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Plane Symmetric Solutions of Gravitational Field Equations in Five Dimensions

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Abstract

We present the effective field equations obtained from a generalized gravity action with Euler-Poincaré term and a cosmological constant in a D dimensional bulk space-time. A class of plane-symmetric solutions that describe a 3-brane world embedded in a $D = 5$ dimensional bulk space-time are given.

1. Introduction

Brane-world theories that receive a lot of interest recently are strictly motivated by string models [1]. They were mainly proposed to provide new solutions to the hierarchy problem and compactification of extra dimensions [2],[3]. The main content of the brane-world idea is that we live in a four dimensional world embedded in a higher dimensional bulk space-time. According to the brane-world scenarios, the gauge fields, fermions and scalar fields of the Standard Model should be localised on a 3-brane, while gravity may freely propagate into the higher dimensional bulk.

In our previous work [4] we derived covariant gravitational field equations on a 3-brane embedded in a five-dimensional bulk space-time with $\mathbb{Z}_2$ symmetry in a generalization that included a dilaton scalar as well as the second order Euler-Poincaré density in the action. We introduced a general ADM-type coordinate setting to show that the effective gravitational field equations on the 3-brane remain unchanged, however, the evolution equations off the brane are significantly modified due to the acceleration of normals to the brane surface in the non-geodesic, ADM slicing of space-time.

In the second part of this paper, using the language of differential forms, we present the field equations of a generalized gravity model with a dilaton 0-form and an axion 3-form in Einstein frame from an action that includes the second order Euler-Poincaré term and a cosmological constant in a D-dimensional bulk space-time. In the third part, we present some plane-symmetric solutions that generalize the well-known domain-wall solution [5].

2. Model

We consider a D-dimensional bulk space-time manifold M equipped with a metric g and a torsion-free, metric compatible connection ∇. We determine our gravitational field equations by a variational principle from a D-dimensional action that includes the second order Euler-Poincarè term and a cosmological constant

$$I[e,\omega,\phi,H] = \int_M \mathcal{L} \tag{1}$$

where in the Einstein frame the Lagrangian density D-form [6]

$$\begin{aligned}\mathcal{L} \;=\;& \frac{1}{2}R^{ab}\wedge *(e_a\wedge e_b) - \frac{\alpha}{2}d\phi\wedge *d\phi + \frac{\beta}{2}e^{-\beta_2\phi}H\wedge *H + \Lambda e^{-\beta_1\phi} *1 \\ &+\frac{\eta}{4}R^{ab}\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d) \\ &+(de^a + \omega^a{}_b\wedge e^b)\wedge\lambda_a + (dH - \frac{\varepsilon}{2}R_{ab}\wedge R^{ab})\wedge\mu\,.\end{aligned} \tag{2}$$

Here λ_a and μ are Lagrange multiplier forms that upon variation impose the zero-torsion and anomaly-freedom constraints.

The final form of the variational field equations to be solved are the Einstein field equations

$$\begin{aligned}\frac{1}{2}R^{ab}\wedge *(e_a\wedge e_b\wedge e_c) =& -\frac{\alpha}{2}\tau_c[\phi] + \frac{\beta}{2}e^{-\beta_2\phi}\tau_c[H] - \Lambda e^{-\beta_1\phi} * e_c \\ &-\frac{\eta}{4}R^{ab}\wedge R^{dg}\wedge *(e_a\wedge e_b\wedge e_d\wedge e_g\wedge e_c) \\ -2\varepsilon\beta D(e^{-\beta_2\phi}\iota_b(R^b{}_c\wedge *H)) &- \frac{\varepsilon\beta}{2}e_c\wedge D(e^{-\beta_2\phi}\iota_s\iota_l(R^{ls}\wedge *H)),\end{aligned} \tag{3}$$

where the dilaton stress-energy forms

$$\tau_a[\phi] = \iota_a d\phi * d\phi + d\phi\wedge\iota_a * d\phi$$

and the axion stress-energy forms

$$\tau_a[H] = \iota_a H\wedge *H + H\wedge\iota_a * H\ ,$$

the dilaton scalar field equation

$$\alpha d(*d\phi) = \frac{\beta_2\beta}{2}e^{-\beta_2\phi}H\wedge *H + \Lambda\beta_1 e^{-\beta_1\phi} * 1, \tag{4}$$

and the axion field equations

$$dH = \frac{\varepsilon}{2}R_{ab}\wedge R^{ab} \quad , \quad d(e^{-\beta_2\phi} * H) = 0. \tag{5}$$

3. Plane symmetric solutions in $D = 5$

We investigate below a class of plane symmetric solutions in 5-dimensions. We consider the metric

$$g = -f^2(t,\omega)dt^2 + u^2(t,\omega)d\omega^2 + g^2(t,\omega)\left(\frac{dx^2+dy^2+dz^2}{\left(1+\frac{kr^2}{4}\right)^2}\right), \tag{6}$$

the dilaton scalar field

$$\phi = \phi(t,\omega) \tag{7}$$

and 3-form gauge field

$$H = h(t,\omega)\frac{dx\wedge dy\wedge dz}{\left(1+\frac{kr^2}{4}\right)^3} \tag{8}$$

in terms of local coordinates

$$x^M : \left\{x^0 = t, x^5 = \omega, x^1 = x, x^2 = y, x^3 = z\right\}.$$

We choose our co-frame 1-forms as

$$e^0 = f(t,\omega)dt, \qquad e^5 = u(t,\omega)d\omega, \qquad e^i = g(t,\omega)\frac{dx^i}{\left(1+\frac{kr^2}{4}\right)}, \quad i = 1,2,3. \tag{9}$$

Then we calculate the Levi-Civita connection 1-forms

$$\omega^0{}_i = \frac{g_t}{fg}e^i, \qquad \omega^i{}_j = \frac{k}{2g}(x^i e^j - x^j e^i), \tag{10}$$

$$\omega^0{}_5 = \frac{u_t}{fu}e^5 + \frac{f_\omega}{fu}e^0, \qquad \omega^i{}_5 = \frac{g_\omega}{ug}e^i. \tag{11}$$

and the corresponding curvature 2-forms

$$R^{ij} = \frac{1}{g^2}\left\{k + \left(\frac{g_t}{f}\right)^2 - \left(\frac{g_\omega}{u}\right)^2\right\}e^i\wedge e^j, \tag{12}$$

$$R^{05} = \frac{1}{fu}\left\{\left(\frac{f_\omega}{u}\right)_\omega - \left(\frac{u_t}{f}\right)_t\right\}e^5\wedge e^0, \tag{13}$$

$$R^{0i} = \frac{1}{fg}\left\{\left(\frac{g_t}{f}\right)_t - \frac{f_\omega g_\omega}{u^2}\right\}e^0\wedge e^i + \frac{1}{ug}\left\{\left(\frac{g_t}{f}\right)_\omega - \frac{u_t g_\omega}{fu}\right\}e^5\wedge e^i, \tag{14}$$

$$R^{i5} = \frac{1}{fg}\left\{\frac{f_\omega g_t}{fu} - \left(\frac{g_\omega}{u}\right)_t\right\}e^i\wedge e^0 + \frac{1}{ug}\left\{\left(\frac{g_\omega}{u}\right)_\omega - \frac{g_t u_t}{f^2}\right\}e^5\wedge e^i. \tag{15}$$

From these expressions we note that $R_{ab}\wedge R^{ab} = 0$. Therefore $dH = 0$ implying that

$$H = \frac{Q}{g^3}e^1\wedge e^2\wedge e^3 \tag{16}$$

where Q may be identified as a magnetic charge. Now, for simplicity, we let $k = 0$ and take the functions g, f and u independent of time. Then we obtain the following system of coupled ordinary differential equations ($'$ denotes derivative with respect to ω):

$$\begin{aligned} 2G - 2C - B - A &= -\eta(2CG - AB) - \frac{\alpha}{2}\left(\frac{\phi'}{u}\right)^2 \\ &\quad -\frac{\beta}{2}\frac{Q^2}{g^6}e^{-\beta_1\phi} + \Lambda e^{-\beta_2\phi}, \end{aligned} \tag{17}$$

$$3A - 3G = 3\eta GA + \frac{\alpha}{2}\left(\frac{\phi'}{u}\right)^2 - \frac{\beta}{2}\frac{Q^2}{g^6}e^{-\beta_2\phi} - \Lambda e^{-\beta_1\phi}, \tag{18}$$

$$3C + 3A = -3\eta CA - \frac{\alpha}{2}\left(\frac{\phi'}{u}\right)^2 - \frac{\beta}{2}\frac{Q^2}{g^6}e^{-\beta_2\phi} - \Lambda e^{-\beta_1\phi}, \tag{19}$$

$$\alpha\left(\frac{\phi' f g^3}{u}\right)' \frac{1}{g^3 f u} = \frac{\beta_2\beta}{2}e^{-\beta_2\phi}\frac{Q^2}{g^6} + \Lambda\beta_1 e^{-\beta_1\phi}. \tag{20}$$

where

$$A = -\left(\frac{g'}{g}\right)^2 \frac{1}{u^2} \qquad B = -\left(\frac{f'}{u}\right)' \frac{1}{fu}, \tag{21}$$

$$C = -\frac{f'g'}{u^2 fg} \qquad G = \left(\frac{g'}{u}\right)' \frac{1}{ug}. \tag{22}$$

We will give below some special classes of solutions:

Case: $\phi = constant$, $H = 0$ and $\eta = 0$.

Here the Euler-Poincaré term is absent, $H = 0$ and the dilaton scalar is constant. We obtain the AdS solution in 5-dimensions that is also known as Randall-Sundrum model [3]:

$$g = d\omega^2 + e^{\mp 2p\omega}(-dt^2 + dx^2 + dy^2 + dz^2). \tag{23}$$

where $p^2 = \frac{\Lambda}{6}$.

Case: $\phi = constant$, $H = 0$.

Here $H = 0$ and the dilaton scalar is constant. Solutions are given by the metric

$$g = d\omega^2 + e^{\mp 2s\omega}(-dt^2 + dx^2 + dy^2 + dz^2) \tag{24}$$

where

$$s^2 = \frac{1 + \sqrt{1 - \frac{\eta\Lambda}{3}}}{\eta} \tag{25}$$

provided that $\Lambda\eta \leq 3$. When $\eta\Lambda = 3$, the solution may alternatively be given in AdS form as

$$g = -4\cosh^2(l\omega)dt^2 + d\omega^2 + 4\sinh^2(l\omega)(dx^2 + dy^2 + dz^2) \tag{26}$$

where $l^2 = \frac{1}{\eta}$.

Case: $\eta = 0$, $H = 0$.

Here the Euler-Poincaré term is absent and $H = 0$. We obtain the following solution:

$$g = e^{\frac{16\alpha}{3\beta_1}\phi(\omega)}d\omega^2 + e^{\frac{4\alpha}{3\beta_1}\phi(\omega)}(-dt^2 + dx^2 + dy^2 + dz^2) \tag{27}$$

with

$$\phi(\omega) = \frac{1}{\left(\frac{\beta_1}{2} - \frac{8\alpha}{3\beta_1}\right)} \ln\left|\sqrt{\frac{2\beta_1\Lambda}{\left(\frac{16\alpha}{3\beta_1} - \beta_1\right)\alpha}}\left(\frac{\beta_1}{2} - \frac{8\alpha}{3\beta_1}\right)\omega + C_0\right| \tag{28}$$

where C_0 is an integration constant. When $\beta_1 = 2$, it reduces to a supersymmetric domain wall solution presented in [5].

Case: $\eta = 0$.

In this case the solution possesses a magnetic charge. It is given by

$$g = e^{\frac{4(\beta_1-\beta_2)}{3}\phi(\omega)}d\omega^2 + e^{\frac{(\beta_1-\beta_2)}{3}\phi(\omega)}(-dt^2 + dx^2 + dy^2 + dz^2) \tag{29}$$

with

$$\phi(\omega) = \frac{6}{4\beta_2-\beta_1}\ln\left|\left(\frac{4\beta_2-\beta_1}{6}\right)\sqrt{\frac{6\left(\frac{\beta_2\beta}{2}Q^2 + \Lambda\beta_1\right)}{(\beta_1-4\beta_2)\alpha}}\omega + C\right| \tag{30}$$

provided that the constants satisfy

$$(\beta_1-\beta_2)\left(\frac{\beta Q^2\beta_2}{2} + \beta_1\Lambda\right) = \left(\frac{\beta Q^2}{2} + 4\Lambda\right)\alpha. \tag{31}$$

C is an integration constant. H is given by

$$H = Qe^{\frac{(\beta_2-\beta_1)}{2}\phi(\omega)}e^1 \wedge e^2 \wedge e^3 \tag{32}$$

We note that when $Q = 0$ and the constants β_1 and β_2 satisfy $\beta_1 - \beta_2 = \frac{4\alpha}{\beta_1}$, the solutions reduce to (27) and (28).

We also note that an electric dual of solutions (29) and (30) may be given by defining a 2-form field

$$F = e^{\beta_2\phi} * H. \tag{33}$$

Then the solutions are identified as electrically charged solutions.

4. Conclusion

We have given a class of solutions to the variational field equations of a generalized theory of gravity in a D dimensional bulk space-time derived from an action that includes the second-order Euler-Poincaré term and a cosmological constant. The theory describes a heterotic type first order effective string model in D dimensions in the Einstein frame. The special class of plane-symmetric solutions of this model in 5-dimensions we gave refer to a 3-brane world also called a domain wall solution in the literature [5].

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