Turk J Phys
31 (2007) , 179 – 190.

Algebraic Treatment of Scattering via the Manning-Rosen Potential Related to the SO(2, 1) Group

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Received 10.01.2006

Abstract

In this study, we consider SO(2,1) ⊃ SO(1,1) subgroup reduction from one dimensional scattering systems related to SO(2,1). As an analytical application, we show via algebraic approach the Manning-Rosen potential belongs to the class of potentials corresponding to the reduction SO(2,1) ⊃ SO(1,1). The wave function and the scattering matrix of the system with this potential were determined.

Key Words: Algebraic approach, Manning Rosen potential, Natanzon potentials, One-dimensional scattering, Scattering theory.

1. Introduction

Some of the potentials used in quantum mechanics are exactly solvable. This means that the energy eigenvalues, the bound-state wave functions and the scattering matrix can be determined in closed analytical form. Most of the solvable potentials are one-dimensional problems, which mean that they are defined on the $(-\infty, \infty)$ domain (or, on some finite sub-interval); they can be reduced to radial problems in higher spatial dimensions, in which case they are confined to the positive real axis [1]. Natanzon has found a general potential structure for solvable Schrödinger equation [2], known as the Natanzon potentials, and are included many of analytical and exactly solvable potentials. Many of the most well known potentials, such Coulomb, harmonic oscillator, Morse, Pöschl-Teller, Manning-Rosen, Hulten and Ginocchio potentials, among others, belong to this structure [2-5]. The Manning-Rosen potential appears in the study of vibrations of diatomic molecules [2-7].

Among the numerical and analytical applications of solvable potentials in quantum physics is the study of scattering problems by algebraic methods [8-15]. The method used in [8] has not applied other scattering problems. Furthermore, it has been rather difficult to derive the S-matrix for the many-body scattering problems by the methods mentioned in [9-15] , since the used theory has not been developed as far as one would wish.

A new way, which allows pure algebraic calculation of S-matrices for the systems whose Hamiltonians are related to the Casimir operator C of some noncompact group G has been proposed by Kerimov [16]. This method has shown that scattering matrices are exactly calculated using directly invariance algebra.

Kerimov and Sezgin in [17] discussed the scattering problems connected with relation $H = f(C)|_{\mathcal{H}}$ to $SO(2,1)$ group, where H is linear on C and $\mathcal{H}$ are the eigensubspaces of the compact generators. It has been show that the scattering problems can be completely solved within the framework of group theory without explicit knowledge of the interaction potentials. Moreover, it has been shown that there are three classes of one-dimensional scattering problems related to SO(2,1) according to $\mathrm{SO}(2,1) \supset \mathrm{SO}(2)$, SO(2,1) $\supset$ SO(1,1) and $\mathrm{SO}(2,1) \supset \mathrm{E}(1)$ subgroup reductions.

There also exist other solvable models while the Hamiltonian of the systems are related to the Casimir operator C as

$$Q(H-E) = (C-p)|_{\mathcal{H}} \tag{1}$$

where q is an eigenvalue of C, Q is some nontrivial operator [18]. It is observed that equation (1) reduces to $H = f(C)|_{\mathcal{H}}$ if $Q = const$ is chosen. It has been determined in [19] that structures of the interaction potentials for one-dimensional scattering systems whose Hamiltonians are related to the Casimir operator C of SO(2,1) as

$$[C - j(j+1)]|_{\mathcal{H}_\mu} = Q(x)(H-E), \quad j = -\frac{1}{2} - i\rho, \quad 0 \leq \rho < \infty, \quad \mu = m, \nu, \lambda, \tag{2}$$

where $\mathcal{H}_\mu$ is a one-dimensional subspace in the reductions $\mathrm{SO}(2,1) \supset \mathrm{SO}(2)$, SO(2,1) $\supset$ SO(1,1) or $\mathrm{SO}(2,1) \supset \mathrm{E}(1)$ of representation of group SO(2,1). The parameters ρ, m, ν and λ specify the irreducible representations of $SO(2,1)$, $SO(2)$, SO(1,1) and $E(1)$, respectively. Furthermore, ρ^2, m^2, ν^2, λ^2 parameters are linear functions of the energy E. Hence, the potentials, which correspond to reductions $\mathrm{SO}(2,1) \supset \mathrm{SO}(2)$, SO(2,1) $\supset$ SO(1,1) and $\mathrm{SO}(2,1) \supset \mathrm{E}(1)$ are classified. Each class includes a certain set of Natanzon potentials. These classes are hypergeometric Natanzon, generalized Ginocchio and confluent hypergeometric Natanzon potentials classes, respectively. In [20], by one of the present authors, using general expressions given in [19], a member of a class of potentials corresponding to the reduction of SO(2,1) $\supset$ SO(1,1) was obtained as the Manning Rosen potentials. The wave function and scattering matrix of the system with this potential have been determined using general expressions given in [19].

In this study, we found the potential corresponding to choice of parameters $a_1 = a_2 = 1$ in the class of potential corresponding to the reduction of SO(2,1) $\supset$ SO(1,1) as an analytical application of [19]. This potential, which is obtained algebraically, is the Manning Rosen potential. The wave function and the scattering matrix of the system with this potential were determined without using general expressions of [19] as different from [20].

2. Class of Potentials Related to SO(2,1) $\supset$ SO(1,1) Reduction

The Hamiltonian operators solving (2) are

$$Cf = j(j+1)f, \quad j = -\frac{1}{2} - i\rho, \quad 0 \leq \rho < \infty. \tag{3}$$

The Hamiltonians H thus found are a reducible quasiregular representation contained the principal series from the unitary irreducible representations (UIRs) of $SO(2,1)$.

We consider a quasiregular representation $T(g)$ of SO(2,1) realized in the Hilbert space of square-integrable functions $f(\xi)$ on an upper sheet of hyperboloid $\Xi = \mathrm{SO}(2,1)/\mathrm{SO}(2)$, $\xi_0^2 - \xi_1^2 - \xi_2^2 = 1$, $\xi_0 > 0$. Generally, one can use the carrier space with any quasi-invariant measure $d\mu(\xi)$ on Ξ for the construction of the quasiregular representation. The representations with different measure are unitarily equivalent. The representation is given by

$$T(g)f(\xi) = [d\mu(\xi g)/d\mu(\xi)]^{1/2} f(\xi g), \tag{4}$$

with the inner product

$$(f, f') = \int \overline{f(\xi)} f'(\xi) d\mu(\xi). \tag{5}$$

The hyperboloid Ξ has invariant measure if $d\mu(\xi) = d\xi$ is supplied for each element of SO(2,1). In this case, $d\xi \equiv d\xi_1 d\xi_2/\xi_0$ is an invariant measure on Ξ and $d\mu(\xi g)/d\mu(\xi)$=1. The representation is denoted as $\breve{T}$.

The quasi-invariant measure $d\mu$ in the reduction of SO(2,1) $\supset$ SO(1,1) is chosen in form to be invariant under the transformations SO(1,1). The measure $d\mu$ is $d\mu = \upsilon(\xi_2)d\xi$, where $\upsilon(\xi_2) = \left(1+\xi_2^2\right)^{-1/2}$ and $\upsilon(\xi_2) \geq 0$. Such defined quasiregular representation, denoted T, is unitarily equivalent to $\breve{T}$. The unitary mapping W, which realizes the equivalence, is given by

$$W : f \to \breve{f} = \upsilon^{1/2} f. \tag{6}$$

The generators of quasiregular representation J_k, $k = 0, 1, 2$ are denoted by

$$J_0 = i\xi_2 \frac{\partial}{\partial \xi_1} - i\xi_1 \left(\frac{1}{2\upsilon} \frac{\partial \upsilon}{\partial \xi_2} + \frac{\partial}{\partial \xi_2} \right),\ J_1 = i\xi_0 \frac{\partial}{\partial \xi_1},\ J_2 = i\xi_0 \left(\frac{\partial}{\partial \xi_2} + \frac{1}{2\upsilon} \frac{\partial \upsilon}{\partial \xi_2} \right), \tag{7}$$

where J_o is the generator corresponding to the rotations in the 1-2 plane while J_1 and J_2 are the generators corresponding to the pure Lorentz transformations along axis 1 and 2, respectively. Note our introduction of variables ξ_0, ξ_1, ξ_2, which are expressed in terms of x, β viz.

$$\xi_0 = \frac{\cosh \beta}{\sqrt{1 - z^2(x)}}, \quad \xi_1 = \frac{\sinh \beta}{\sqrt{1 - z^2(x)}}, \quad \xi_2 = \frac{z(x)}{\sqrt{1 - z^2(x)}} \tag{8}$$

with $-\infty < \beta < \infty$, $-\infty < x < \infty$, where $z(x)$ is a differentiable function on R with values in the range $[-1,\ 1]$. Generators J_k are denoted by

$$\begin{aligned} J_o &= i \left[-\frac{(1 - z^2)}{\dot{z}} \sinh \beta \frac{\partial}{\partial x} + z \cosh \beta \frac{\partial}{\partial \beta} - \left(\frac{\ddot{z}(1 - z^2)}{2\dot{z}^2} + \frac{z}{2} \right) \sinh \beta \right], \\ J_1 &= i \frac{\partial}{\partial \beta}\ , \\ J_2 &= i \left[\tfrac{(1-z^2)}{\dot{z}} \cosh \beta \tfrac{\partial}{\partial x} - z \sinh \beta \tfrac{\partial}{\partial \beta} + \left(\tfrac{\ddot{z}(1-z^2)}{2\dot{z}^2} + \tfrac{z}{2} \right) \cosh \beta \right], \end{aligned} \tag{9}$$

and J_k are the Hermitian operators and satisfy the commutation relations

$$[J_+, J_-] = -2J_0, \quad [J_0, J_+] = J_+, \quad [J_0, J_-] = -J_-, \tag{10}$$

where $J_\pm = J_1 \pm iJ_2$ are ladder operators and dots represent derivates with respect to x, i.e. $\dot{z} = dz/dx$, $\ddot{z} = d^2z/dx^2$, etc. The Casimir operator $C = J_0(J_0 - 1) - J_+J_-$ has the form

$$C = \frac{\left(1 - z^2\right)^2}{\dot{z}^2} \left[\frac{\partial^2}{\partial x^2} + \frac{1}{2} \frac{\dddot{z}}{\dot{z}} - \frac{3}{4} \left(\frac{\ddot{z}}{\dot{z}} \right)^2 + \frac{\dot{z}^2 (2 + z^2)}{4\left(1 - z^2\right)^2} + \frac{\dot{z}^2}{1 - z^2} \frac{\partial^2}{\partial \beta^2} \right], \tag{11}$$

where the conditions $\dot{z} > 0$ and $1 - z^2 > 0$ are satisfied.

The basis functions $f_{\nu\tau}$ corresponding to the reduction of SO(2,1) $\supset$ SO(1,1) of the principal series of SO(2,1) group are the eigenfunctions of the set of operators C and J_1

$$C f_{\nu\tau} = \left(-\frac{1}{4} - \rho^2 \right) f_{\nu\tau}, \quad J_1 f_{\nu\tau} = \nu\ f_{\nu\tau}, \tag{12}$$

where $J_1 = i\frac{\partial}{\partial \beta}$ and C is given by (11).

Denote by C_ν a restriction of C on the one-dimensional subspace H_ν spanned by $f_{\nu\tau}$ with fixed ν and τ, where ν determine the irreducible representations of SO(1,1) group and $\tau = \pm 1$. Then,

$$C_\nu = \frac{(1-z^2)^2}{\dot{z}^2}\left[\frac{\partial^2}{\partial x^2} + \frac{1}{2}\frac{\dddot{z}}{\dot{z}} - \frac{3}{4}\left(\frac{\ddot{z}}{\dot{z}}\right)^2 + \frac{\dot{z}^2(2+z^2)}{z^2(1-z^2)^2} - \frac{\dot{z}^2\nu^2}{1-z^2}\right]. \tag{13}$$

For this class of Hamiltonians, equation (2) is written as

$$C_\nu - j(j+1) = Q(x)\,[H-E]\,, \quad j = -\frac{1}{2} - i\rho, \quad 0 \le \rho < \infty. \tag{14}$$

Equation (14) is satisfied by the choice of ν^2 and ρ^2 as

$$1+\nu^2 = a_1 E + b_1, \quad 1+\rho^2 = a_2 E + b_2, \tag{15}$$

where ν^2 and ρ^2 are a linear function of E. Then, by using the simplified units $2m = \hbar = 1$, the Hamiltonians in (14) have the potentials structure

$$V(x) = \frac{b_1 z^2(1-z^2) - \frac{3(1-z^2)}{4} - b_2 z^2 + 1}{R} + \frac{z^4(1-z^2)^2}{R^2}\left(a_1 + \frac{a_1 + a_2(2z^2-1)}{z^2(z^2-1)} - \frac{5\Delta}{4R}\right), \tag{16}$$

where $R(z) = a_1 z^4 + (a_2 - a_1)\, z^2$ and $\Delta = (a_1 - a_2)^2$. Potentials (16) are related to the parameters of dimension a_1, a_2, b_1, b_2. The Hamiltonians H are related to the Casimir operator of $\mathrm{SO}(2,1)$ in the sense that $\left(C_\nu + \rho^2 + 1/4\right) = Q(x)\,[H-E]$, provided

$$\dot{z}\,(x) = z\left(1-z^2\right)\Big/\sqrt{R\,(z)}, \tag{17}$$

where $R\,(z) = a_1 z^4 + (a_2 - a_1)\, z^2$ and $Q(x) = -(1-z^2)^2/\dot{z}^2$.

Equation (17) gives relationship between z and x. It is possible to find different $z(x)$ solutions according to various choices of parameters a_1, a_2 in (17), thus giving different structures of potentials. For instance, when putting $a_1 = (1-\gamma^2)/\gamma^4$, $a_2 = 1/\gamma^4$, $b_1 = \delta\,(\delta+1) + 3/4$, $b_2 = 1$ with $\gamma \le 1$, $\delta\,(\delta+1) \ge 1/4$, and introducing Ginocchio's variable $y = z/\sqrt{z^2 + \gamma^2\,(1-z^2)}$, the Ginocchio potentials [5, 19], which are a sub class of Natanzon potentials, are obtained as

$$V(x) = \left\{\, -\gamma^2\delta\,(\delta+1) + \left[\left(1-\gamma^2\right)/4\right]\ \left[5\left(1-\gamma^2\right)y^4 - \left(7-\gamma^2\right)y^2 + 2\right]\,\right\}\ \left(1-\gamma^2\right).$$

The Wave Functions and Scattering Matrix for Class of Potentials Related to the Reduction SO$(2,1) \supset$ SO$(1,1)$

Wave functions of the potentials in (16) are related to the basis functions $f_{\nu\tau}(\xi)$, where ν determines the irreducible representations of $\mathrm{SO}(1,1)$ group and $\tau = \pm 1$. The basis functions $f_{\nu\tau}(\xi)$ are found with help of the integral representation for the basis functions of the series representations [19]. The integral representation for the basis functions of the principal series representations induced by (4) is

$$\breve{f}_{\nu\tau}\,(\xi) = \int_{-\infty}^{\infty} (\xi_0 \cosh\alpha - \xi_1 \sinh\alpha - \xi_2\tau)^{-1-j}\, e^{-i\nu\alpha} d\alpha. \tag{18}$$

Using equations (6), (8) and (18), one can then obtain $f_{\nu\tau}\,(\xi)$ as

$$f_{\nu\tau}\,(\xi) = \frac{\left(1-z^2\right)^{1/4}}{\sqrt{\dot{z}}} e^{-i\nu\beta} \int \left(\frac{\cosh\alpha}{\sqrt{1-z^2}} - \frac{z\tau}{\sqrt{1-z^2}}\right)^{-1-j} e^{-i\nu\alpha} d\alpha. \tag{19}$$

The wave functions for the class of potential given in (16) have the form

$$\Psi_\tau(x) \propto (\dot{z})^{-\frac{1}{2}}\left(1-z^2\right)^{\frac{2j+3}{4}}$$

$$\times\left\{\Gamma\left(\frac{j+i\nu+1}{2}\right)\Gamma\left(\frac{j-i\nu+1}{2}\right){}_2F_1\left(\frac{j+i\nu+1}{2},\frac{j-i\nu+1}{2};\frac{1}{2};z^2\right)\right. \tag{20}$$

$$\left.+\;2\tau\; z\Gamma\left(\frac{j+i\nu+2}{2}\right)\Gamma\left(\frac{j-i\nu+2}{2}\right)\cdot{}_2F_1\left(\frac{j+i\nu+2}{2},\frac{j-i\nu+2}{2};\frac{3}{2};z^2\right)\right\}$$

[19]. Reference [21] gives the relationship between ${}_2F_1$ hypergeometric functions and Γ functions. It should be noted that the potentials functions of this class have a double degeneracy of the wave function for every positive value of energy. Therefore, one may construct wave packets that are partly transmitted and partly reflected by the potential. Here, the function Ψ_{-1} represents the wave incident from the left. Reflection occurs at the potential barrier; meanwhile there is also transmission to the right. However, the function Ψ_{+1} represents a wave incident from the right. Reflection occurs at the potential barrier, but there is also transmission to the left at the same time. In this study, we considered $\tau = -1$.

The S matrix for one-dimensional scattering problems related to the SO(2,1) $\supset$ SO(1,1) is a unitary 2×2 matrix and it is given by

$$S_\nu = \begin{pmatrix} R_\nu & T_\nu \\ T_\nu & R_\nu \end{pmatrix} \tag{21}$$

$$R_\nu = c\left(\rho\right)\frac{1}{\pi}\cosh\left(\pi\nu\right)\Gamma\left(\frac{1}{2}-i\rho+i\nu\right)\Gamma\left(\frac{1}{2}-i\rho-i\nu\right),$$

$$T_\nu = -c\left(\rho\right)\frac{1}{\pi}\sinh\left(\pi\rho\right)\Gamma\left(\frac{1}{2}-i\rho+i\nu\right)\Gamma\left(\frac{1}{2}-i\rho-i\nu\right),$$

where $c\left(\rho\right)$ is an arbitrary phase factor; and ρ and ν specify the irreducible representations of SO(2,1) and SO(1,1), respectively [15]. One should note that all potentials in (17) are repulsive and do not support bound states. This is because only the principle series representations appear in the decomposition of the quasiregular representation realized in the space of functions on the upper sheet of hyperboloid Ξ.

The Manning-Rosen Potential

In this part, we find the potential corresponding to election of parameters $a_1 = a_2 = 1$ in the class of potential corresponding to the reduction of SO(2,1) $\supset$ SO(1,1) as an analytical application of ref. [19] by algebraic approach. The wave function of the system with this potential will be determined with help the integral representation for the basis functions of the series representations. The scattering matrix of system will be determined investigating asymptotic behavior of the wave function.

By choice of values for a_1, a_2 in (17), it is possible to find different solutions of $z(x)$and thus different potentials structures.

It is possible to find different $z(x)$ solutions according to various choices of parameters a_1, a_2 in (17). Then, different potentials structures corresponding to the various parameter choices can be obtained. With this in mind, we choose $a_1 = a_2 = 1$, with which (17) gives

$$\dot{z}\left(x\right) = z\left(1-z^2\right)\Big/\sqrt{R(z)}, \quad R(z) = z^4. \tag{22}$$

Equation (22) can be written as $\pm dx = \frac{zdz}{1-z^2}$. Thus, we found $\left|1-z^2\right| = e^{\mp 2x}$. From this, we obtain eight different solutions expressed as a relationship between z and x. However, the valid solutions for us are solutions with the conditions $\dot{z} > 0$ and $1 - z^2 > 0$. Thus, z becomes

$$z(x) = \begin{cases} -\sqrt{1-e^{+2x}} & -\infty < x < 0\ ,\ -1 \leq z < 0 \\ \\ +\sqrt{1-e^{-2x}} & 0 < x < \infty\ ,\ 0 < z \leq 1 \end{cases} . \tag{23}$$

Using equations (9) and (10), we find the following:

a) for $z = -\sqrt{1-e^{+2x}}$, $-\infty < x < 0$

$$J_\pm = \mp\sqrt{1-e^{2x}}\cosh\beta\frac{\partial}{\partial x} + \left(i \mp \sqrt{1-e^{2x}}\sinh\beta\right)\frac{\partial}{\partial\beta} \mp \frac{\cosh\beta}{2\sqrt{1-e^{2x}}}. \tag{24}$$

b) for $z = \sqrt{1-e^{-2x}}$, $0 < x < \infty$

$$J_\pm = \mp\sqrt{1-e^{-2x}}\cosh\beta\frac{\partial}{\partial x} + \left(i \pm \sqrt{1-e^{-2x}}\sinh\beta\right)\frac{\partial}{\partial\beta} \pm \frac{\cosh\beta}{2\sqrt{1-e^{-2x}}}. \tag{25}$$

We compute the Casimir operator $C = J_0(J_0-1) - J_+J_-$ as

a) for $z = -\sqrt{1-e^{+2x}}$, $-\infty < x < 0$

$$C = \left(1-e^{2x}\right)\left[\frac{\partial^2}{\partial x^2} + \frac{e^{2x}}{1-e^{2x}}\frac{\partial^2}{\partial\beta^2} + \frac{e^{2x}}{\left(1-e^{2x}\right)^2} - \frac{1}{4\left(1-e^{2x}\right)^2}\right], \tag{26}$$

b) for $z = \sqrt{1-e^{-2x}}$, $0 < x < \infty$

$$C = \left(1-e^{-2x}\right)\left[\frac{\partial^2}{\partial x^2} + \frac{e^{-2x}}{1-e^{-2x}}\frac{\partial^2}{\partial\beta^2} + \frac{e^{-2x}}{\left(1-e^{-2x}\right)^2} - \frac{1}{4\left(1-e^{-2x}\right)^2}\right]. \tag{27}$$

Considering equations (12) and (14), using units with $2m = \hbar = 1$, the Hamiltonian is found as

$$H = -\frac{d^2}{dx^2} + V(x),$$

with

$$V(x) = -\frac{3/4}{4\sinh^2 x} + 4\coth x5, \quad -\infty < x < 0, \tag{28}$$

$$V(x) = -\frac{3/4}{4\sinh^2 x} - 4\coth x - 5, \quad 0 < x < \infty, \tag{29}$$

and choosing parameters ν^2 and ρ^2 as

$$1 + \nu^2 = E + 2, \quad 1 + \rho^2 = E + 10, \tag{30}$$

where ν^2 and ρ^2 are linear functions of E due to (1). For simplicity, we choose $b_1 = 2$, $b_2 = 10$ as arbitrary in (15). Hence, we algebraically obtain the Hamiltonian of the Manning-Rosen potential. Figure shows the Manning Rosen potential per equations (28)–(30). The Hamiltonian H is related to SO(2, 1) in the sense that

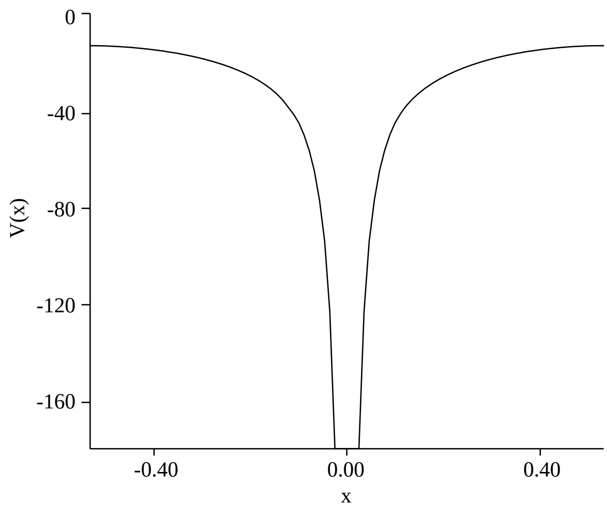

Figure. The Manning Rosen potential.

$$\left(C+\rho^2+\frac{1}{4}\right)=Q(x)\left[H-E\right], \quad -\infty<x<0 \tag{31}$$

$$\left(C+\rho^2+\frac{1}{4}\right)=Q(x)\left[H-E\right], \quad 0<x<\infty. \tag{32}$$

provided the condition specified in (22), where $Q(x)=2\sinh x\left(\sinh x+\cosh x\right)\ -\infty<x<0$, $Q(x)=2\sinh x\left(\sinh x-\cosh x\right)\quad 0<x<\infty$.

Wave Function and Scattering Matrix of a System with Manning-Rosen Potential

The basis functions $f_{\nu\tau}\left(\xi\right)$ for the Manning-Rosen potential are found with help of the integral representation for the basis functions of the series representation [19]. We determine the basis function for the two cases $z=-\sqrt{1-e^{+2x}}$, $-\infty<x<0$; and $z=\sqrt{1-e^{-2x}}$, $0<x<\infty$.

Case A: $z=-\sqrt{1-e^{+2x}}$, $-\infty<x<0$. When we write equation (18) using (8), we get

$$\breve{f}_{\nu(-1)}\left(\xi\right)=\int_{-\infty}^{\infty}\left[e^{-x}\left(\cosh\alpha-\sqrt{1-e^{2x}}\right)\right]^{-1-j}e^{-i\nu(\alpha+\beta)}d\alpha. \tag{33}$$

When we consider the unitary mapping $W:f\to\breve{f}=e^{x/2}(1-e^{2x})^{-1/4}f$, it becomes

$$f_{\nu(-1)}\left(\xi\right)=e^{-x/2}\left(1-e^{2x}\right)^{1/4}e^{-i\nu\beta}\int_{-\infty}^{\infty}\left[e^{-x}\left(\cosh\alpha-\sqrt{1-e^{2x}}\right)\right]^{-1-j}e^{-i\nu\alpha}d\alpha. \tag{34}$$

The basis functions $f_{\nu(-1)}\left(\xi\right)$ are written as

$$f_{\nu(-1)}\left(\xi\right)=e^{-i\nu\beta}\psi_{(-1)}(x), \tag{35}$$

$$\psi_{(-1)}(x)=e^{-x/2}\left(1-e^{2x}\right)^{1/4}\int_{-\infty}^{\infty}\left[e^{-x}\left(\cosh\alpha-\sqrt{1-e^{2x}}\right)\right]^{-1-j}e^{-i\nu\alpha}d\alpha. \tag{36}$$

Benefiting from [22], and the relation

$$\begin{aligned}&\int_{-\infty}^{\infty}\left[e^{-x}\left(\cosh\alpha+\sqrt{1-e^{2x}}\right)\right]^{-1-j}e^{-i\nu\alpha}d\alpha=\\&2\int_{0}^{\infty}\left[e^{-x}\left(\cosh\alpha-\sqrt{1-e^{2x}}\right)\right]^{-1-j}\cosh i\nu\alpha d\alpha,\end{aligned}\tag{37}$$

we find

$$\begin{aligned}f_{\nu(-1)}\left(\xi\right)&=\left(e^{-x}\right)^{-\frac{1}{2}-j}\left(1-e^{2x}\right)^{1/4}2e^{-i\nu\beta}\\&\times\sum_{\kappa=0}^{\infty}\frac{\Gamma\left(1+j+\kappa\right)}{\Gamma\left(1+j\right)\kappa!}\left(\sqrt{1-e^{2x}}\right)^{\kappa}\int_{0}^{\infty}\left(\cosh\alpha\right)^{-1-j-\kappa}\cosh i\nu\alpha d\alpha.\end{aligned}\tag{38}$$

When we use formulas 3.512 and 8.384-1 of [23], we write

$$\int_{0}^{\infty}\frac{\cosh i\nu\alpha d\alpha}{\left(\cosh\alpha\right)^{1+j+\kappa}}=2^{j+\kappa-1}\Gamma\left(\frac{1+j+\kappa+i\nu}{2}\right)\Gamma\left(\frac{1+j+\kappa-i\nu}{2}\right)/\Gamma\left(1+j+\kappa\right).\tag{39}$$

Putting equation (39) into (38), we get

$$f_{\nu(-1)}\left(\xi\right)=\left(e^{-x}\right)^{-\frac{1}{2}-j}\left(1-e^{2x}\right)^{1/4}\frac{e^{-i\nu\beta}2^{j}}{\Gamma\left(1+j\right)}A,\tag{40}$$

where

$$A=\sum_{\kappa=0}^{\infty}\frac{\left(2\sqrt{1-e^{2x}}\right)^{\kappa}}{\kappa!}\Gamma\left(\frac{1+j+\kappa+i\nu}{2}\right)\Gamma\left(\frac{1+j+\kappa-i\nu}{2}\right).$$

A can be written as

$$\begin{aligned}A=&\sum_{m=0}^{\infty}\frac{\left(2\sqrt{1-e^{2x}}\right)^{2m}}{(2m)!}\Gamma\left(\frac{1+j+i\nu}{2}+m\right)\Gamma\left(\frac{1+j-i\nu}{2}+m\right)\\&+\sum_{m=0}^{\infty}\frac{\left(2\sqrt{1-e^{2x}}\right)^{2m+1}}{(2m+1)!}\Gamma\left(1+\frac{j+i\nu}{2}+m\right)\Gamma\left(1+\frac{j-i\nu}{2}+m\right).\end{aligned}\tag{41}$$

When we use [21] and formulae 8.335 of [23], equation (41) becomes

$$\begin{aligned}A=&\Gamma\left(\frac{1+j+i\nu}{2}\right)\Gamma\left(\frac{1+j-i\nu}{2}\right){}_{2}F_{1}\left[\frac{1+j+i\nu}{2},\frac{1+j-i\nu}{2};\frac{1}{2};1-e^{2x}\right]\\&+2\sqrt{1-e^{2x}}\Gamma\left(1+\frac{j+i\nu}{2}\right)\Gamma\left(1+\frac{j-i\nu}{2}\right){}_{2}F_{1}\left[1+\frac{j+i\nu}{2},1+\frac{j-i\nu}{2};\frac{3}{2};1-e^{2x}\right].\end{aligned}\tag{42}$$

Putting equation (42) into (40), we get

$$\begin{aligned}f_{\nu(-1)}\left(\xi\right)=&\left(e^{x}\right)^{\frac{1}{2}+j}\left(1-e^{2x}\right)^{1/4}\frac{2^{j}}{\Gamma\left(1+j\right)}e^{-i\nu\beta}\\&\times\left\{\Gamma\left(\frac{1+j+i\nu}{2}\right)\Gamma\left(\frac{1+j-i\nu}{2}\right){}_{2}F_{1}\left[\frac{1+j+i\nu}{2},\frac{1+j-i\nu}{2};\frac{1}{2};1-e^{2x}\right]\right.\\&\left.+2\sqrt{1-e^{2x}}\Gamma\left(1+\tfrac{j+i\nu}{2}\right)\Gamma\left(1+\tfrac{j-i\nu}{2}\right){}_{2}F_{1}\left[1+\tfrac{j+i\nu}{2},1+\tfrac{j-i\nu}{2};\tfrac{3}{2};1-e^{2x}\right]\right\}.\end{aligned}\tag{43}$$

Hence, for the wave function of the Manning-Rosen potential given in (28), we have

$$\begin{aligned}\Psi_{-1}(x) &= e^{-i\rho x}\left(1-e^{2x}\right)^{1/4}\frac{2^{-\frac{1}{2}-i\rho}}{\Gamma\left(\frac{1}{2}-i\rho\right)}\left\{\Gamma\left[\frac{1+2i(\nu-\rho)}{4}\right]\Gamma\left[\frac{1-2i(\nu+\rho)}{4}\right]\right.\\ &\times {}_2F_1\left[\frac{1+2i(\nu-\rho)}{4},\frac{1-2i(\nu+\rho)}{4};\frac{1}{2};1-e^{2x}\right]+\ 2\sqrt{1-e^{2x}}\Gamma\left[\frac{3+2i(\nu-\rho)}{4}\right]\\ &\times\Gamma\left[\tfrac{3-2i(\nu+\rho)}{4}\right]{}_2F_1\left[\tfrac{3+2i(\nu-\rho)}{4},\tfrac{3-2i(\nu+\rho)}{4};\tfrac{3}{2};1-e^{2x}\right]\left.\right\},\end{aligned}\tag{44}$$

where $2m=\hbar=1$, $k^2=2mE/\hbar^2$ and $0\le\rho<\infty$. ν and ρ are determined as $\nu=\pm\sqrt{k^2+1}$, $\rho=\sqrt{k^2+9}$, $0\le\rho<\infty$ from (30).

Case B: $z=\sqrt{1-e^{-2x}}$, $0<x<\infty$. When we write equation (18) using (8), we get

$$\breve{f}_{\nu(-1)}(\xi)=\int_{-\infty}^{\infty}\left[e^x\left(\cosh\alpha+\sqrt{1-e^{-2x}}\right)\right]^{-1-j}e^{-i\nu(\alpha+\beta)}d\alpha.\tag{45}$$

When we consider the unitary mapping $W: f\to\breve{f}=e^{-x/2}(1-e^{-2x})^{-1/4}f$ in (45), it becomes

$$f_{\nu(-1)}(\xi)=e^{x/2}\left(1-e^{-2x}\right)^{1/4}e^{-i\nu\beta}\int_{-\infty}^{\infty}\left[e^x\left(\cosh\alpha+\sqrt{1-e^{-2x}}\right)\right]^{-1-j}e^{-i\nu\alpha}d\alpha.\tag{46}$$

The basis functions $f_{\nu(-1)}(\xi)$ are given by (35), but where

$$\psi_{(-1)}(x)=e^{x/2}\left(1-e^{-2x}\right)^{1/4}\int_{-\infty}^{\infty}\left[e^x\left(\cosh\alpha+\sqrt{1-e^{-2x}}\right)\right]^{-1-j}e^{-i\nu\alpha}d\alpha.\tag{47}$$

Benefiting from [21] and (37), we found

$$\begin{aligned}f_{\nu(-1)}(\xi) &= \left(e^{-x}\right)^{\frac{1}{2}+j}\left(1-e^{-2x}\right)^{1/4}2e^{-i\nu\beta}\\ &\times\sum_{\kappa=0}^{\infty}\frac{\Gamma(1+j+\kappa)}{\Gamma(1+j)\,\kappa!}\left(-\sqrt{1-e^{-2x}}\right)^{\kappa}\int_0^{\infty}(\cosh\alpha)^{-1-j-\kappa}\cosh i\nu\alpha d\alpha.\end{aligned}\tag{48}$$

Putting equation (39) into (48), we get

$$f_{\nu(-1)}(\xi)=\left(e^{-x}\right)^{\frac{1}{2}+j}\left(1-e^{-2x}\right)^{1/4}e^{-i\nu\beta}\frac{2^j}{\Gamma(1+j)}B,\tag{49}$$

where

$$B=\sum_{\kappa=0}^{\infty}\frac{\left(-2\sqrt{1-e^{-2x}}\right)^{\kappa}}{\kappa!}\Gamma\left(\frac{1+j+\kappa+i\nu}{2}\right)\Gamma\left(\frac{1+j+\kappa-i\nu}{2}\right).$$

We can write B as

$$\begin{aligned}B &= \sum_{m=0}^{\infty}\frac{\left(-2\sqrt{1-e^{-2x}}\right)^{2m}}{(2m)!}\Gamma\left(\frac{1+j+i\nu}{2}+m\right)\Gamma\left(\frac{1+j-i\nu}{2}+m\right)\\ &+\sum_{m=0}^{\infty}\frac{\left(-2\sqrt{1-e^{-2x}}\right)^{2m+1}}{(2m+1)!}\Gamma\left(1+\frac{j+i\nu}{2}+m\right)\Gamma\left(1+\frac{j-i\nu}{2}+m\right).\end{aligned}\tag{50}$$

When we use [21] and formulae 8.335 of [23], equation (50) becomes

$$\begin{aligned}B &= \Gamma\left(\frac{1+j+i\nu}{2}\right)\Gamma\left(\frac{1+j-i\nu}{2}\right){}_2F_1\left[\frac{1+j+i\nu}{2},\frac{1+j-i\nu}{2};\frac{1}{2};1-e^{-2x}\right]\\ &-2\sqrt{1-e^{-2x}}\Gamma\left(1+\frac{j+i\nu}{2}\right)\Gamma\left(1+\frac{j-i\nu}{2}\right){}_2F_1\left[1+\frac{j+i\nu}{2},1+\frac{j-i\nu}{2};\frac{3}{2};1-e^{-2x}\right].\end{aligned}\tag{51}$$

Putting equation (51) into (49), we get

$$
\begin{aligned}
f_{\nu(-1)}(\xi) &= \left(e^{-x}\right)^{\frac{1}{2}+j}\left(1-e^{-2x}\right)^{1/4}\frac{e^{-i\nu\beta}2^{j}}{\Gamma(1+j)} \\
&\times\left\{\Gamma\left(\frac{1+j+i\nu}{2}\right)\Gamma\left(\frac{1+j-i\nu}{2}\right){}_2F_1\left[\frac{1+j+i\nu}{2},\frac{1+j-i\nu}{2};\frac{1}{2};1-e^{-2x}\right]\right. \\
&\left.-2\sqrt{1-e^{-2x}}\Gamma\left(1+\frac{j+i\nu}{2}\right)\Gamma\left(1+\frac{j-i\nu}{2}\right){}_2F_1\left[1+\frac{j+i\nu}{2},1+\frac{j-i\nu}{2};\frac{3}{2};1-e^{-2x}\right]\right\}.
\end{aligned}
\tag{52}
$$

Hence, for the wave function of the Manning-Rosen potential given in (29) we have

$$
\begin{aligned}
\Psi_{-1}(x) &= e^{i\rho x}\left(1-e^{-2x}\right)^{1/4}\frac{2^{-\frac{1}{2}-i\rho}}{\Gamma\left(\frac{1}{2}-i\rho\right)}\left\{\Gamma\left[\frac{1+2i(\nu-\rho)}{4}\right]\Gamma\left[\frac{1-2i(\nu+\rho)}{4}\right]\right. \\
&\times{}_2F_1\left[\frac{1+2i(\nu-\rho)}{4},\frac{1-2i(\nu+\rho)}{4};\frac{1}{2};1-e^{-2x}\right]-\ 2\sqrt{1-e^{-2x}}\Gamma\left[\frac{3+2i(\nu-\rho)}{4}\right] \\
&\left.\times\Gamma\left[\frac{3-2i(\nu+\rho)}{4}\right]{}_2F_1\left[\frac{3+2i(\nu-\rho)}{4},\frac{3-2i(\nu+\rho)}{4};\frac{3}{2};1-e^{-2x}\right]\right\},
\end{aligned}
\tag{53}
$$

where $2m=\hbar=1$, $k^2=2mE/\hbar^2$ and $0\le\rho<\infty$. ν and ρ are determined as $\nu=\pm\sqrt{k^2+1}$, $\rho=\sqrt{k^2+9}$ from (30).

The scattering matrix of the system with the Manning-Rosen potential is determined investigating asymptotic behavior of the wave functions (44) and (53)

$$
\lim_{x\to-\infty}\Psi_{-1}(x)=\frac{2^{-\frac{1}{2}-i\rho}\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-i\rho\right)}\left[2\Gamma(-i\rho)\,e^{i\rho x}+\Gamma(i\rho)\,R\;e^{-i\rho x}\right],
\tag{54}
$$

$$
\lim_{x\to+\infty}\Psi_{-1}(x)=\frac{2^{-\frac{1}{2}-i\rho}\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-i\rho\right)}\,\Gamma(i\rho)\;T\;e^{i\rho x},
\tag{55}
$$

where

$$
R=\frac{\Gamma\left[\frac{1+2i(\nu-\rho)}{4}\right]\Gamma\left[\frac{1-2i(\nu+\rho)}{4}\right]}{\Gamma\left[\frac{1-2i(\nu-\rho)}{4}\right]\Gamma\left[\frac{1+2i(\nu+\rho)}{4}\right]}+\frac{\Gamma\left[\frac{3+2i(\nu-\rho)}{4}\right]\Gamma\left[\frac{3-2i(\nu+\rho)}{4}\right]}{\Gamma\left[\frac{3-2i(\nu-\rho)}{4}\right]\Gamma\left[\frac{3+2i(\nu+\rho)}{4}\right]},
$$

$$
T=\frac{\Gamma\left[\frac{1+2i(\nu-\rho)}{4}\right]\Gamma\left[\frac{1-2i(\nu+\rho)}{4}\right]}{\Gamma\left[\frac{1-2i(\nu-\rho)}{4}\right]\Gamma\left[\frac{1+2i(\nu+\rho)}{4}\right]}-\frac{\Gamma\left[\frac{3+2i(\nu-\rho)}{4}\right]\Gamma\left[\frac{3-2i(\nu+\rho)}{4}\right]}{\Gamma\left[\frac{3-2i(\nu-\rho)}{4}\right]\Gamma\left[\frac{3+2i(\nu+\rho)}{4}\right]}
$$

(see [20] for $\tau=+1$). Now R and T can be written as

$$
R=2^{1+2i\rho}\frac{1}{\pi}\cosh(\pi\nu)\,\Gamma\left[\frac{1}{2}+i(\nu-\rho)\right]\Gamma\left[\frac{1}{2}-i(\nu+\rho)\right],
\tag{56}
$$

$$
T=-\frac{2^{1+2i\rho}}{\pi}\sinh(\pi\rho)\,\Gamma\left[\frac{1}{2}+i(\nu-\rho)\right]\Gamma\left[\frac{1}{2}-i(\nu+\rho)\right].
\tag{57}
$$

We find $c(\rho)=2^{1+2i\rho}$ if we compare (21) with (56) and (57). Using [24], the reflection and transmission coefficients are obtained by

$$
|R|^2=\frac{\cosh^2\pi\nu}{\cosh^2\pi\nu+\sinh^2\pi\rho}\ ,\ |T|^2=\frac{\sinh^2\pi\rho}{\cosh^2\pi\nu+\sinh^2\pi\rho}\ ,
\tag{58}
$$

respectively.

3. Conclusion

In this paper, we considered the SO(2,1) $\supset$ SO(1,1) subgroup reduction from one-dimensional scattering systems, which provided the relation (2) between Hamiltonians H and the Casimir operator C of SO(2,1) group. As an analytical application, we showed in an algebraic way that the Manning-Rosen potential belong to the class of potentials corresponding to the reduction SO(2,1) $\supset$ SO(1,1) with $a_1 = a_2 = 1$ in (17). The Manning-Rosen potential was given by (28) for $-\infty < x < 0$ and (29) for $0 < x < \infty$. By helping the integral representation for the basis functions of the series representations, the wave function of system with this potential was found as (44) for $-\infty < x < 0$ and (53) for $0 < x < \infty$. We saw that obtained results harmonized with the wave function and the scattering matrix given in [19].

Acknowledgement

The authors would like to thank G.A.Kerimov for useful discussion.

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