = \frac{1}{2}\left[\boldsymbol{AB} - \left(\overline{\boldsymbol{AB}}\right)\right]. \tag{11}$$

The norm of the real octonion $\boldsymbol{A}$ is obtained by multiplying the octonion and its octonionic conjugate; the result is a real number.

$$N\left(\boldsymbol{A}\right) = \boldsymbol{A}\bar{\boldsymbol{A}} = \bar{\boldsymbol{A}}\boldsymbol{A} = \sum_{n=0}^{7} a_n^2 \tag{12}$$

The norm process for 2 real octonions is multiplicative, as follows:

$$N\left(\boldsymbol{AB}\right) = N\left(\boldsymbol{A}\right)N\left(\boldsymbol{B}\right). \tag{13}$$

Nonzero octonions also have a multiplicative inverse. The inverse of the real octonion, $\boldsymbol{A}$, is denoted by $\boldsymbol{A}^{-1}$, and this term can be obtained by the norm and the conjugate of the octonion.

$$\boldsymbol{A}^{-1} = \frac{\bar{\boldsymbol{A}}}{N\left(\boldsymbol{A}\right)} \tag{14}$$

Let $\mathbf{X}$ be a complex octonion. It can be expressed by a linear combination of 2 real octonions, $\boldsymbol{A}$ and $\boldsymbol{A}'$, with a complex unit i:

$$\mathbf{X} = \boldsymbol{A} + \mathrm{i}\boldsymbol{A}', \tag{15}$$

$$\begin{aligned}\mathbf{X} &= (a_0\boldsymbol{e}_0 + a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_4\boldsymbol{e}_4 + a_5\boldsymbol{e}_5 + a_6\boldsymbol{e}_6 + a_7\boldsymbol{e}_7)\\ &\quad +\mathrm{i}\,(a'_0\boldsymbol{e}_0 + a'_1\boldsymbol{e}_1 + a'_2\boldsymbol{e}_2 + a'_3\boldsymbol{e}_3 + a'_4\boldsymbol{e}_4 + a'_5\boldsymbol{e}_5 + a'_6\boldsymbol{e}_6 + a'_7\boldsymbol{e}_7)\,,\end{aligned} \tag{16}$$

$$\begin{aligned}\mathbf{X} &= \sum_{n=0}^{7}(a_n + \mathrm{i}a'_n)\,\boldsymbol{e}_n = (a_0 + \mathrm{i}a'_0)\,\boldsymbol{e}_0 + (a_1 + \mathrm{i}a'_1)\,\boldsymbol{e}_1 + (a_2 + \mathrm{i}a'_2)\,\boldsymbol{e}_2 + (a_3 + \mathrm{i}a'_3)\,\boldsymbol{e}_3\\ &\quad + (a_4 + \mathrm{i}a'_4)\,\boldsymbol{e}_4 + (a_5 + \mathrm{i}a'_5)\,\boldsymbol{e}_5 + (a_6 + \mathrm{i}a'_6)\,\boldsymbol{e}_6 + (a_7 + \mathrm{i}a'_7)\,\boldsymbol{e}_7,\end{aligned} \tag{17}$$

$$\mathbf{X} = \sum_{n=0}^{7}\mathbf{x}_n\boldsymbol{e}_n = \mathbf{x}_0\boldsymbol{e}_0 + \mathbf{x}_1\boldsymbol{e}_1 + \mathbf{x}_2\boldsymbol{e}_2 + \mathbf{x}_3\boldsymbol{e}_3 + \mathbf{x}_4\boldsymbol{e}_4 + \mathbf{x}_5\boldsymbol{e}_5 + \mathbf{x}_6\boldsymbol{e}_6 + \mathbf{x}_7\boldsymbol{e}_7. \tag{18}$$

Here, $\mathbf{x}_n$'s are complex numbers and i denotes the complex unit $\left(\mathrm{i} = \sqrt{-1}\right)$. While complex octonions have similar algebraic properties to real ones, they differ in having 16 components and an additional complex unit i. This means that there exists an additional complex conjugate of a complex octonion. Octonion conjugate $\bar{\mathbf{X}}$ and complex conjugate $\mathbf{X}^*$ are written as shown below [25].

$$\begin{aligned}\bar{\mathbf{X}} &= (a_0 + \mathrm{i}a'_0)\,\boldsymbol{e}_0 - (a_1 + \mathrm{i}a'_1)\,\boldsymbol{e}_1 - (a_2 + \mathrm{i}a'_2)\,\boldsymbol{e}_2 - (a_3 + \mathrm{i}a'_3)\,\boldsymbol{e}_3\\ &\quad - (a_4 + \mathrm{i}a'_4)\,\boldsymbol{e}_4 - (a_5 + \mathrm{i}a'_5)\,\boldsymbol{e}_5 - (a_6 + \mathrm{i}a'_6)\,\boldsymbol{e}_6 - (a_7 + \mathrm{i}a'_7)\,\boldsymbol{e}_7\end{aligned} \tag{19}$$

$$\begin{aligned}\mathbf{X}^* &= (a_0 - \mathrm{i}a'_0)\,\boldsymbol{e}_0 + (a_1 - \mathrm{i}a'_1)\,\boldsymbol{e}_1 + (a_2 - \mathrm{i}a'_2)\,\boldsymbol{e}_2 + (a_3 - \mathrm{i}a'_3)\,\boldsymbol{e}_3\\ &\quad + (a_4 - \mathrm{i}a'_4)\,\boldsymbol{e}_4 + (a_5 - \mathrm{i}a'_5)\,\boldsymbol{e}_5 + (a_6 - \mathrm{i}a'_6)\,\boldsymbol{e}_6 + (a_7 - \mathrm{i}a'_7)\,\boldsymbol{e}_7\end{aligned} \tag{20}$$

The other algebraic properties, such as multiplication, norm, inverse, are similar processes to those of the real octonions.

3. EM and GEM equations for energy conservation

Maxwell equations are central for the description of classical electromagnetism, electromagnetic wave theory, and optics. Maxwell equations are presented in the SI unit system [30] as shown below.

$$\begin{aligned}&\vec{\nabla}\cdot\vec{E} = \rho_e\\ &\vec{\nabla}\cdot\vec{H} = 0\\ &\vec{\nabla}\times\vec{E} = -\tfrac{\partial\vec{H}}{\partial t}\\ &\vec{\nabla}\times\vec{H} = \vec{J}_e + \tfrac{\partial\vec{E}}{\partial t}\end{aligned} \tag{21}$$

Here, $\vec{E}$, $\vec{H}$, ρ_e, and $\vec{J}_e$ are the electric field, magnetic field, electrical charge density, and electrical current density, respectively. The constitutive equations of $\vec{D} = \varepsilon_0\vec{E}$ and $\vec{B} = \mu_0\vec{H}$ equalities are given for EM in isotropic media. Here, the terms $\vec{D}$, $\vec{B}$, ε_0, and μ_0 represent electrical displacement, magnetic flux density, permittivity, and permeability constants for free space, respectively. There is a usual assumption that $\varepsilon_0 = \mu_0 = 1$ for theoretical studies in physics. Eq. (21) in EM has been derived from electric charges. This state is related to the Coulomb law in electricity.

Maxwell equations are popular and fundamental elements for physical systems. They are also invariant under the Lorentz and duality transformations. However, Eq. (21) does not satisfy duality transformations. In order to obtain both symmetry and duality invariance of Maxwell equations, Dirac proposed the magnetic monopole term [31,32]. Thus, generalized Dirac–Maxwell equations gain a new form in vectorial notation as follows:

$$\begin{aligned}
&\vec{\nabla}\cdot\vec{E}=\rho_e\\
&\vec{\nabla}\cdot\vec{H}=\rho_m\\
&\vec{\nabla}\times\vec{E}=-\vec{J}_m-\tfrac{\partial\vec{H}}{\partial t}\\
&\vec{\nabla}\times\vec{H}=\vec{J}_e+\tfrac{\partial\vec{E}}{\partial t},
\end{aligned}\tag{22}$$

where ρ_m and $\vec{J}_m$ are magnetic charge and magnetic current densities, respectively.

Duality transformations are given by the following expressions in the literature [10–12]:

$$\vec{E}\rightarrow\vec{E}cos\theta+\vec{H}sin\theta \quad , \quad \vec{H}\rightarrow-\vec{E}sin\theta+\vec{H}cos\theta.\tag{23}$$

For a special case of theta angle as $\theta=\pi/2$, these transformations can be rewritten as:

$$\vec{E}\rightarrow\vec{H} \quad , \quad \vec{H}\rightarrow-\vec{E}.\tag{24}$$

Duality transformations can be also rearranged by using matrix notations as below:

$$\begin{bmatrix}\vec{E}\\ \vec{H}\end{bmatrix}\rightarrow\begin{bmatrix}cos\theta & sin\theta\\ -sin\theta & cos\theta\end{bmatrix}\begin{bmatrix}\vec{E}\\ \vec{H}\end{bmatrix},\tag{25}$$

$$\begin{bmatrix}\vec{E}\\ \vec{H}\end{bmatrix}\rightarrow\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\vec{E}\\ \vec{H}\end{bmatrix}.\tag{26}$$

These transformations are also applied for source and current densities as follows:

$$\begin{aligned}
&\rho_e\rightarrow\rho_m \quad , \quad \rho_m\rightarrow-\rho_e\\
&\vec{J}_e\rightarrow\vec{J}_m \quad , \quad \vec{J}_m\rightarrow-\vec{J}_e.
\end{aligned}\tag{27}$$

In order to prevent the symmetry disappearance, Dirac used these transforms, and then Maxwell equations were written as in Eq. (22). It is easily seen that the fields, charge, and current densities can be converted to one another in Eq. (22) by using duality processes.

Recently, similarly to Maxwell equations in EM, some equations have been set in linear gravity. This can be connected to Newton's law as a body mass in a gravitational field. In addition, the moving matter (mass current) generates a gravitomagnetic field according to Einstein's general relativity just as a magnetic field is produced in Maxwell's equations by the moving charges (electric current). Hence, the set of governing equations related to gravity can be expressed in a similar form with their electromagnetic counterparts. This theoretical analogy between the Maxwell equations and gravity was first proposed by Heaviside in 1893 [15,16]. The new equation set is named as Maxwell-like equations for linear gravity, and they can be presented as shown below.

$$\begin{aligned}
&\vec{\nabla}\cdot\tilde{E}=-g_e\\
&\vec{\nabla}\cdot\tilde{H}=-g_m\\
&\vec{\nabla}\times\tilde{E}=\tilde{J}_m-\tfrac{\partial\tilde{H}}{\partial t}\\
&\vec{\nabla}\times\tilde{H}=-\tilde{J}_e+\tfrac{\partial\tilde{E}}{\partial t}
\end{aligned}\tag{28}$$

Here, $\tilde{E}$, $\tilde{H}$, g_e, g_m, $\tilde{J}_e$, and $\tilde{J}_m$ terms are the gravitoelectric field, gravitomagnetic field, gravitoelectric charge density, gravitomagnetic charge density, gravitoelectric current density, and gravitomagnetic current density, respectively. Due to the similarity to Maxwell equations, it is seen that these equations are also invariant under duality transformations. Therefore, the following transformations can be given:

$$\begin{aligned} &\tilde{E} \to \tilde{H} \quad , \quad \tilde{H} \to -\tilde{E} \\ &g_e \to g_m \quad , \quad g_m \to -g_e \\ &\tilde{J}_e \to \tilde{J}_m \quad , \quad \tilde{J}_m \to -\tilde{J}_e. \end{aligned} \tag{29}$$

From the third and fourth Maxwell equations in Eq. (22), the expression for the electromagnetic energy equation (termed the Poynting theorem) may be derived from:

$$-\vec{E}\cdot\left(\vec{\nabla}\times\vec{H}\right)+\vec{H}\cdot\left(\vec{\nabla}\times\vec{E}\right)+\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}+\vec{H}\cdot\frac{\partial\vec{H}}{\partial t}=-\vec{E}\cdot\vec{J}_e-\vec{H}\cdot\vec{J}_m. \tag{30}$$

Using the vectorial identities as $\vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right)$, this equation may then be described as the conservation law for electromagnetic energy as

$$\frac{\partial \mathrm{u}}{\partial t}+\vec{\nabla}\cdot\vec{S}=-\vec{E}\cdot\vec{J}_e-\vec{H}\cdot\vec{J}_m, \tag{31}$$

where

$$\vec{S}=\vec{E}\times\vec{H} \tag{32}$$

is termed the Poynting vector. The changing of the energy density $\frac{\partial \mathrm{u}}{\partial t}$ is defined as [25,30]:

$$\frac{\partial \mathrm{u}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}\left(E^2+H^2\right)=\frac{1}{2}\frac{\partial}{\partial t}\left(\vec{E}\cdot\vec{E}+\vec{H}\cdot\vec{H}\right). \tag{33}$$

Additionally, $\left(\vec{E}\cdot\vec{J}_e\right)$ and $\left(\vec{H}\cdot\vec{J}_m\right)$ denote the power or change of work depending on both electric and magnetic fields.

All calculations in terms of the energy conservation equation for EM can also be adapted for a GEM system. In a similar way, the gravitoelectromagnetic energy equation (termed the gravito-Poynting theorem) may be derived from the third and fourth Maxwell-like equations in Eq. (28).

$$-\tilde{E}\cdot\left(\vec{\nabla}\times\tilde{H}\right)+\tilde{H}\cdot\left(\vec{\nabla}\times\tilde{E}\right)+\tilde{E}\cdot\frac{\partial\tilde{E}}{\partial t}+\tilde{H}\cdot\frac{\partial\tilde{H}}{\partial t}=\tilde{E}\cdot\tilde{J}_e+\tilde{H}\cdot\tilde{J}_m \tag{34}$$

With the necessary mathematical operations, the Poynting theorem can be obtained for GEM as:

$$\vec{\nabla}\cdot\tilde{S}+\frac{\partial\tilde{\mathrm{u}}}{\partial t}=\tilde{E}\cdot\tilde{J}_e+\tilde{H}\cdot\tilde{J}_m, \tag{35}$$

where $\tilde{S}$, $\tilde{\mathrm{u}}$, $\left(\tilde{E}\cdot\tilde{J}_e\right)$, and $\left(\tilde{H}\cdot\tilde{J}_m\right)$ are gravitoelectromagnetic energy flux, gravitoelectromagnetic energy density, gravitoelectric power, and gravitomagnetic power, respectively. Here, gravitoelectromagnetic Poynting vectors $\tilde{S}$ and $\tilde{\mathrm{u}}$ are denoted as:

$$\tilde{S}=\tilde{E}\times\tilde{H}, \tag{36}$$

$$\frac{\partial\tilde{\mathrm{u}}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}\left(\tilde{E}\cdot\tilde{E}+\tilde{H}\cdot\tilde{H}\right). \tag{37}$$

4. Complex octonionic representation of energy conservations

The complex octonionic differential operator and its octonionic conjugate are defined in terms of Cayley–Dickson notation in the literature as below [22–25]:

$$\mathbf{D_t} = \mathrm{i}\frac{\partial}{\partial t}\boldsymbol{e}_0 + \frac{\partial}{\partial x}\boldsymbol{e}_5 + \frac{\partial}{\partial y}\boldsymbol{e}_6 + \frac{\partial}{\partial z}\boldsymbol{e}_7, \tag{38}$$

$$\bar{\mathbf{D}}_\mathbf{t} = \mathrm{i}\frac{\partial}{\partial t}\boldsymbol{e}_0 - \frac{\partial}{\partial x}\boldsymbol{e}_5 - \frac{\partial}{\partial y}\boldsymbol{e}_6 - \frac{\partial}{\partial z}\boldsymbol{e}_7. \tag{39}$$

The multiplication of these operators is commutative and the result is equal to

$$\mathbf{D_t}\bar{\mathbf{D}}_\mathbf{t} = \bar{\mathbf{D}}_\mathbf{t}\mathbf{D_t} = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \Delta - \frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{\partial^2}{\partial t^2}, \tag{40}$$

where the symbol Δ denotes the Laplacian operator in Cartesian coordinates.

In order to obtain energy conservation equations for EM and GEM, a field $\mathbf{F}$ should be suggested in terms of electric, magnetic, gravitoelectric, and gravitomagnetic fields. The new complex octonionic field equation can be defined by the following expressions:

$$\begin{aligned} \mathbf{F} &= \boldsymbol{E} + \mathrm{i}\boldsymbol{H} \\ &= \left(\vec{\boldsymbol{E}} + \mathrm{i}\vec{\boldsymbol{H}}\right) + \left(\tilde{\boldsymbol{E}} + \mathrm{i}\tilde{\boldsymbol{H}}\right) \\ &= \left(\tilde{\boldsymbol{E}} + \vec{\boldsymbol{E}}\right) + \mathrm{i}\left(\vec{\boldsymbol{H}} + \tilde{\boldsymbol{H}}\right), \end{aligned} \tag{41}$$

$$\begin{aligned} \mathbf{F} = {} & \left(\tilde{E}_x\boldsymbol{e}_1 + \tilde{E}_y\boldsymbol{e}_2 + \tilde{E}_z\boldsymbol{e}_3 + E_x\boldsymbol{e}_5 + E_y\boldsymbol{e}_6 + E_z\boldsymbol{e}_7\right) \\ & + \mathrm{i}\left(H_x\boldsymbol{e}_1 + H_y\boldsymbol{e}_2 + H_z\boldsymbol{e}_3 + \tilde{H}_x\boldsymbol{e}_5 + \tilde{H}_y\boldsymbol{e}_6 + \tilde{H}_z\boldsymbol{e}_7\right). \end{aligned} \tag{42}$$

It should be noticed that the last field equation includes 12 components and each field term has different basis elements. By operating a complex octonionic differential operator on this field term, the following equality is obtained in terms of Maxwell equations for EM and GEM as shown below.

$$\begin{aligned} \mathbf{D_t F} = {} & \boldsymbol{e}_0\left(-\vec{\nabla}\cdot\vec{E}\right) \\ & + \boldsymbol{e}_1\left[-\tfrac{\partial H_x}{\partial t} - \left(\vec{\nabla}\times\vec{E}\right)_x\right] + \boldsymbol{e}_2\left[-\tfrac{\partial H_y}{\partial t} - \left(\vec{\nabla}\times\vec{E}\right)_y\right] + \boldsymbol{e}_3\left[-\tfrac{\partial H_z}{\partial t} - \left(\vec{\nabla}\times\vec{E}\right)_z\right] \\ & + \boldsymbol{e}_4\left(\vec{\nabla}\cdot\tilde{E}\right) \\ & + \boldsymbol{e}_5\left[-\tfrac{\partial \tilde{H}_x}{\partial t} - \left(\vec{\nabla}\times\tilde{E}\right)_x\right] + \boldsymbol{e}_6\left[-\tfrac{\partial \tilde{H}_y}{\partial t} - \left(\vec{\nabla}\times\tilde{E}\right)_y\right] + \boldsymbol{e}_7\left[-\tfrac{\partial \tilde{H}_z}{\partial t} - \left(\vec{\nabla}\times\tilde{E}\right)_z\right] \\ & + \mathrm{i}\boldsymbol{e}_0\left(-\vec{\nabla}\cdot\tilde{H}\right) \\ & + \mathrm{i}\boldsymbol{e}_1\left[\tfrac{\partial \tilde{E}_x}{\partial t} - \left(\vec{\nabla}\times\tilde{H}\right)_x\right] + \mathrm{i}\boldsymbol{e}_2\left[\tfrac{\partial \tilde{E}_y}{\partial t} - \left(\vec{\nabla}\times\tilde{H}\right)_y\right] + \mathrm{i}\boldsymbol{e}_3\left[\tfrac{\partial \tilde{E}_z}{\partial t} - \left(\vec{\nabla}\times\tilde{H}\right)_z\right] \\ & + \mathrm{i}\boldsymbol{e}_4\left(\vec{\nabla}\cdot\vec{H}\right) \\ & + \mathrm{i}\boldsymbol{e}_5\left[\tfrac{\partial E_x}{\partial t} - \left(\vec{\nabla}\times\vec{H}\right)_x\right] + \mathrm{i}\boldsymbol{e}_6\left[\tfrac{\partial E_y}{\partial t} - \left(\vec{\nabla}\times\vec{H}\right)_y\right] + \mathrm{i}\boldsymbol{e}_7\left[\tfrac{\partial E_z}{\partial t} - \left(\vec{\nabla}\times\vec{H}\right)_z\right] \end{aligned} \tag{43}$$

The last equation can be reduced to charge and current densities.

$$\begin{aligned} \mathbf{J} = {} & -\rho_e\boldsymbol{e}_0 + J_{mx}\boldsymbol{e}_1 + J_{my}\boldsymbol{e}_2 + J_{mz}\boldsymbol{e}_3 - g_e\boldsymbol{e}_4 - \tilde{J}_{mx}\boldsymbol{e}_5 - \tilde{J}_{my}\boldsymbol{e}_6 - \tilde{J}_{mz}\boldsymbol{e}_7 \\ & + \mathrm{i}g_m\boldsymbol{e}_0 + \mathrm{i}\tilde{J}_{ex}\boldsymbol{e}_1 + \mathrm{i}\tilde{J}_{ey}\boldsymbol{e}_2 + \mathrm{i}\tilde{J}_{ez}\boldsymbol{e}_3 + \mathrm{i}\rho_m\boldsymbol{e}_4 - \mathrm{i}J_{ex}\boldsymbol{e}_5 - \mathrm{i}J_{ey}\boldsymbol{e}_6 - \mathrm{i}J_{ez}\boldsymbol{e}_7 \end{aligned} \tag{44}$$

This expression is called a complex octonionic source equation. It can be easily seen that the relationship between the complex octonionic field and source equation is summarized in a basic and economical form, as follows:

$$\mathbf{D_t F} = \mathbf{J}. \tag{45}$$

In order to arrive at the Poynting theorem for both EM and GEM, the result of $\mathbf{F}^* \cdot (\mathbf{D_t F})$ or $\mathbf{F}^* \cdot \mathbf{J}$ equalities should be found. These equalities are related to Eq. (10). Namely, they must be confirmed by the scalar product rule of octonions. This state was explained in previous studies by Kansu et al. [25] and Tanışlı [26,27]. Therefore, the equalities depending on differential operator and source terms can be written as:

$$\begin{aligned} \mathbf{F}^* \cdot (\mathbf{D_t F}) &= \mathbf{F}^* \cdot \mathbf{J} \\ -\frac{1}{2}\left[\mathbf{F}^*(\mathbf{D_t F}) + \overline{\mathbf{F}^*(\mathbf{D_t F})}\right] &= -\frac{1}{2}\left[\mathbf{F}^*\mathbf{J} + \overline{\mathbf{F}^*\mathbf{J}}\right], \end{aligned} \tag{46}$$

where $\mathbf{F}^*$ represents the complex conjugate of the complex octonionic field term as follows:

$$\begin{aligned} \mathbf{F}^* = &\left(\tilde{E}_x \boldsymbol{e}_1 + \tilde{E}_y \boldsymbol{e}_2 + \tilde{E}_z \boldsymbol{e}_3 + E_x \boldsymbol{e}_5 + E_y \boldsymbol{e}_6 + E_z \boldsymbol{e}_7\right) \\ &-\mathrm{i}\left(H_x \boldsymbol{e}_1 + H_y \boldsymbol{e}_2 + H_z \boldsymbol{e}_3 + \tilde{H}_x \boldsymbol{e}_5 + \tilde{H}_y \boldsymbol{e}_6 + \tilde{H}_z \boldsymbol{e}_7\right). \end{aligned} \tag{47}$$

First, by multiplying $\mathbf{F}^*(\mathbf{D_t F})$ and $\overline{\mathbf{F}^*(\mathbf{D_t F})}$ in octonion forms and with the help of the necessary mathematical processes, the following equality is obtained.

$$\begin{aligned} \mathbf{F}^*(\mathbf{D_t F}) = \overline{\mathbf{F}^*(\mathbf{D_t F})} = \; &\boldsymbol{e}_0\Big[\tilde{E} \cdot \tfrac{\partial \vec{H}}{\partial t} + \tilde{E} \cdot \left(\vec{\nabla} \times \vec{E}\right) - \tilde{H} \cdot \tfrac{\partial \vec{E}}{\partial t} + \tilde{H} \cdot \left(\vec{\nabla} \times \vec{H}\right) \\ &+ \vec{E} \cdot \tfrac{\partial \tilde{H}}{\partial t} + \vec{E} \cdot \left(\vec{\nabla} \times \tilde{E}\right) - \vec{H} \cdot \tfrac{\partial \tilde{E}}{\partial t} + \vec{H} \cdot \left(\vec{\nabla} \times \tilde{H}\right)\Big] \\ &+ \mathrm{i}\boldsymbol{e}_0\Big[-\tilde{E} \cdot \tfrac{\partial \tilde{E}}{\partial t} + \tilde{E} \cdot \left(\vec{\nabla} \times \tilde{H}\right) - \tilde{H} \cdot \tfrac{\partial \tilde{H}}{\partial t} - \tilde{H} \cdot \left(\vec{\nabla} \times \tilde{E}\right) \\ &- \vec{E} \cdot \tfrac{\partial \vec{E}}{\partial t} + \vec{E} \cdot \left(\vec{\nabla} \times \vec{H}\right) - \vec{H} \cdot \tfrac{\partial \vec{H}}{\partial t} - \vec{H} \cdot \left(\vec{\nabla} \times \vec{E}\right)\Big] \end{aligned} \tag{48}$$

It is easily seen that the result is found in terms of only the scalar component. In other words, the vectorial components zeroize each other. This means that the energy is a scalar quantity. Similar octonionic multiplication can be also applied for $\mathbf{F}^*\mathbf{J}$ and $\overline{\mathbf{F}^*\mathbf{J}}$ on the right side of Eq. (46). The result of this product can be obtained as:

$$\begin{aligned} \mathbf{F}^*\mathbf{J} = \overline{\mathbf{F}^*\mathbf{J}} = \; &\boldsymbol{e}_0\left[-\tilde{E} \cdot \vec{J}_m + \vec{E} \cdot \tilde{J}_m - \vec{H} \cdot \tilde{J}_e + \tilde{H} \cdot \vec{J}_e\right] \\ &+ \mathrm{i}\boldsymbol{e}_0\left[-\tilde{E} \cdot \tilde{J}_e + \vec{E} \cdot \vec{J}_e + \vec{H} \cdot \vec{J}_m - \tilde{H} \cdot \tilde{J}_m\right]. \end{aligned} \tag{49}$$

Similarly to Eq. (48), all components in the same bases without $\boldsymbol{e}_0$ scalar bases zeroize one another. It is clear that there is no contribution to the energy conservation equations from $\boldsymbol{e}_0$ bases of Eqs. (48) and (49) Because, the field components of EM and GEM are mixed with each other and they have no physical meaning. Therefore, the components with $\mathrm{i}\boldsymbol{e}_0$ bases for Eqs. (48) and (49) are valid. By using electromagnetic energy flux and density, Eqs. (48) and (49) can be equaled together as follows:

$$\left[\vec{\nabla} \cdot \vec{S} + \frac{\partial(\mathrm{u})}{\partial t} + \vec{\nabla} \cdot \tilde{S} + \frac{\partial(\tilde{\mathrm{u}})}{\partial t}\right] = \left[-\vec{E} \cdot \vec{J}_e - \vec{H} \cdot \vec{J}_m + \tilde{E} \cdot \tilde{J}_e + \tilde{H} \cdot \tilde{J}_m\right], \tag{50}$$

where $\vec{S}$, u, $\tilde{S}$, and ũ represent electromagnetic energy flux, electromagnetic energy density, gravitoelectromagnetic energy flux, and gravitoelectromagnetic energy density, respectively. As a result, the Poynting theorems for EM and GEM are separately obtained using octonion algebra once again as below:

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial (\mathrm{u})}{\partial t} = -\vec{E} \cdot \vec{J}_e - \vec{H} \cdot \vec{J}_m, \tag{51}$$

$$\vec{\nabla} \cdot \tilde{S} + \frac{\partial (\tilde{\mathrm{u}})}{\partial t} = \tilde{E} \cdot \tilde{J}_e + \tilde{H} \cdot \tilde{J}_m. \tag{52}$$

5. Results and conclusions

The energy conservation law is one of the most important laws for physics and its applications. It is known that there is an analogy between EM and GEM in the literature. The main purpose of this study is to generalize Poynting theorems for both EM and GEM in higher dimensions. For this process, an algebraic structure with 12 components is needed. Therefore, octonion algebra is used in this study. However, real octonions were not used due to their number of components and their not providing the same results with mathematical operations. Namely, complex octonions are more suitable than real ones for this study with their algebraic structures. Octonions are one of the members of higher dimensional algebra. They are useful favorable, alternative tools for representing physical systems in theory. Although sedenions have 16 components like complex octonions, octonion algebra has alternative division rings due to its dimensions. That is, octonions are the last member of verifying division algebra in hypernumber systems. The Poynting theorem is an energy conservation equality for EM. In this study, as distinct from previous studies [25,26], Maxwell equations have been chosen with the source terms, and then the Poynting theorem has been verified in this sense. In addition, this state has also been adapted for linear gravity.

The new field term, consisting of both electromagnetic and gravitoelectromagnetic components, has first been defined in terms of octonion basis elements. Then, by applying a complex octonionic differential operator to this field, the complex octonionic source equation has been written in a more elegant, useful, and simple form. By using the scalar product rule of octonions, the energy conservation equation has been obtained with the source terms for both EM and GEM for the first time. Although electromagnetic and gravitoelectromagnetic field components have been written in a different octonion basis, the Poynting theorem for both EM and GEM has been obtained in the same scalar basis. This case indicates that the energy is a scalar quantity in physics. As a result, both electromagnetic and gravitoelectromagnetic Poynting theorems have been combined under the same scalar elements by using nonassociative algebra. This study shows that the field term denoted in Eq. (42) is similar to our previous field [25] without the gravity field components. Our previous study relying on energy conservation [25] does not include electromagnetic equations with sources.

In the following stages of work, similar processes for EM and GEM will be applied by using Lorentz transformations. In addition, it is planned to investigate whether these applications could be rearranged or not with different higher dimensional algebras such as octons, sedenions, or sedeons.

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