

Phase Distribution of the Output of Jaynes-Cummings Model with the Superposition of Squeezed Displaced Fock States

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Abstract

The Wigner quasi-probability function for the superposition of squeezed displaced Fock states (SDFS's) is reviewed. The interaction of these states with a two-level atom in cavity with the presence of additional Kerr medium is studied. Exact general matrix elements of the time-dependent operators of a Jaynes-Cummings model (JCM), in the presence of a Kerr medium, with these states are derived. We have obtained the phase distribution by two different ways: one is by Pegg-Barnett formalism, the second is by integration of the Wigner function over the radial variable. Results of these two approaches are compared. The Wigner phase distributions for some values of parameters are illustrated. The behaviors of the distributions have been shown as a function of the squeeze parameter in JCM.

Key Words: Non-classical field states; squeezed displaced Fock states; Wigner quasi-probability function; phase distribution, JCM.

1. Introduction

The squeezed displaced Fock states (SDFS's) have been introduced and studied in [1]. These states generalize squeezed coherent states, squeezed number states, and displaced Fock states [2]. They exhibit both number squeezing in the strong sense and the quadrature squeezing. One of the most fundamental features of quantum mechanics is the well known linear superposition principle which gives rise to amplitude interference effects as its direct consequence. This motivated us to study the phase distribution of interaction of superposition of SDFS's [3] with two-level atom with additional Kerr medium.

Recently there has been much interest in the interaction of various forms of non-classical radiation with atoms. Real systems are often approximated by simple models, which can be solved exactly. One such model is the Jaynes-Cummings model (JCM), i.e., a system where a single two-level atom interacts with a single cavity mode [4]. Its simplicity allows us to apply the fundamental laws of quantum electrodynamics to it and still be able to solve it analytically. The dynamics of the JCM by assuming the field to be initially in different quantum states have been studied in refs. [5, 6]. The evolution of JCM in the presence of a Kerr medium is investigated by [7].

The quantum description of phase and angle is more complicated than its classical counterpart and met considerable difficulties [8, 9]. These difficulties have received much theoretical interest and have been

described by a number of authors [8–12]. Pegg and Barnett [10] have introduced a new Hermitian phase formalism which successfully overcomes the troubles inherent in the Susskind-Glogower [11] phase formalism and enables one to study finer details of the phase properties of quantum fields. Such quantities, as expectation values and variances of the Hermitian phase operators or phase distribution functions, are now available for investigations.

This work is organized as follows. In section 2, the model of the interaction of two-level atom is demonstrated. In section 3, the s-parameterized phase distribution has been investigated. In section 4, we consider the phase distribution behavior.

2. The Model Hamiltonian

The JCM is realized in the ion trap by the application of a laser tuned to the first upper vibrational sideband. Tuning to the first lower vibrational sideband yields the counter-rotating terms in Jaynes-Cummings model. Tuning to the k^{th} sidebands gives rise to k -photon vibrational analogue of multi-photon generalizations to the Jaynes-Cummings discussed in the quantum optics literature [4–8]. This is useful in making quantum non-demolition measurements of an ion and in generating quantum superposition of coherent states [7].

In the rotating wave approximation, the Hamiltonian operator for a two-level system coupled to a single cavity mode with an additional non-linear term [7] takes the form ($\hbar = 1$),

$$H = \omega_c a^+ a + \chi (a^+)^2 a^2 + \frac{\omega_a \sigma_z}{2} + g(a\sigma_+ + a^+ \sigma_-), \quad (1)$$

where a^+, a are the creation and annihilation operators of the single mode cavity field with frequency; ω_c, χ describes the strength of the quadratic non-linearity modeling the Kerr medium; ω_a is the frequency of the single two-level atom; and g is the strength of the atom-field coupling. The operators σ_z and σ_{\pm} are the z component of the Pauli matrix and the spin-flip matrices for the two-level system which satisfy $[\sigma_z, \sigma_{\pm}] = \pm 2\sigma_{\pm}$, and $[\sigma_+, \sigma_-] = \sigma_z$. We assume that initially the two-level atom is in its upper (excited) state and that the cavity field is in the superposition of SDFS's $|\Psi_m(0)\rangle$. The SDFS [1], $|z, \alpha, n\rangle$ is defined by

$$|z, \alpha, n\rangle = S(z)D(\alpha)|n\rangle. \quad (2)$$

The displacement operator $D(\alpha)$ and the squeeze operator $S(z)$ are given by

$$\begin{aligned} D(\alpha) &= \exp(\alpha a^+ - \alpha^* a); \quad \alpha = |\alpha| \exp(i\theta) \\ S(z) &= \exp\left(\frac{z}{2} a^2 - \frac{z^*}{2} a^{+2}\right); \quad z = r \exp(i\varphi), \end{aligned} \quad (3)$$

where $a, (a^+)$ is the annihilation (creation) operator of the boson fields. Here, r is known as the squeeze parameter and φ is the direction of squeezing and it is taken to be zero.

The initial field is assumed to be the superposition of SDFS's (Schrödinger cat) $|\Psi_m(0)\rangle$ which has the form [3]

$$|\Psi_m(0)\rangle = A^{-\frac{1}{2}} (|z, \alpha_0, m\rangle + K|z, -\alpha_0, m\rangle) \quad (4)$$

with A being the normalization constant given by

$$A = \{ 1 + |K|^2 + (K + K^*) \exp(-2|\bar{\alpha}_0|^2) L_m(4|\bar{\alpha}_0|^2) \}, \quad (5)$$

where $\bar{\alpha} = \mu\alpha_0 + \nu\alpha_0^*$, $\mu = \cosh r$, $\nu = \exp(i\varphi) \sinh r$ and $L_m(\cdot)$ is the Laguerre polynomial. For $K = 0$ we have the SDFS's, but for $K = 1$ or -1 the resulting states depend on m : if m is an even number and $K = 1$, we have superposition of even states and while odd states are obtained when $K = -1$. But when m is an odd number, the result is reversed.

To begin, we set that the initial field state is in the form

$$|\Psi_m(0)\rangle = \sum_{l=0}^{\infty} \Theta_l |l\rangle, \quad (6)$$

where

$$\Theta_l = A^{-\frac{1}{2}} [C_l^+(m) + KC_l^-(m)], \quad (7)$$

with $C_l^\pm(m) = \langle l|z, \pm\alpha_0, m\rangle$ being the photon number amplitudes of SDFS, which is given in [1]. By assuming the atom to start in its excited state, then the atom-field state at $t > 0$ is

$$|\Psi(t)\rangle = |\Psi_1(t)\rangle|e\rangle + |\Psi_2(t)\rangle|g\rangle, \quad (8)$$

where the excited (upper) and ground states are represented by $|e\rangle$ and $|g\rangle$, respectively, with

$$|\Psi_1(t)\rangle = \sum_{n=0}^{\infty} \Theta_n A_n |n\rangle, \quad |\Psi_2(t)\rangle = \sum_{n=0}^{\infty} \Theta_n B_n |n+1\rangle, \quad (9)$$

and

$$A_n = \exp(-i\chi n^2 t) [\cos(\delta_n t) - i \frac{(\omega_a - \omega_c - 2\chi n)}{2\delta_n} \sin(\delta_n t)] \quad (10)$$

$$B_n = -i\sqrt{n+1} \exp(-i\chi n^2 t) \frac{g}{\delta_n} \sin(\delta_n t), \quad (11)$$

$$\delta_n = \sqrt{\left(\frac{\omega_a - \omega_c}{2} - \chi n\right)^2 + g(n+1)}. \quad (12)$$

To study the dynamics of the field we need the density operator. The density operator of the composite system is given by

$$\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|. \quad (13)$$

The reduced density operator of the field is

$$\rho^f(t) = Tr_a \rho(t) = |\Psi_1(t)\rangle\langle\Psi_1(t)| + |\Psi_2(t)\rangle\langle\Psi_2(t)|. \quad (14)$$

From this we can derive the density matrix for an atom starting from its excited state as

$$\rho_{nm}^f(t) = [\Theta_n A_n \Theta_m^* A_m^* + \Theta_{n-1} B_{n-1} \Theta_{m-1}^* B_{m-1}^*]. \quad (15)$$

We shall be using these density matrix elements to calculate some statistical phase distribution properties of the output field in the next sections.

3. s-Parameterized Phase Distribution

Analytical forms for the s-parameterized quasi-probability function (QDF) such as the P-function, Wigner function and Q function for the SDFS's allow for clear interpretations in phase space. The fact that the states are phase dependent makes it interesting to study their phase properties which, to our knowledge, have not been studied so far. The s-parameterized QDF is the Fourier transformation of the s-parameterized characteristic function [1]. It is well known that such a parameter is associated with the ordering of the bosonic field operators. We consider the s-parameterized QDF for our field states in the form

$$F(\beta, s) = \frac{2}{\pi(1-s)} \exp \left[\frac{-2}{(1-s)} |\beta|^2 \right] \sum_{k,j}^{\infty} (-1)^k \sqrt{\frac{k!}{j!}} \times \frac{(1+s)^k}{(1-s)^j} (2\beta^*)^{k-j} L_j^{k-j} \left(\frac{4|\beta|^2}{1-s^2} \right) \rho_{kj}^f(t). \quad (16)$$

In Figure 1, the Wigner functions for some values of the photon number m , the displacement parameter α_0 , and squeeze parameter r , are shown.

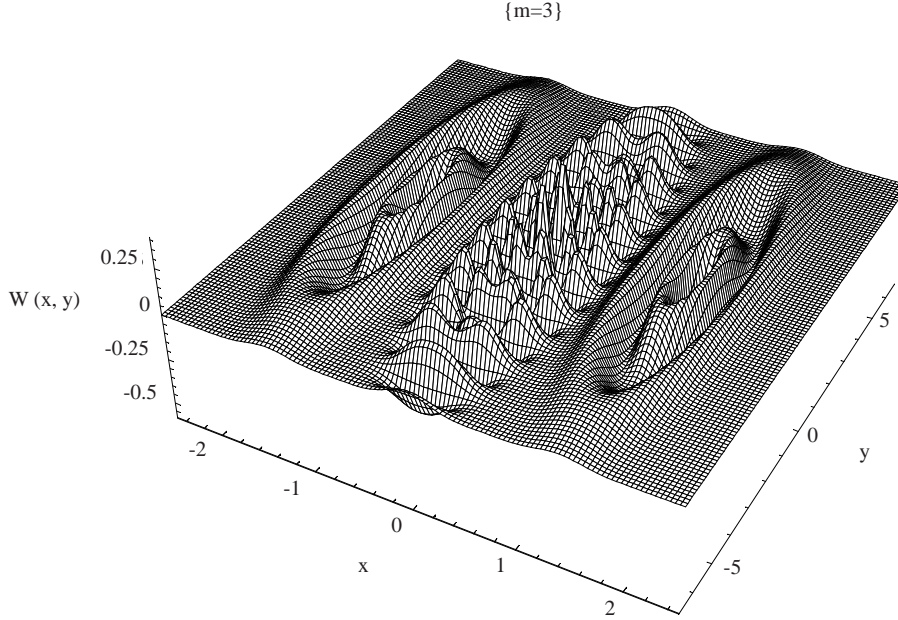


Figure 1. The Wigner function for SDFS's superposition state, with the parameters $K = 1, t = 0, \alpha_0 = 1.4, m = 3, r = 1.2$.

The phase distribution [9] can be obtained by integrating the s-parameterized functions over the radial variable. The s-parameterized phase distribution can be obtained by integrating s-parameterized quasi-probability function over the radial variable.

$$P(\theta, s) = \int_0^{\infty} F(\beta, s) |\beta| d|\beta|. \quad (17)$$

Therefore the phase distribution can be calculated in a straightforward manner [9]. The phase distribution $P(\theta, 0)$ associated with the Wigner function illustrated in Figure 2, and $P(\theta, -1)$ associated with the Q-function has been computed earlier [3]. The behavior of the PB phase distributions of output of JCM with SDFS's superposition as input field will be discussed.

For the situation here, however, it is more convenient to find a phase-like quasi-probability distribution instead.

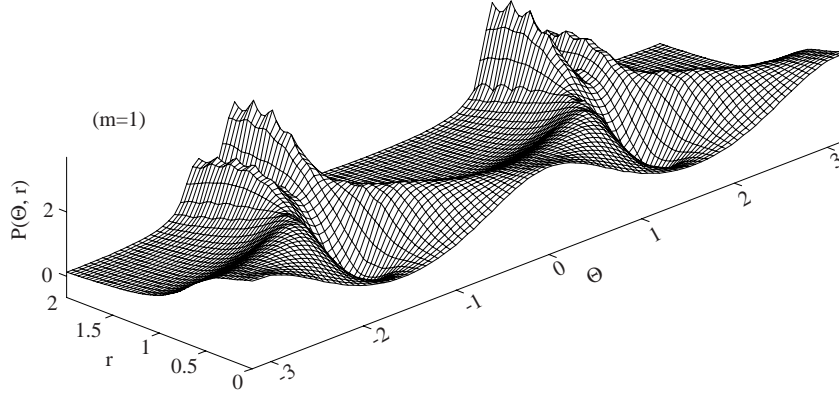


Figure 2. The Wigner phase distribution function $P(\theta, 0)$, with the parameters $K = 1$, $t = 0$, $\alpha_0 = 0.5$, $m = 1$.

This is the shape of the phase distribution obtained by integrating the Wigner function (see Figure 2) for the input field states, i.e., $t = 0$.

4. Barnett-Pegg Phase Distribution

Recently, Barnett and Pegg defined a Hermitian phase operator in a finite dimensional state space [10]. They used the fact that, in this state space, one can define phase states rigorously. The phase operator is then defined as the projection operator on the particular phase state multiplied by the corresponding value of the phase. The main idea of the Pegg-Barnett formalism consists in evaluation of all expectation values of physical variables in a finite dimensional Hilbert space. These give real numbers, which depend parametrically on the dimension of the Hilbert space. Because a complete description of the harmonic oscillator involves an infinite number of states to be taken, a limit is taken only after the physical results are evaluated. This leads to proper limits which correspond to the results obtainable in ordinary quantum mechanics. It can be used for investigation of the phase properties of quantum states of the single mode of the electromagnetic field [8–10]. Therefore, we will study phase properties in the JCM with a light field initially prepared in a superposition of SDFS's using the Pegg-Barnett phase formalism.

The Pegg-Barnett phase distribution $P(\eta)$ is defined through the infinite sum [10]

$$P(\eta) = \frac{1}{2\pi} \sum_{j,k}^{\infty} \rho_{jk}^f(t) \exp[i(k-j)(\eta - \eta_0)], \quad (18)$$

Where the angle η_0 is the phase reference angle and we take it to be zero. The phase distribution can be written as

$$P(\eta) = \frac{1}{2\pi} \left(1 + 2Re \sum_{j,k,k>j}^{\infty} \rho_{jk}^f(t) \cos [i(k-j)\eta] \right) \quad (19)$$

In Figure 3 the phase distribution $P(\eta)$ is plotted against η and time t . The results calculated numerically show how, in such a model, the squeezing parameter can affect phase properties. We shall consider only the case of resonance for which $\Delta = \omega_a - \omega_c = 0$. In our computations, we have taken $\alpha_0 = 3$, $K = 1$, $\chi = 0$ and the squeeze parameter has $r = 1$, $m = 0$.

For $r = 0$, the case of superposition of two coherent states is shown in [5]. First note the difference between the figure here, where $t = 0$, a peak in the middle at $\theta = \pi$, and that two wings at $\theta = 0$ and $\theta = 2\pi$ appear; while in the case of one coherent state, only a peak in the middle occurs [9]. As time t increases, the peak in the middle splits into two, diverging away from the middle towards the wings while the two wings converge to the middle. The phases of the two states seem to develop in time differently. When $r \neq 0$, the peak is rather broad, as shown in Figure 1, one of the arms exhibits larger amplitude than the other arm, which is due to the effect of squeezing. In addition, as the squeeze parameter r is not equal to zero, the bifurcation of the phase distribution appears [12, 13].

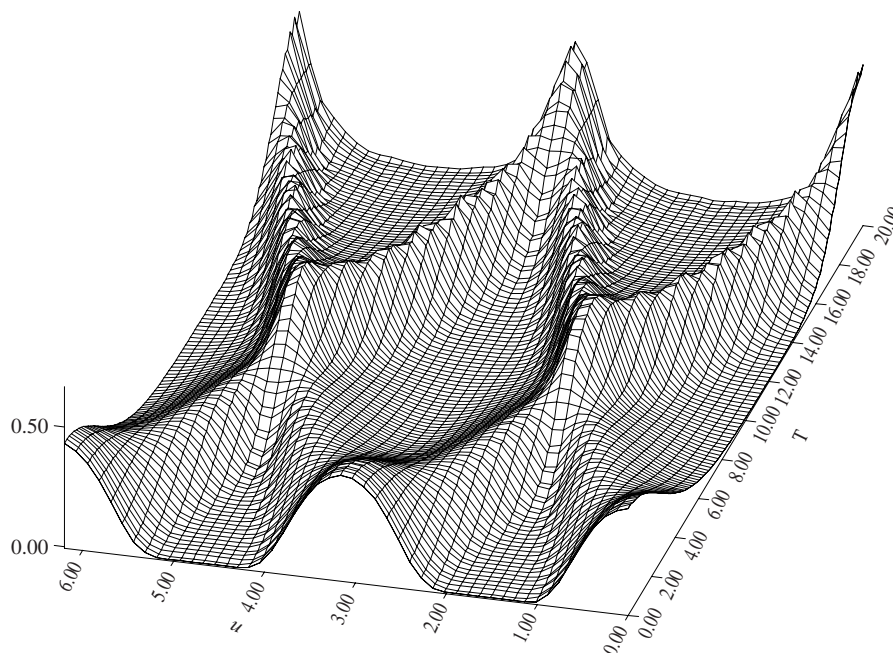


Figure 3. The phase distribution $P(\eta, t)$, with the parameters assumed as: $\alpha_0 = 3$, $K = 1$, $\Delta = 0$, $\chi = 0$ and the squeeze parameter has the value $r = 1$.

Numerical calculations show that, the phase distributions are exhibited in the double peak, with their splitting appearing when $t = 0$. It is clear that the phase graph with two peaks is moving and the peaks become broader as the variable t increases to t_R (revival time). Also, the four peaks may be found around the time $t = 2t_R$. At $t = t_R$ the distributions move and change its shape by $\pi/2$ from the states $t = 0$ and $t = 2t_R$. However, it is apparent that as t increases two sharper peaks appear distinctly for both even and odd SDFS's inputs, as numerical calculations show, associated with phase distributions. The larger the squeezing parameter is, the more pronounced the phase distribution splits and interference. For any value of the squeeze parameter, we may establish the correspondence between the phase properties and the collapses and revivals of the atomic inversion.

5. Conclusion

In this work we have studied two aspects of the interaction between the two-level atom and the field initially in an SDFS's superposition, namely the evolution of the output field statistics. We have obtained the solution of the generalized JCM with Kerr medium. The SDFS's superpositions have been used as input field state. The output fields of that system have been studied. A similar behavior for the output fields whenever the atom is assumed to be in its excited or ground state has been shown. We have discussed the

Pegg-Barnett phase distribution and also plotted with some parameters. Notice that a similar bifurcation of phase distribution has been obtained for large squeezing. Bifurcation of the phase distribution appears in the interval of collapse. We have found that the distribution of the field reflect the collapses and revivals of the level occupation probabilities in most situations.

The Wigner function for the SDFS's has been reviewed. The three dimensional plots of the Wigner function for some parameters has been illustrated. We have discussed the phase properties for the SDFS's. We have obtained the phase distribution by two different ways: one of them is by Pegg-Barnett formalism, the second is by integration of the Wigner function over the radial variable. The resulting phase distributions are very useful and generalize results in the field of "coherent, squeezed, displaced Fock" states. The behavior of the distributions has been shown as a function of the squeeze parameter. Our present work was motivated by the desire to realize physically certain specific quantum states (superposition of SDFS's). It is hoped that the superposition of SDFS's will find application in quantum optics.

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