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Thermodynamic properties of the noncommutative black hole in (z = 3)-Hořava–Lifshitz gravity

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Abstract: In this work, we investigate the effects of noncommutative spaces on the Hořava–Lifshitz black hole. We construct the black hole solutions in the noncommutative space of **(z = 3)**-Hořava–Lifshitz gravity**.** We calculate the horizon and the thermodynamic properties such as the Hawking temperature, the ADM-Mass, and entropy, which reduce to their commutative limits when the noncommutativity parameter tends to zero.

Key words: Noncommutative black hole, Hořava–Lifshitz gravity, Hawking temperature, ADM-Mass, entropy

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1. Introduction

Recently Hořava proposed a renormalizable theory of gravity at a Lifshitz point [1], which may be regarded as a UV complete candidate for general relativity. However, it is reduced to Einstein's general relativity at large distances [2]. Noting that Hořava–Lifshitz theory goes to standard general relativity if the coupling λ has the specific value $\lambda = 1$.

However, the static solutions with the spherical symmetry black hole in Hořava–Lifshitz theory were studied by Lu et al. [2], while the topological black hole solutions and their thermodynamic properties were discussed in detail by Cai et al. [3].

In the ADM formalism, the 4-dimensional metric of general relativity is parameterized as follows [4]:

$$ds^2 = -N^2dt^2 + g_{ij}\left(dx^i - N^i dt\right)\left(dx^i - N^j dt\right). \tag{5}$$

The $z = 3$-Hořava–Lifshitz action with a parameter λ proposed by Hořava is given by [1]:

$$S_{HL} = \int (L_0 + L_1) \tag{6}$$

with

$$L_0 = \sqrt{g}N\left\{\frac{2}{k^2}\left(K_{ij}K^{ij} - \lambda K^2\right) + \frac{k^2\mu^2\left(\Lambda_W R - 3\Lambda^2{}_W\right)}{8\left(1-3\lambda\right)}\right\} \tag{7}$$

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$$L_1 = \sqrt{g}N\left\{\frac{k^2\mu^2(1-4\lambda)}{32(1-3\lambda)}R^2 + \frac{k^2}{2w^4}\left(C_{ij} - \frac{\mu w^2}{2}R_{ij}\right)\left(C^{ij} - \frac{\mu w^2}{2}R^{ij}\right)\right\} \tag{8}$$

where λ, k, μ, w, and W are constant parameters, and C^{ij} is the Cotton tensor, defined by

$$C^{ij} = \varepsilon^{ikl}\nabla_k\left(R^j{}_l - \frac{1}{4}R\delta^j{}_l\right). \tag{9}$$

In order to understand Lifshitz black holes we will consider the 2 conditions $N^2 = \hat{N}^2 f(r)$ and $N^i = 0$. A spherically symmetric solution could be obtained with a metric ansatz proposed by Lu–Mei–Pope (LMP) [2]

$$ds^2_{LMP} = -\hat{N}^2(r)f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{10}$$

which implies that

$$K_{ij} = C_{ij} = 0 \tag{11}$$

If we take only the Lagrangian L_0 we obtain the Schwarzschild-AdS (SAdS) [5] black hole whose metric function is given by

$$f(r) = 1 - \frac{\Lambda_W}{2}r^2 - \frac{m}{r} \quad \text{and} \quad \hat{N}^2 = 1. \tag{12}$$

The relevance of noncommutative geometry was developed when it was shown that field theories become noncommutative if the matter is coupled to gravity, and the spacetime induces algebra of noncommutative coordinates. The noncommutativity is introduced by means of [6]

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \tag{1}$$

where θ^{ij} is an antisymmetric tensor, $D \times D$ matrix, where D is the dimension of the spacetime. It has dimensions of (length) 2 and it parameterizes the spacetime. The noncommutative relation given by (1) induces quantum mechanical fluctuations in the metric $g_{\mu\nu}$. The spacetime is quantized and the coordinates become noncommutative.

In an approach to noncommutative geometry, the multiplication of 2 fields in the Lagrangian is replaced by the star (or Moyal) product given in [7] by

$$(f * g)(\hat{x}) = \exp\left[\frac{i}{2}\theta^{ab}\frac{\partial}{\partial\hat{x}^a}\frac{\partial}{\partial\hat{x}^b}\right] f(\hat{x})g(\hat{x}), \tag{2}$$

where f and g are functions of the spaces coordinates. It was noted by Chaichian et al. [8] that at the replacement

$$\hat{x}_i = x_i + \frac{1}{2}\theta_{ij}p_j, \quad \hat{p}_j = p_j \tag{3}$$

we get to the standard commutation relationships

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \qquad [\hat{x}_i, p_j] = i\delta_{ij}, \qquad [p_i, p_j] = 0 \tag{4}$$

where x_i is the related commutative coordinates $[x_i, x_j] = 0$.

In our paper, we investigate the effects of noncommutative spaces on the Hořava–Lifshitz black hole. We construct the black hole solutions in the noncommutative space of **(z = 3)**-Hořava–Lifshitz gravity. We calculate the horizon and the thermodynamic properties such as the Hawking temperature, the ADM-Mass, and entropy, which reduce to their commutative limits when the noncommutativity parameter tends to zero.

The remainder of this work is organized as follows. In section 2, we construct the solution of the noncommutative Hořava–Lifshitz black hole, starting with the simple case of Schwarzhild–AdS and then generalizing to LMP models. Section 3 is devoted to finding the thermodynamics properties of black holes, namely the entropy and Hawking temperature, and section 4 contains the concluding remarks.

2. Hořava–Lifshitz black hole in noncommutative space

2.1. Schwarzhild–AdS formulation

To understand this construction, we will start with the Schwarzhild–AdS case, which corresponds to the Lagrangian L_0; thus we have the following solution of a black hole:

$$ds^2 = \left(1 - \frac{\Lambda_W}{2}\hat{r}\hat{r} - \frac{m}{\sqrt{\hat{r}\hat{r}}}\right) dt^2 + \frac{d\hat{r}^2}{\left(1 - \frac{\Lambda_W}{2}\hat{r}\hat{r} - \frac{m}{\sqrt{\hat{r}\hat{r}}}\right)} + \hat{r}\hat{r}\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{13}$$

The horizon of this black hole is localized at

$$\left(1 - \frac{\Lambda_W}{2}\hat{r}\hat{r} - \frac{m}{\sqrt{\hat{r}\hat{r}}}\right) = 0 \tag{14}$$

By insertion of formula (3) and developing we find the following fourth-degree equation

$$r^4 + \frac{\Lambda_w\left(p^2\theta^2 - \left(\vec{p}.\vec{\theta}\right)^2\right) - 3\Lambda_w\vec{L}.\vec{\theta} - 16}{8\Lambda_w}r^2 + \frac{2m}{\Lambda_w}r + \frac{16\vec{L}.\vec{\theta} + 2\Lambda_w\left(\vec{L}.\vec{\theta}\right)^2 - \Lambda_w\vec{L}.\vec{\theta}\left(p^2\theta^2 - \left(\vec{p}.\vec{\theta}\right)^2\right)}{64\Lambda_w} = 0 \tag{15}$$

Here $\theta_{ij} = \frac{1}{2}\varepsilon_{ijk}\theta^k$ and $L = r \times p$, Eq. (15) is equivalent to

$$r^4 + Ar^2 + Br + C = 0 \tag{16}$$

where the coefficients are given by

$$A = \frac{\Lambda_w\left(p^2\theta^2 - \left(\vec{p}.\vec{\theta}\right)^2\right) - 3\Lambda_w\vec{L}.\vec{\theta} - 16}{8\Lambda_w}, \quad B = \frac{2m}{\Lambda_w},$$

$$C = \frac{16\vec{L}.\vec{\theta} + 2\Lambda_w\left(\vec{L}.\vec{\theta}\right)^2 - \Lambda_w\vec{L}.\vec{\theta}\left(p^2\theta^2 - \left(\vec{p}.\vec{\theta}\right)^2\right)}{64\Lambda_w} \tag{17}$$

Applying the Descarte method [9] in Eq. (16), we find the following solutions:

$$r_1 = \frac{-a - \sqrt{-\left(a^2 + 2A + \frac{2B}{a}\right)}}{2}, \quad r_2 = \frac{-a + \sqrt{-\left(a^2 + 2A + \frac{2B}{a}\right)}}{2}$$

$$r_3=\frac{a-\sqrt{-\left(a^2+2A-\frac{2B}{a}\right)}}{2},\qquad r_4=\frac{a+\sqrt{-\left(a^2+2A-\frac{2B}{a}\right)}}{2} \tag{18}$$

with

$$\begin{aligned} a&=\pm\left[\left(-\frac{q}{2}-\frac{1}{2}\sqrt{\frac{27q^2+4p^3}{27}}\right)^{1/3}+\left(-\frac{q}{2}+\frac{1}{2}\sqrt{\frac{27q^2+4p^3}{27}}-\frac{2A}{3}\right)^{1/3}\right]^{1/2}\\ b&=\frac{1}{2}\left(a^2+A+\frac{B}{a}\right),\qquad c=\frac{1}{2}\left(a^2+A-\frac{B}{a}\right)\\ p&=-4C-\frac{A^2}{3},\qquad q=\frac{27A}{3}\left(A^2+36C\right)+C \end{aligned} \tag{19}$$

Here the horizon corresponds to a positive choice of solutions

$$r_H=\frac{a+\sqrt{-\left(a^2+2A-\frac{2B}{a}\right)}}{2}. \tag{20}$$

2.2. LMP formulation: general case

To get to the general case, we use the LMP formulation of Hořava–Lifshitz black hole in noncommutative space, which is obtained by introducing a newly radial coordinate $x=\sqrt{-\Lambda_W}r$. As a consequence, we have LMP black hole solutions where f and $\hat{N}$ are determined to be

$$\begin{aligned} f=&\quad 1+x^2-\left(\sqrt{-\Lambda_W}mx\right)^{p\pm(\lambda)},\\ \hat{N}=&\quad x^{q\pm(\lambda)}, \end{aligned} \tag{21}$$

where

$$p_{\pm}(\lambda)=\frac{2\lambda\pm\sqrt{6\lambda-2}}{\lambda-1},\quad q\pm(\lambda)=-\frac{1+3\lambda\pm2\sqrt{6\lambda-2}}{\lambda-1} \tag{22}$$

In order to understand this situation, we will deal with 2 different cases, namely $\lambda=3$ and $\lambda=1$. For $\lambda=3$: we have

$$f=1+x^2-\sqrt{-\Lambda_W}mx \tag{23}$$

and

$$\begin{aligned} \hat{N}^2f &= x^{-2}\left(1+x^2-\sqrt{-\Lambda_W}mx\right)\\ &= x^{-2}+1-\sqrt{-\Lambda_W}mx^{-1} \end{aligned} \tag{24}$$

In the noncommutative case, we have $\hat{N}^2f=0$; therefore,

$$\hat{x}^{-2}+1-\sqrt{-\Lambda_W}m\hat{x}^{-1}=0 \tag{25}$$

Expanding this equation we find the following equation:

$$x^4-\frac{\sqrt{-\Lambda_W}m}{4}x^3+\frac{3\left(p^2\theta^2-\left(\vec{p}.\ \vec{\theta}\right)^2\right)-6\left(\vec{L}.\vec{\theta}\right)+16}{32}x^2+\frac{\sqrt{-\Lambda_W}m\left[4\left(\vec{L}.\vec{\theta}\right)-3\left(p^2\theta^2-\left(\vec{p}.\ \vec{\theta}\right)^2\right)\right]}{16}x+\frac{\left(p^2\theta^2-\left(\vec{p}.\ \vec{\theta}\right)^2\right)-4\left(\vec{L}.\vec{\theta}\right)}{32}=0 \tag{26}$$

We set

$$\begin{aligned} A' &= -\tfrac{\sqrt{-\Lambda_W}m}{2}, & B' &= \tfrac{3\left(p^2\theta^2-\left(\vec{p}.\ \vec{\theta}\right)^2\right)}{16} - \tfrac{3\left(\vec{L}.\vec{\theta}\right)}{8} + 1 \\ C' &= \sqrt{-\Lambda_W}m\left[\tfrac{\left(\vec{L}.\vec{\theta}\right)}{2} - \tfrac{3\left(p^2\theta^2-\left(\vec{p}.\ \vec{\theta}\right)^2\right)}{8}\right], & D' &= \tfrac{\left(p^2\theta^2-\left(\vec{p}.\ \vec{\theta}\right)^2\right)}{16} - \tfrac{\left(\vec{L}.\vec{\theta}\right)}{4} \end{aligned} \tag{27}$$

Therefore, Eq. (26) becomes

$$2x^4 + A'x^3 + B'x^2 + C'x + D' = 0. \tag{28}$$

To solve this equation we make the following change in variable

$$x = X - \frac{A'}{8} \tag{29}$$

Thus, Eq. (28) becomes

$$x^4 + Ux^2 + Vx + S = 0 \tag{30}$$

where

$$\begin{aligned} U &= \frac{-3A'}{32} + \frac{B}{2}, \qquad V = \frac{\left(\frac{A'}{2}\right)^3}{8} - \frac{A'B}{8} + \frac{C}{2} \\ S &= -\left(\frac{A'}{8}\right)^4 + \frac{B\left(\frac{A'}{2}\right)^2}{8} - \frac{A'C}{16} + \frac{D}{2} \end{aligned} \tag{31}$$

Using again the Descartes method [9] we find the following solutions:

$$\begin{aligned} r_1^{'} &= \frac{-r - \sqrt{-\left(r^2 + 2U + \frac{2V}{t}\right)}}{2}, & r_2^{'} &= \frac{-t + \sqrt{-\left(t^2 + 2U + \frac{2V}{t}\right)}}{2} \\ r_3^{'} &= \frac{s - \sqrt{-\left(t^2 + 2U - \frac{2V}{t}\right)}}{2}, & r_4^{'} &= \frac{t + \sqrt{-\left(t^2 + 2U - \frac{2V}{t}\right)}}{2} \end{aligned} \tag{32}$$

Here the horizon corresponds also to a positive choice of solutions

$$r_H' = \frac{t + \sqrt{-\left(t^2 + 2U - \frac{2V}{t}\right)}}{2} \tag{33}$$

If we take $Q^2 = \frac{1}{-\Lambda_W}$ we find again the Reissner–Nordstrom black hole and when $\theta \to 0$ we meet the horizon of a Reissner–Nordstrom black hole.

For $\lambda = 1$: the noncommutative version of metric function is given by

$$f = 1 + x^2 - \sqrt{-\Lambda_W}mx, \quad \hat{N} = 1, \tag{34}$$

which gives

$$\begin{aligned} f &= 1+\left(x_i-\frac{1}{2}\theta_{ij}p_j\right)\left(x_i-\frac{1}{2}\theta_{ik}pk\right)-\sqrt{-\Lambda_W m}\left(\sqrt{\left(x_i-\frac{1}{2}\theta_{ij}p_j\right)\left(x_i-\frac{1}{2}\theta_{ij}p_j\right)}\right) \\ &= \frac{32x^3-\sqrt{-\Lambda_W m}x^2+\left(4\vec{L}.\vec{\theta}+2p^2\theta^2-2\left(\vec{p}.\ \vec{\theta}\right)^2+32\right)x}{32x} \\ &\quad +\frac{\sqrt{-\Lambda_W m}\left(\left(\vec{L}.\vec{\theta}\right)+p^2\theta^2-\left(\vec{L}.\vec{\theta}\right)^2\right)}{32x} \end{aligned} \tag{35}$$

If we take $f=0$, we have the following solution:

$$\begin{aligned} y &= \sqrt[3]{-\frac{q}{2}-\frac{1}{2}\sqrt{\frac{27q^2+4p^3}{27}}}+\sqrt[3]{-\frac{q}{2}+\frac{1}{2}\sqrt{\frac{27q^2+4p^3}{27}}} \\ r &= \sqrt[3]{-\frac{q}{2}-\frac{1}{2}\sqrt{\frac{27q^2+4p^3}{27}}}+\sqrt[3]{-\frac{q}{2}+\frac{1}{2}\sqrt{\frac{27q^2+4p^3}{27}}}-\frac{\left(\sqrt{-\Lambda_w}m\right)}{3} \end{aligned} \tag{36}$$

where

$$\begin{aligned} p &= \left(\sqrt{-\Lambda_w}m\right)^2-1+\frac{\left(8\vec{L}.\vec{\theta}\right)-2p^2\theta^2+2\left(\vec{p}.\vec{\theta}\right)^2}{32} \\ q &= \frac{1}{3}\left(\sqrt{-\Lambda_w}m\right)+\frac{\sqrt{-\Lambda_w}m}{32}\left[-\frac{20}{3}\vec{L}.\vec{\theta}+\frac{5}{3}p^2\theta^2-\frac{5}{3}\left(\vec{p}.\vec{\theta}\right)^2\right] \end{aligned} \tag{37}$$

3. Thermodynamic properties

3.1. Hawking temperature

We compute the Hawking temperature as follows [10]:

$$T=\left.\frac{\left(\hat{N}^2 f\right)'}{4\pi}\sqrt{-g^{tt}g^{rr}}\right|_{x=x_+}=\frac{\left(\hat{N}^2 f\right)'}{4\pi}\sqrt{-g^{tt}g^{rr}} \tag{38}$$

In the case of the AdS–Schwarszchild black hole

$$T=\frac{\sqrt{-\Lambda_W}}{8\pi}\frac{3x^2{}_++2}{x_+} \tag{39}$$

Here x_+ is given from (20) by

$$x_+=r_{4_+}=\frac{a+\sqrt{-\left(a^2+2A-\frac{2B}{a}\right)}}{2} \tag{40}$$

We find

$$T=\frac{\sqrt{-\Lambda_W}}{4\pi}\frac{\frac{3}{4}\left(a+\sqrt{-\left(a^2+2A-\frac{2B}{a}\right)}\right)^2+2}{a+\sqrt{-\left(a^2+2A-\frac{2B}{a}\right)}} \tag{41}$$

When we have $\lambda=3$ we find

$$T=\frac{x^{-1}\left[(2-1)\ x^2-1\right]}{4\pi r}=\frac{x^{-1}\left[x^2-1\right]}{4\pi r}=\frac{8M}{16\pi x_h^2-4\pi(\vec{L}.\vec{\theta})+\pi p^2\theta^2-\pi(\vec{p}.\ \vec{\theta})^2} \tag{42}$$

which is the temperature of the Reisner–Nordstrom black hole in the commutative space $\theta=0$.

3.2. ADM-Mass

We derive the ADM-Mass as from Cai et al. [11]:

$$M=\frac{\pi k^2\mu^2\left(-\Lambda\right)}{2\sqrt{6\lambda-2}}\frac{\left(1+x^2{}_+\right)^2}{x_+^{2p(\lambda)}} \tag{43}$$

In the case of $\lambda=3$ we find

$$M=\frac{\pi k^2\mu^2\left(-\Lambda\right)}{8}\frac{\left(1+x^2{}_+\right)^2}{x_+^2} \tag{44}$$

In noncommutative space we find

$$M=\frac{\pi k^2\mu^2\left(-\Lambda\right)}{8}\frac{\left(1+x_+^2-\frac{\left(\vec{L}.\vec{\theta}\right)}{4}+\frac{p^2\theta^2-\left(\vec{p}.\ \vec{\theta}\right)^2}{16}\right)^2}{x_+^2-\frac{\left(\vec{L}.\vec{\theta}\right)}{4}+\frac{p^2\theta^2-\left(\vec{p}.\ \vec{\theta}\right)^2}{16}} \tag{45}$$

3.3. Entropy

Now we calculate the entropy of this black hole. The first law of thermodynamics gives

$$dM=TdS. \tag{46}$$

It means that

$$S=\frac{\sqrt{2}\left(-\Lambda\right)}{\sqrt{3\lambda-2}}\left(\pi r_+^2-\frac{\pi}{\Lambda}\ln\frac{A}{4}-S_0\right) \tag{47}$$

In noncommutative space we have the following expression for black hole entropy:

$$S=\frac{\sqrt{2}\left(-\Lambda\right)}{\sqrt{3\lambda-2}}\left(\pi r_+^2-\frac{\pi}{\Lambda}\ln\frac{A}{4}-S_0-\frac{\pi\left(\vec{L}.\vec{\theta}\right)}{4}+\frac{\pi\left(p^2\theta^2-\left(\vec{p}.\ \vec{\theta}\right)^2\right)}{16}\right) \tag{48}$$

If $\lambda=3$, we have the following expression:

$$S = \frac{\sqrt{2}\,(-\Lambda)}{\sqrt{7}} \left(\pi \frac{A}{4} - \frac{\pi}{\Lambda} \ln \frac{A}{4} - S_0 \right), \tag{49}$$

where

$$A = \pi r^2{}_+ = \pi \left(\frac{t + \sqrt{-\left(t^2 + 2U - \frac{2V}{t}\right)}}{2} \right)^2. \tag{50}$$

4. Conclusion

We investigated the effects of noncommutative spaces on the Hořava–Lifshitz black hole. We constructed the black hole solutions in the noncommutative space of (z = 3)-Hořava–Lifshitz gravity starting with the simple case of Schwarzhild–AdS and generalizing to LMP models. We calculated the horizon and the thermodynamic properties such as the Hawking temperature [Eq. (41)], the ADM-Mass [Eq. (45)], and entropy [Eq. (48)].

We have given detailed exact formulae of thermodynamic properties and the important point is that we have recovered all results discussed in the commutative case when the noncommutativity parameter θ tends to zero. We have in this case the following thermodynamic results for the Hořava–Lifshitz black hole:

Metric function: $dsf(r) = 1 - \frac{\Lambda_W}{2} r^2 - \frac{m}{r}$,

Hawking temperature: $ds\frac{M}{2\pi x_h^2}$,

ADM Mass: $ds\frac{\pi k^2 \mu^2 (-\Lambda)}{8} \frac{\left(1 + x_+^2\right)^2}{x_+^2}$,

Entropy: $ds\frac{\sqrt{2}(-\Lambda)}{\sqrt{3\lambda - 2}} \left(\pi r_+^2 - \frac{\pi}{\Lambda} \ln \frac{A}{4} - S_0 \right)$.

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