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Dynamical behavior of traveling wave solutions for the $K(2,2)$ equation

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Abstract

In this paper, we discuss the qualitative behavior of degenerate singular points for the $K(2,2)$ equation. By analyzing the different dynamical behaviors of the singular system and its associated regular system, we come to the conclusion that for the singular system the smoothness property of its orbits, which have intersection point(s) with the singular straight line, will not be destroyed. Meanwhile, the corresponding smooth traveling wave solutions for the $K(2,2)$ equation are obtained.

Key Words: $K(2,2)$ equation, singular traveling wave system, nilpotent, smooth periodic wave

1. Introduction

The one-dimensional motion of solitary waves of inviscid and incompressible fluids has been the subject of research for more than a century [1]. Derivation of the well-known Korteweg-de Vries (KdV) equation

$$u_t + 6\mu u u_x + u_{xxx} = 0, \quad \mu = \pm 1 \tag{1.1}$$

was probably one of the most important results in the study of solitary waves. The KdV equation was originally formulated to model unidirectional propagation of shallow water waves in one spatial dimension [2].

To study the role of nonlinear dispersion in the formation of patterns in liquid drops, Rosenau and Hyman [3] introduced a family of fully nonlinear KdV equations $K(m,n)$,

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m > 0,\ 1 < n \leq 3, \tag{1.2}$$

and presented a class of solitary waves with compact support (called compactons) that are solutions of the $K(2,2)$ equation

$$u_t + (u^2)_x + (u^2)_{xxx} = 0. \tag{1.3}$$

Recently, Yin [4] obtained a variety of traveling wave solutions for the $K(2,2)$ equation (1.3) by applying the qualitative analysis method given by [5–6]. In passing, note that most of these solutions, including peakons, cuspons, loopons and stumpons, are of the composite waves, i.e., they consist of two or more branches of non-smooth solutions.

As far as we are concerned, the qualitative behavior of traveling wave solutions for equation (1.3) has not been investigated in the literature. In this paper, we will study the qualitative behavior of traveling wave solutions for equation (1.3) in the case of degenerate singular points by applying the qualitative theory of polynomial differential systems.

To investigate the traveling wave solutions of equation (1.3), let $u(x,t) = u(x+ct) = \phi(\xi)$, where c is the wave speed and $\xi = x + ct$. Substituting it into (1.3) and integrating (1.3) once with respect to ξ leads to

$$c\phi + \phi^2 + 2\phi_\xi^2 + 2\phi\phi_\xi = g, \tag{1.4}$$

where g is an integral constant. Let $\phi_\xi = y$, then, we have the traveling wave system

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \frac{-\phi^2 - 2y^2 - c\phi + g}{2\phi} \tag{1.5}$$

with the first integral

$$H(\phi, y) = \phi^2(y^2 + \phi^2/4 + c\phi/3 - g/2) = h. \tag{1.6}$$

System (1.5) is called a singular traveling wave system since the second equation of system (1.5) is not continuous on the singular straight line $\phi = 0$. Making the transformation $d\xi = 2\phi d\tau$, for $\phi \neq 0$, system (1.5) becomes

$$\frac{d\phi}{d\tau} = 2\phi y, \qquad \frac{dy}{d\tau} = -\phi^2 - 2y^2 - c\phi + g, \tag{1.7}$$

which is called the associated regular system of system (1.5). Obviously, the singular line $\phi = 0$ now becomes an invariant straight line solution of system (1.7).

We point out that system (1.5) has the same topological phase portrait as system (1.7), except on the singular straight line $\phi = 0$. In other words, if an orbit of system (1.5) is far away from the singular straight line $\phi = 0$, then, the transformation $d\xi = 2\phi d\tau$ does not affect the smoothness property of $\phi(\xi)$. However, if an orbit of system (1.5) is close to or intersects with the straight line $\phi = 0$, then the smoothness property of orbit of the corresponding singular system (1.5) with respect to the "time variable," ξ should be carefully studied, for the straight line $\phi = 0$ is not an orbit of system (1.5). Even though we have some exact explicit parametric representations of traveling wave solutions for equation (1.3). It is important to figure out the dynamical behavior of traveling wave solutions governed by singular traveling wave equations. We will discuss the above interesting problems in this work. The method of the phase plane [7-9] plays an important role in our study.

2. Qualitative behavior of traveling wave system

In this section, we study the qualitative behavior of system (1.7). Throughout the paper, we assume that $c > 0$ since (1.7) is invariant under the transformation $\phi \to -\phi,\ c \to -c$. We first introduce a lemma.

Lemma 2.1 *(See [10].) Let $(0,0)$ be a nilpotent singular point of the vector field $(y + F(x,y), G(x,y))$, where F and G are analytic functions in a neighborhood of the origin at least with quadratic terms in the variables x*

and y. Let $y = f(x)$ *be the solution of the equation* $y + F(x,y) = 0$ *in a neighborhood of* $(0,0)$*. Assume that the development of the function* $G(x, f(x))$ *is of the form* $\alpha x^k + o(x^k)$ *and* $\Phi(x) \equiv (\partial F/\partial x + \partial G/\partial y)(x, f(x)) = \beta x^n + o(x^n)$ *with* $\alpha \neq 0, k \geq 2$ *and* $n \geq 1$*, Then the following statements hold.*

(1) If k *is even and*

(1.a) $k > 2n+1$*, then the origin is a saddle-node. Moreover, the saddle-node has one separatrix tangent to the semi-axis* $x < 0$*, and other two separatrices tangent to the semi-axis* $x > 0$*.*

(1.b) $k < 2n+1$ *or* $\Phi \equiv 0$*, then the origin is a cusp, i.e. a singular point formed by the union of two hyperbolic sectors. Moreover, the cusp has two separatrices tangent to the positive* x*-axis.*

(2) If k *is odd and* $\alpha > 0$*, then the origin is a saddle. Moreover, the saddle has two separatrices tangent to the semi-axis* $x < 0$*, and other two separatrices tangent to the semi-axis* $x > 0$*.*

(3) If k *is odd and* $\alpha < 0$ *and*

(3.a) n *even,* $k = 2n+1$ *and* $\beta^2 + 4(n+1)\alpha \geq 0$*, or* n *even and* $k > 2n+1$*, then the origin is a stable (unstable) node if* $L < 0$ $(L > 0)$*, having all the orbits tangent to the* x*-axis at* $(0,0)$*.*

(3.b) n *odd,* $k = 2n+1$ *and* $\beta^2 + 4(n+1)\alpha \geq 0$*, or* n *odd and* $k > 2n+1$*, then the origin is an elliptic-saddle, i.e. a singular point formed by the union of one hyperbolic sector and one elliptic sector. Moreover, one separatrix of the elliptic-saddle is tangent to the semi-axis* $x < 0$*, and the other to the semi-axis* $x > 0$*.*

(3.c) $k = 2n+1$ *and* $\beta^2 + 4(n+1)\alpha < 0$*, or* $k < 2n+1$*, then the origin is a focus or a center, and if* $\Phi(x) \equiv 0$ *then the origin is a center.*

Define $\Delta = c^2 + 4g$. Obviously, there are two equilibrium points $E_{1,2}(\phi_{1,2}, 0)$ on the ϕ-axis, where $\phi_{1,2} = \frac{-c\pm\sqrt{\Delta}}{2}$, if $\Delta > 0$. When $g > 0$, there exist two equilibrium points $N_{\pm}(0, Y_{\pm})$ on the straight line $\phi = 0$, where $Y_{\pm} = \pm\sqrt{g/2}$. Let $M(\phi_e, y_e)$ be the coefficient matrix of the linearized system (1.7) at singular point (ϕ_e, y_e), J be its Jacobin determinant and T be its trace. By the theory of planar dynamical systems, we know that for a singular point (ϕ_e, y_e) of a planar integrable system, (ϕ_e, y_e) is a saddle if $\Delta < 0$, a node if $T^2 > 4J > 0$ (stable if $T < 0$, unstable if $T > 0$), a center if $T = 0 < \Delta$, a nilpotent singular point if $\Delta = T = 0$ but $M(\phi_e, y_e)$ is not identically zero. Here, we emphasize the qualitative behavior in the case of degenerate singular point. Based on Lemma 2.1, after simple computation, we know that for $g = -c^2/4$, $E_{1,2}(-c/2, 0)$ is a cusp, i.e. the cusp is formed by the union of two hyperbolic sectors. Moreover, the cusp has two separatrices tangent to the semi-axis $x < 0$. When $g = 0$, the origin is a saddle, i.e. the saddle has two separatrices tangent to the semi-axis $y < 0$, and other two separatrices tangent to the semi-axis $y > 0$. The phase portraits of (1.7) for $c > 0$ are shown in Figures 1(a)–(d).

3. Different dynamical behaviors of orbits of vector fields defined by (1.7) and (1.5)

In this section, we study the dynamical behaviors of (1.7) and (1.5). Let $h_s = H(0, Y_{\pm})$ and $h_{1,2} = H(\phi_{1,2}, 0)$, where $H(\phi, y)$ is defined by (1.6). We know that if an orbit of (1.5) is far from the singular straight line $\phi = 0$, the transformation $d\xi = 2\phi d\tau$ does not affect the smoothness property of $\phi(\xi)$. As a direct consequence, we can see from Figure 1(b) that the homoclinic orbit given by $H(\phi, y) = h_1$ gives rise to a smooth solitary wave solution of equation (1.3). The family of periodic orbits defined by $H(\phi, y) = h$, $h \in (h_2, h_1)$

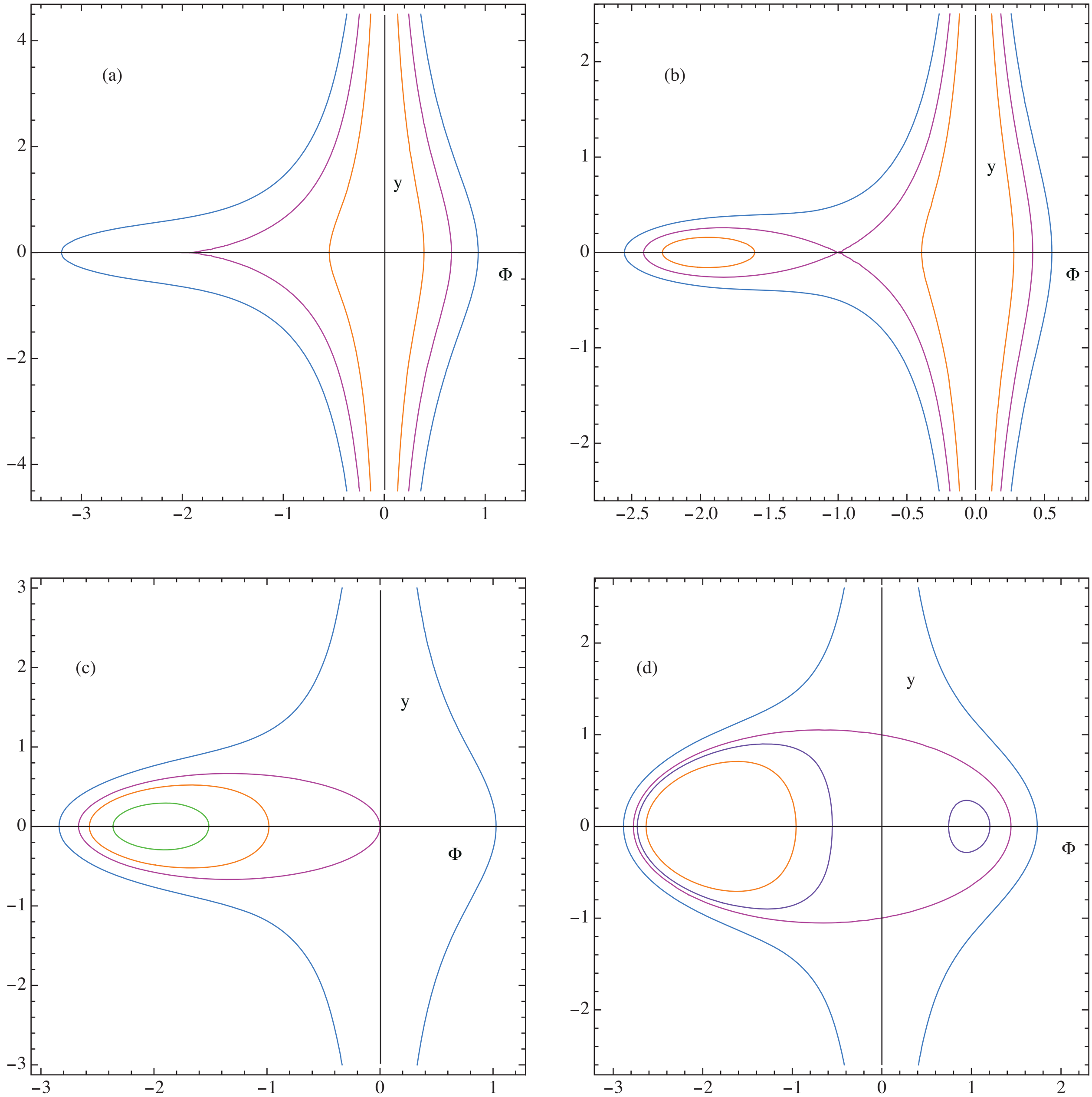

Figure 1. Phase portraits of system (1.7). (1a) $g = -c^2/4$. (1b) $-c^2/4 < g < 0$. (1c) $g = 0$. (1-d) $g > 0$.

enclosed by the homoclinic orbit gives rise to a family of smooth periodic wave solution of equation (1.3). But what if an orbit of (1.5) has intersection point(s) with the singular straight line $\phi = 0$? In the following, let us solve this problem.

Notice that the orbits of system (1.7) have the following dynamical behavior:

(1) Corresponding to the curves defined by $H(\phi, y) = h_s$ (see Figure 2(a)), there are two heteroclinic orbits connecting the two saddle points $N_{\pm}(0, Y_{\pm})$ on the left and right sides of the straight line $\phi = 0$.

(2) Corresponding to the curves defined by $H(\phi, y) = h_s$ (see Figure 2(b)), there is a homoclinic orbit

which is tangent with the y-axis at the origin.

By theory of planar dynamical systems, it is known that as the "time variable" τ is varied, a point (ϕ, y) in an orbit of (1.7) tends to the saddle point N_+ or N_-, only if $\tau \to +\infty$ or $-\infty$.

Differing from (1.7), for system (1.5), on the left side of the straight line $\phi = 0$, the direction of orbits of the vector field defined by (1.5) is just the inverse direction of the orbits of the vector field defined by (1.7). In addition, similar to Theorem 2.4 [8], we have the following lemma.

Lemma 3.1 *Let $(\phi(\xi), y = \phi'(\xi))$ be the parametric representation of an orbit γ of system (1.5) and $(0, \pm\sqrt{X})$ be two points on the singular straight line $\phi = 0$. Suppose that one of the following two conditions holds:*
(A) $X > 0$ and, along the orbit γ, as ξ increases or (and) decreases, the phase point $(\phi(\xi), y(\xi))$ tends to the points $(0, \pm\sqrt{X})$, respectively.
(B) $X = 0$ and, along the orbit γ, as ξ increases or (and) decreases, the phase point $(\phi(\xi), y(\xi))$ tends to the point $(0,0)$ and γ is in contact with the y-axis at the point $(0,0)$.
Then, there is a finite value $\xi = \widetilde{\xi}$ such that $\lim_{\xi\to\widetilde{\xi}} \phi(\xi) = 0$.

Remark 3.1 *Lemma 3.1 tells us that the solution $(\phi(\xi), y(\xi)))$ of system (1.5), which starts from an initial phase point $(\phi_0(\xi_0), y_0(\xi_0))$ on the orbit defined by $H(\phi, y) = h_s$, can reach the point N_+ or N_- or the origin in a finite time interval of ξ. In other words, the point N_+ or N_- or the origin, which lies on the singular straight line $\phi = 0$, is no longer the equilibrium point of system (1.5).*

Lemma 3.1 and the vector field defined by (1.5) imply the following conclusion.

Theorem 3.2 *(1) The ellipse $12y^2 + 3u^2 + 4cu = 6g$ in Figure 2(c), which connect two points $N_\pm(0, Y_\pm)$ on the left and right sides of the straight line $\phi = 0$, is a smooth periodic orbit of system (1.5). It gives rise to a smooth periodic wave solution of the $K(2,2)$ equation (1.3).*
(2) The ellipse $12y^2 + 3u^2 + 4cu = 0$ in Figure 2(d), which is in contact with the y-axis at the origin on the left side of the straight line $\phi = 0$, is a smooth periodic orbit of (1.5). It gives rise to a smooth periodic wave solution of the $K(2,2)$ equation (1.3).

4. Two smooth traveling wave solutions

In this section, we give two exact traveling wave solutions for the $K(2,2)$ equation (1.3) when $g \geq 0$.

(i) For $g = 0$, the curves defined by $H(u, y) = h_s$ has the algebraic equation

$$y^2 = -\frac{1}{4}u^2 - \frac{c}{3}u. \tag{4.1}$$

By using the first equation of system (1.5) to do integration, we obtain the representation of a smooth periodic wave solution of equation (1.3):

$$\phi(\xi) = -\frac{2}{3}c(1 + \cos\frac{\xi}{2}). \tag{4.2}$$

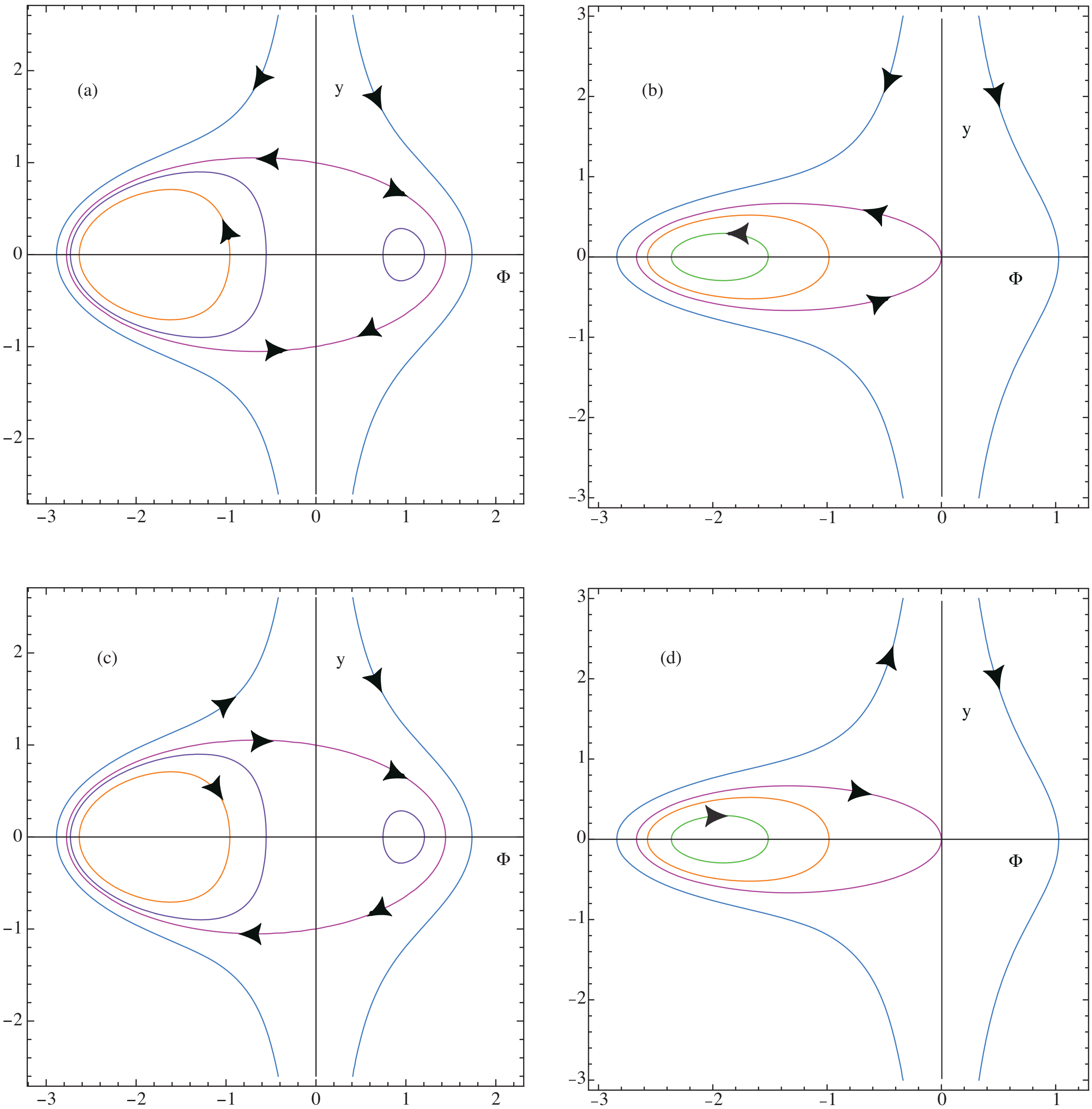

Figure 2. *The different phase portraits of systems (1.7) and (1.5). (a) The orbits of the vector field defined by (1.7) when $g>0$. (b) The orbits of the vector field defined by (1.7) when $g=0$. (c) The orbits of the vector field defined by (1.5) when $g>0$. (d) The orbits of the vector field defined by (1.5) when $g=0$.*

(ii) For $g>0$, the curves defined by $H(u,y)=h_s$ has the algebraic equation

$$y^2=\frac{(r_1-u)(u-r_2)}{4}, \tag{4.3}$$

where $r_{1,2}=\frac{-2c\pm\sqrt{2(2c+9g)}}{3}$ and $r_2<r_1$. By using the first equation of system (1.5) to do integration, we

obtain the representation of a smooth periodic wave solution of equation (1.3)

$$\phi(\xi) = \frac{r_1 + r_2}{2} - \frac{r_1 - r_2}{2} \cos \frac{\xi}{2}. \tag{4.4}$$

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