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Green's Function for Particle of Spin 0 and 1/2 in the Field of Volkov Plane Wave

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Abstract

The Green's functions for relativistic particles, with and without spin in interaction with an electromagnetic plane wave, are obtained via Parisi-Wu stochastic quantization method after having solved the corresponding Langevin equation.

Key Words: Green's function; Langevin equation; stochastic quantization.
P.A.C.S.: 03.65.Ca Formalism
P.A.C.S.: 03.65.Pm Relativistic wave equations
P.A.C.S.: 5.10.Gg Stochastic analysis methods

1. Introduction

In this paper, we propose to determine the correlation products of two fields, using Green's functions involving a Klein-Gordon (KG) particle, and Dirac particle, both moving in a Volkov wave field, using the Parisi-Wu stochastic quantization method (SQM) [1, 2].

The wave function, described by a 4-potential, $A_\mu \equiv A_\mu(\varphi)$, is a function only of the product $\varphi = kx$, where the wave vector k is such that $k^2 = 0$, and satisfies the Lorentz gauge condition

$$\partial_\mu A^\mu = 0, \tag{1}$$

which is equivalent to

$$kA = 0. \tag{2}$$

This interaction has been the subject of various studies via the algebraic approach [3] and the Feynman path integral method [4, 5].

Briefly, the Parisi-Wu stochastic quantum method [1] is formulated as follows. First, one adds a fictitious time u to the real time. The field $\phi(x)$ becomes $\phi(x,u)$ for which the evolution is given by the Langevin equation

$$\frac{\partial\phi(x,u)}{\partial u} = i\frac{\delta S}{\delta\phi(x,u)} + \eta(x,u), \tag{3}$$

where

$$S(\phi) = \int dx^4 \mathcal{L}(\phi, \partial_\mu \phi) = \int d^4x \mathcal{L}\left(\phi, \partial_0\phi, \vec{\nabla}\phi\right) \tag{4}$$

is the classical action, which describes the motion of the system; and η is a white noise, where

$$\begin{aligned} \langle \eta(x,u) \rangle &= 0, \\ \langle \eta(x,u)\eta(x',u') \rangle &= 2\delta^4(x-x^{'})\delta(u-u'). \end{aligned} \tag{5}$$

The stochastic average is expressed as a functional integral:

$$\langle \phi(x_1,u)\phi(x_2,u)... \rangle = \int D\phi\phi(x_1)\phi(x_2)...P[\phi,u], \tag{6}$$

where P is the probability distribution which satisfies the Fokker–Planck equation

$$\frac{\partial}{\partial u}P[\phi,u] = \int D\phi(x)\frac{\delta}{\delta\phi(x)}\left(\frac{\delta}{\delta\phi(x)} - i\frac{\delta S}{\delta\phi(x)}\right)P[\phi,u], \tag{7}$$

and is normalized as

$$\langle 1 \rangle = \int \mathcal{D}\phi P[\phi,u] = 1. \tag{8}$$

On this level, let us specify that each path in the path integral formalism is affected by the complex weight $\exp(iS(\phi))$, and the propagator, or the averages, for example, are in general ill-defined. To remedy it, we can carry out a Wick rotation, causing the weight to become real and have the value $\exp(-S_E)$

In our case, the space in which the particle moves is Minkowski space and its movement is governed by the Langevin equation containing a complex drift term [6]. With the Wick rotation, we can also modify the term $i\frac{\delta S}{\delta\phi(x)}$ into $-\frac{\delta S_E}{\delta\phi(x)}$, wherein the Langevin and FP equations become real; and at thermal equilibrium $(u \to \infty)$, the probability $P \to \exp(-S_E)$ is a stationary solution; and the average becomes the standard average of the formalism path integral (with weight $\exp(-S_E)$), where

$$S_E(\phi) = -\int d^4x \mathcal{L}\left(\phi, i\partial_0\phi, \vec{\nabla}\phi\right) \tag{9}$$

is the euclidean action.

However, it was shown in [6] that one can use the complex Langevin equation in Minkowski space by implicitly modifying in the action (or the Lagrangian) the mass by an imaginary term $(m^2 \to m^2 - i0)$, to ensure the convergence of the stochastic process and the propagator; thus with which average expressions of standard quantum mechanics are obtained at thermal equilibrium $(u \to \infty)$.

In addition let us note that we can, by separating the real and imaginary parts of the Langevin equation, calculate physical quantities by using the notion of real and positive probability [7] and thus, at equilibrium $(u \longrightarrow \infty)$, the correlation function becomes exactly the same as that defined by the Feynman path integral formalism:

$$\lim_{u\longrightarrow\infty} \langle \phi(x_1,u)\phi(x_2,u)... \rangle = \frac{\int D\phi\phi(x_1)\phi(x_2)\cdots e^{iS}}{\int D\phi e^{iS}} \tag{10}$$

$$= \langle 0| T(\phi(x_1)\phi(x_2)) |0\rangle. \tag{11}$$

To determine the Green's functions for Klein-Gordon and Dirac equations by the stochastic approach, it is thus necessary to solve as a preliminary the Langevin equation.

Thus we consider the problem of the KG particle.

2. Klein Gordon particle in Volkov's wave field

For a spinless particle moving in Volkov's wave field, the Green's function provides a solution to the KG equation

$$(\pi_b^2 - m^2)\Delta(x_b, x_a) = \delta^4(x_b - x_a), \tag{12}$$

where

$$\pi_\mu = i\partial_\mu - eA_\mu \ .$$

In SQM, this Green's function $\Delta(x_b, x_a)$ is the correlation function of the field ϕ and its conjugation ϕ^* in thermal equilibrium:

$$\Delta(x_b, x_a) = \lim_{u_a = u_b \to \infty} < \phi(x_b, u_b)\phi^*(x_a, u_a) >, \tag{13}$$

where u is fictitious time.

Therefore, it is necessary to know the field ϕ, which solves the Langevin equation. First, we give the action for a charged KG field:

$$S = \int d^4x \ \left[(\partial_\mu \phi + ieA_\mu \phi) (\partial^\mu \phi^* - ieA^\mu \phi^*) - m^2 \phi\phi^* \right]. \tag{14}$$

Next, we apply the SQM by introducing a new fictitious time u, i.e.

$$\phi(x) \longrightarrow \phi(x, u), \quad \phi^*(x) \longrightarrow \phi^*(x, u), \tag{15}$$

and write the Langevin equations which govern the evolution of the field ϕ and its conjugation ϕ^*:

$$\begin{cases} \dfrac{\partial \phi\,(x,u)}{\partial u} = -\dfrac{\delta S}{\delta \phi^*(x,u)} + \eta(x,u) \\ \dfrac{\partial \phi^*\,(x,u)}{\partial u} = -\dfrac{\delta S}{\delta \phi(x,u)} + \eta^*(x,u) \end{cases}, \tag{16}$$

where the noise fulfill

$$< \eta(x,u) >=< \eta^*(x,u) >= 0 \tag{17}$$

$$< \eta(x,u)\eta^*(x^{'}, u^{'}) >= 2\delta^4(x - x^{'})\delta(u - u^{'}). \tag{18}$$

Explicitly, these two equations are

$$\begin{cases} \dfrac{\partial \phi(x,u)}{\partial u} = \left[\partial^\mu \partial_\mu \phi + 2ieA^\mu \left(\partial_\mu \phi \right) + \left(m^2 - e^2 A^2 \right) \phi \right] + \eta(x,u) \\ \dfrac{\partial \phi^*\,(x,u)}{\partial u} = \left[\partial^\mu \partial_\mu \phi^* - 2ieA^\mu \left(\partial_\mu \phi^* \right) + \left(m^2 - e^2 A^2 \right) \phi^* \right] + \eta^*(x,u) \end{cases}. \tag{19}$$

It is obvious that it is sufficient to solve only one equation since these two equations are conjugations of one another.

The general solution for the first equation is

$$\phi(x,u) = \phi_0(x,u) + \int_{-\infty}^{\infty} du' \int d^4x\prime G(x - x'; u - u')\eta(x', u'),$$

where $\phi_0 = \phi(x,u)_{u=0} = 0$ is a particular solution and $G(x-x';u-u')$ is the Green's function solution of

$$\left[\frac{\partial}{\partial u} - \partial^\mu \partial_\mu - 2ieA^\mu \partial_\mu - \left(m^2 - e^2 A^2\right)\right] G(x-x';u-u') \\ = \delta(u-u')\delta^4(x-x'). \tag{20}$$

To obtain G, it suffices to: (a) find the solution of equation (20) without the δ functions of the second member; and (b) add to this solution function $\theta(u)$, since $\dfrac{d\theta(u)}{du} = \delta(u)$.

Thus, let us solve this equation. To this end, we carry out the replacement $G \to \Phi$ to reduce the notation:

$$\left[\frac{\partial}{\partial u} - \partial^\mu \partial_\mu - 2ieA^\mu \partial_\mu - \left(m^2 - e^2 A^2\right)\right] \Phi(x,u) = 0, \tag{21}$$

where we define a new function $\Phi(x,u) \to \tilde{\phi}(p,\varphi,u)$ via Fourier transform:

$$\Phi(x,u) = i \int \frac{d^4p}{(2\pi)^4} \exp\left(-ipx\right) \tilde{\phi}(p,\varphi,u). \tag{22}$$

Equation (21) then becomes

$$\frac{\partial \tilde{\phi}(p,\varphi,u)}{\partial u} = \Bigg[-\left(p^2 - m^2\right)\tilde{\phi}(p,\varphi,u) - \left[2e\left(Ap\right) + \left(e^2 A^2\right)\right]\tilde{\phi}(p,\varphi,u) \\ - 2i\left(pk\right)\frac{\partial \tilde{\phi}(p,\varphi,u)}{\partial \varphi}\Bigg], \tag{23}$$

where we used the relations $k^2 = 0$, $kA = 0$.

After some arrangements, we get the final expressions of $G(x;u)$ and $G^*(x;u)$:

$$\begin{cases} G(x;u) = \theta(u) \displaystyle\int \frac{d^4p}{(2\pi)^4} \exp\left\{-\left(p^2 - m^2\right)u - i\left(px\right) - \frac{i}{(pk)}\int^{kx} d\varphi' \left[e\left(Ap\right) - \frac{\left(e^2A^2\right)}{2}\right]\right\} \\ G^*(x;u) = \theta(u) \displaystyle\int \frac{d^4p}{(2\pi)^4} \exp\left\{-\left(p^2 - m^2\right)u + i\left(px\right) + \frac{i}{(pk)}\int^{kx} d\varphi' \left[e\left(Ap\right) - \frac{\left(e^2A^2\right)}{2}\right]\right\} \end{cases} \tag{24}$$

Consequently, the correlation function at the equilibrium limit ($u_2 = u_1 \to \infty$) takes the form

$$\Delta(x_b,x_a) = \lim_{u_2=u_1\to\infty} < \phi(x_b,u_b)\phi^*(x_a,u_a) > \\ = \frac{1}{(2\pi)^4}\int \frac{d^4p}{p^2 - m^2} \exp\left[-ip\left(x_b - x_a\right) - \frac{i}{pk}\int_{kx_a}^{kx_b}\left[e\left(Ap\right) - \frac{\left(e^2A^2\right)}{2}\right] d\varphi\right]. \tag{25}$$

This is exactly the Green's function related to the relativistic particle of spin 0 subjected to the action of the field of an electromagnetic plane wave, identical to that obtained via path integral formalism [4], or that recently obtained by the approach of stochastic mechanics [8].

3. Spinning particle in Volkov's wave field

In this part, we study the same problem for a particle with spin, which differs only by the coupling term σF. The Green's function $S(x_b,x_a)$ of this problem is the solution of the Dirac equation

$$\left(\hat{\pi}_b - m\right) S(x_b,x_a) = \delta^4(x_b - x_a), \tag{26}$$

or rather its square:

$$\left(\pi_b^2 - m^2 + \frac{e}{2}(\sigma F_b)\right) S_1(x_b, x_a) = \delta^4(x_b - x_a). \tag{27}$$

The passage from S_1 to S is obtained through the change of function

$$S(x_b, x_a) = (\hat{\pi}_b + m)\, S_1(x_b, x_a), \tag{28}$$

where we have used the usual notations:

$$\begin{aligned} \hat{\pi} &= \pi_\mu \gamma^\mu, \qquad \sigma.F = \sigma^{\mu\nu} F_{\mu\nu}\ , \\ \sigma^{\mu\nu} &= \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right], \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial \nu A_\mu, \qquad \mu, \nu = \overline{0,3}. \end{aligned} \tag{29}$$

To calculate this Green's function by SQM, we consider two fields: ψ, and its conjugate $\overline{\psi} = \psi^+ \gamma^0$. Next we calculate the correlation function following Parisi-Wu stochastic quantization method:

$$S_1(x_b, x_a) = \lim_{u_a = u_b \to \infty} \left\langle \psi(x_b, u_b) \overline{\psi}(x_a, u_a) \right\rangle. \tag{30}$$

In contrast to the KG case, the fields ψ and its conjugates $\overline{\psi}$ are not scalars variables but Grassmann variables, with their evolutions governed by the Langevin equations

$$\begin{cases} \dfrac{\partial \psi\,(x, u)}{\partial u} = i \dfrac{\delta S}{\delta \overline{\psi}(x, u)} + \eta(x, u) \\ \dfrac{\partial \overline{\psi}\,(x, u)}{\partial u} = i \dfrac{\delta S}{\delta \psi(x, u)} + \overline{\eta}(x, u) \end{cases}, \tag{31}$$

with η and $\bar{\eta}$ two Grassmanian noises having the properties

$$\int d\eta(x, u) = \int d\overline{\eta}(x, u) = 0, \qquad \int d\eta \eta(x, u) = \int d\overline{\eta}\overline{\eta}(x, u) = 1, \tag{32}$$

$$\eta^2(x, u) = \overline{\eta}^2(x, u) = 0, \qquad \eta(x, u)\overline{\eta}(x, u) = -\overline{\eta}(x, u)\eta(x, u), \tag{33}$$

and

$$\begin{cases} \langle \eta(x, u) \rangle = \langle \overline{\eta}(x, u) \rangle =< \eta^2(x, u) >=< \overline{\eta}^2(x, u) >= 0 \\ \langle \eta(x, u) \overline{\eta}(x', u') \rangle = 2\delta^4(x - x')\delta(u - u') \end{cases}. \tag{34}$$

Following the same method for the previous $\Delta(x_b, x_a)$ calculation, where we have in addition, in the action, the coupling spin-field term, the Green's function $S_1(x_b, x_a)$ is easily obtained. The result of this Green's function can be written in standard notation:

$$\begin{aligned} S_1(x_b, x_a) &= \frac{1}{(2\pi)^4} \int \frac{d^4p}{p^2 - m^2} \exp\left\{-ip\,(x_b - x_a) - \frac{i}{pk} \int_{kx_a}^{kx_b} \left[e\,(pA) - \frac{e^2 A^2}{2} + \frac{e\sigma F}{4}\right] d\varphi\right\} \\ &= \frac{1}{(2\pi)^4} \int \frac{d^4p}{p^2 - m^2} \exp\left\{-ip\,(x_b - x_a) - \frac{i}{pk} \int_{kx_a}^{kx_b} \left[e\,(pA) - \frac{e^2 A^2}{2}\right] d\varphi\right\} \\ &\times \exp\left\{\frac{e}{2pk} \hat{k} \left[\hat{A}\,(kx_b) - \hat{A}\,(kx_a)\right]\right\}. \end{aligned}$$

The Green's function $S(x_b, x_a)$ is deduced from equation (28):

$$\begin{aligned} S(x_b, x_a) &= (\hat{\pi}_b + m)\, S_1(x_b, x_a) \\ &= \int \frac{d^4p}{(2\pi)^4} \left[1 + \frac{e\hat{k}\hat{A}_b}{2pk}\right] \left[\frac{\hat{p} + m}{p^2 - m^2}\right] \left[1 - \frac{e\hat{k}\hat{A}_a}{2pk}\right] \\ &\times \exp\left\{-ip\,(x_b - x_a) - \frac{i}{pk} \int_{kx_a}^{kx_b} \left[e\,(Ap) - \frac{e^2 A^2}{2}\right] d\varphi\right\}, \end{aligned} \tag{35}$$

where we used the relations

$$\exp\left[\frac{e\hat{k}\hat{A}}{2pk}\right] = 1 + \frac{e}{2pk}\hat{k}\hat{A}, \qquad \hat{A}\hat{B} + \hat{B}\hat{A} = 2\,(AB)\,. \tag{36}$$

Result (35) is equivalent to that given through the path integral approach [4].

4. Conclusion

By solving the Langevin equations we could determine at thermal equilibrium, and in a direct way, the Green's function for a relativistic particles of spin 0 and 1/2 in interaction with an electromagnetic plane wave field. Calculations are given in an analytical and exact way, thanks to the plane wave properties. The result is compared with those existing calculations and in particular agrees with that obtained via the path integral approach [4], [5].

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