NEAR ULTRAFILTERS AND \textit{LUC}-COMPACTIFICATION OF REAL NUMBERS

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Abstract

In this work we will investigate some of the topological properties of the \textit{LUC}-compactification of real numbers $\mathbb{R}$ in terms of the concept of near ultrafilters.

1. Introduction

By a compactification of the topological space $\mathbb{R}$, we shall mean a compact Hausdorff space $K$ with an embedding $e : \mathbb{R} \to K$ with $e[\mathbb{R}]$ is dense in $K$. We will usually identify $\mathbb{R}$ with $e(\mathbb{R})$ and consider $\mathbb{R}$ as a subspace.

The topological space $\mathbb{R}$ has a compactification $\hat{\mathbb{R}}$ with the property that $C(\hat{\mathbb{R}})$ is isomorphic to the algebra $\textit{LUC}(\mathbb{R})$ of bounded real-valued uniformly continuous functions defined on $\mathbb{R}$. $\hat{\mathbb{R}}$ is the spectrum of $\textit{LUC}(\mathbb{R})$ furnished with the Gelfand topology (i.e., weak topology from $\textit{LUC}(\mathbb{R})^\ast$) (see [1]). As is well known, this compactification has the property that a bounded continuous function $f$ from $\mathbb{R}$ to $\hat{\mathbb{R}}$ has a continuous extension $\hat{f} : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ if and only if $f$ is uniformly continuous (see [2]).

The compactification $\hat{\mathbb{R}}$ was constructed in terms of the concept of near ultrafilters (see [4]). We shall say that a subset $\eta$ of $\mathcal{P}(\mathbb{R})$ has the near finite intersection property if $\eta$ is non-empty and if, for every finite subset $F$ of $\eta$ and every $W \in \mathcal{B}$, $\bigcap_{Y \in F} (W + Y) \neq \emptyset$.

We say that $\eta$ is a near ultrafilter if $\eta$ is maximal subject to being a subset of $\mathcal{P}(\mathbb{R})$ with the near finite intersection property. It is clear that every ultrafilter on $\mathbb{R}$ is a near ultrafilter. We take $\hat{\mathbb{R}}$ to be set of near ultrafilters on $\mathbb{R}$. For each $Y \subseteq \mathbb{R}$, let $C_Y = \{ \eta \in \hat{\mathbb{R}} : Y \in \eta \}$. Then $\hat{\mathbb{R}}$ is made into a topological space by taking the family of all sets $C_Y$ as a base for the closed sets. With this topology $\hat{\mathbb{R}}$ is a compact Hausdorff space and the mapping $e : \mathbb{R} \to \hat{\mathbb{R}}$ is defined by $e(x) = \{ Y \subseteq \mathbb{R} : x \in Y \}$ for each $x \in \mathbb{R}$ is an embedding with $e(\mathbb{R})$ dense in $\hat{\mathbb{R}}$ (see [4]). We identify a subset $Y$ of $\mathbb{R}$ with $e(Y)$.

If $X$ is a topological space, $\beta X$ will denote the Stone-Čech compactification of $X$ and $X^\ast$ will denote the growth $\beta X \setminus X$. $\mathcal{B}$ will denote the set of all neighborhoods of $0$.
in $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^-$ denote the set of nonnegative reals and nonpositive reals, respectively. $\gamma\mathbb{R}$ will denote $\hat{\mathbb{R}}\setminus\mathbb{R}$.

We give some of the properties of near ultrafilters that we will use in this paper. More details about near ultrafilters can be found in [4],[2].

1.1. Some Properties of Near Ultrafilters

Let $\xi$ be a near ultrafilter,

1) If $F$ be a finite subset of $\xi$, then $\bigcap_{Y \in F} (W + Y) \in \xi$ for all $W \in \mathcal{B}$.

2) $Y \in \xi$ if and only if $(W + Y) \cap Z \neq \emptyset$ for every $Z \in \xi$ and every $W \in \mathcal{B}$ if and only if $Y \cap (Z + W) \neq \emptyset$ for every $Z \in \xi$ and every $W \in \mathcal{B}$.

3) $Y \in \xi$ if and only if $(Y + W) \in \xi$ for every $W \in \mathcal{B}$. Furthermore, this is the case if and only if $cl_{\hat{\mathbb{R}}} Y \in \xi$.

4) If $Y_1, Y_2 \subseteq \mathbb{R}$, then $Y_1 \cup Y_2 \in \xi$ if and only if $Y_1 \cap Y_2 \in \xi$.

5) $Y \in \xi$ if and only if $\xi \cap cl_{\hat{\mathbb{R}}} Y$.

2. Some Properties of the Space $\hat{\mathbb{R}}$

Lemma 2.1. Let $\xi \in \hat{\mathbb{R}}$. For any $X \in \xi$ and any $W \in \mathcal{B}$, the set $C_{X+W}$ is a neighborhood of $\xi$, and the sets of this form provide a basis for the neighborhoods of $\xi$.

Proof. Let $X \in \xi$ and $W \in \mathcal{B}$ and let $(X + W)^r = \mathbb{R} \setminus (X + W)$ and $C_Y^r = \hat{\mathbb{R}} \setminus C_Y$ for a subset $y$ of $\mathbb{R}$. Then it is clear that $\xi \in C_{X+W}^r$, and that $C_{X+W}^r \subseteq C_{X+W}$. Therefore, $C_{X+W}$ is a neighborhood of $\xi$.

Now suppose that $Y \subseteq \mathbb{R}$ and that $Y \not\in \xi$. Then $Y \cap (W + X) = \emptyset$ for some $X \in \xi$ and some $W \in \mathcal{B}$. Let $W_1 \in \mathcal{B}$ symmetric and $W_1 + W_1 \subseteq W$. Then clearly $C_{X+W_1} \subseteq \hat{\mathbb{R}} \setminus C_Y$ since $(W_1 + X) \cap (W_1 + Y) = \emptyset$. □

We remind that a topological space $X$ is called an $F$-space if for each $f \in C(X)$, the sets $Negf = \{ x \in X : f(x) < 0 \}$ and $Posf = \{ x \in X : f(x) > 0 \}$ are completely separated, that is, there exists a mapping $h \in C(X)$ such that $h(x) = 0$ if $x \in Posf$ and $h(x) = 1$ if $x \in Negf$.

Theorem 2.1 $\gamma\mathbb{R}$ is not an $F$-space.

Proof. Let $f(x) = \sin x$, and let $\xi \in \gamma\mathbb{R}$ such that $\xi \in cl_{\hat{\mathbb{R}}} \{ 2\pi n \}_{n \in \mathbb{N}}$. Because of the fact that $f$ is uniformly continuous it extends to a continuous function $\tilde{f}$ from $\hat{\mathbb{R}}$ to $\hat{\mathbb{R}}$. Clearly, any neighborhood of $\xi$ contains a point $\eta \in cl_{\hat{\mathbb{R}}} \{ 2\pi n + \delta \}_{n \in \mathbb{N}} \setminus \mathbb{R}$ and a point $\zeta \in cl_{\hat{\mathbb{R}}} \{ 2\pi n - \delta \}_{n \in \mathbb{N}} \setminus \mathbb{R}$ for some $\delta \in \{ 0, \pi \}$. Since $\tilde{f}(\eta) > 0$ and $\tilde{f}(\zeta) < 0$, $\xi \in cl_{\hat{\mathbb{R}}} \{ \mu \in \gamma\mathbb{R} : \tilde{f}(\mu) > 0 \} \cap cl_{\hat{\mathbb{R}}} \{ \mu \in \gamma\mathbb{R} : \tilde{f}(\mu) < 0 \}$ which is a contradiction. □

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Theorem 2.2 Let $\xi, \eta, \zeta \in \mathbb{R}^+ \setminus \mathbb{R}$ and $Y \in \xi$. Then for any $k > 0$, there is a sequence $(y_r) \subseteq Y$ such that $y_{r+1} - y_r > k$ for every $r \in \mathbb{N}$ and $\{y_r : r \in \mathbb{N}\} \in \xi$.

Proof. Let $m \in \mathbb{N}$ with $m > k$. Then, either

$$\bigcup_{n \in 2\mathbb{N} - 1} [nm, (n + 1)m] \in \xi \quad \text{or} \quad \bigcup_{n \in 2\mathbb{N}} [nm, (n + 1)m] \in \xi.$$ 

Suppose that $X_1 = \bigcup_{n \in 2\mathbb{N} - 1} [nm, (n + 1)m] \in \xi$, and that $A$ denote $2\mathbb{N} - 1$, then either

$$\bigcup_{n \in A} [nm, (n + \frac{1}{2})m] \in \xi \quad \text{or} \quad \bigcup_{n \in A} [(n + \frac{1}{2})m, (n + 1)m] \in \xi.$$ 

Suppose that $X_2 = \bigcup_{n \in A} [nm, (n + \frac{1}{2})m] \in \xi$. If we proceed in this way, we can define a sequence of sets $(X_n)$ with the following properties:

i) $X_n \in \xi$;

ii) each $X_n$ can be written as $\bigcup_{r=1}^{\infty} I_{n,r}$, where $I_{n,r}$ is a closed interval of length $\frac{m}{2^{n-1}}$;

iii) for each $n$, $d(I_{n,r}, I_{n,r'}) > m$ if $r \neq r'$;

iv) for each $n$ and $r$, $I_{n+1,r} \subseteq I_{n,r}$;

v) for each $r = 1, 2, 3, \ldots$, there will be a unique point $x_r \in \bigcap_{n=1}^{\infty} I_{n,r}$.

Let $X = \{x_r : r \in \mathbb{N}\}$. It is clear that $x_r < x_{r+1}$ holds for each $r \in \mathbb{N}$. We claim that $X \in \xi$. To see this, let $Z \in \xi$ and let $\epsilon > 0$. Choose $n \in \mathbb{N}$ so that $\frac{m}{2^n-1} < \frac{\epsilon}{2}$. Since $X_n \in \xi$, there will be a point $x \in X_n$ such that $d(x, Z) < \frac{\epsilon}{2}$. If $x \in I_{n,r}$, then $d(x, x_r) < \frac{\epsilon}{2}$. Hence, $d(x, Z) < \epsilon$. Thus, $(X + (-\epsilon, \epsilon)) \cap Z = \emptyset$ and so $X \in \xi$.

Let $\delta \in \mathbb{R}$ and $0 < \delta < \frac{\epsilon}{8}$. Then

$$V = \{x_r : d(x_r, Y) < \delta\} \in \xi.$$ 

For otherwise we should have

$$V' = \{x_r : d(x_r, Y) \geq \delta\} \in \xi.$$ 

This is impossible since $((-\frac{\delta}{8}, \frac{\delta}{8}) + V') \cap Y = \emptyset$. Now, since the finite set $\{x_r : r \leq \frac{1}{\delta}\} \notin \xi$,

$$X_\delta = \{x_r : x_r \in V, \frac{1}{r} < \delta\} \in \xi.$$ 

Now for each $r$, choose $y_r \in Y$ with $d(x_r, y_r) < d(x_r, Y) + \frac{1}{r}$. We shall show that

$$Y_\delta = \{y_r : x_r \in X_\delta\} \in \xi.$$ 

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Let $0 < \epsilon < \delta$. Then if $x_r \in X_r$, we have that $y_r \in Y_\delta$ and $d(y_r, x_r) < 2\delta$. Thus, $X_r \subseteq (Y_\delta + (-2\delta, 2\delta))$ and so $(Y_\delta + (-2\delta, 2\delta)) \in \xi$ and that $Y_\delta \in \xi$. Since $x_{r+1} - x_r > m$, $d(y_{r+1}, x_{r+1}) < 2\delta < \frac{m-k}{2}$ and $d(y_{r+1}, x_{r+1}) < \frac{m-k}{2}$, we have $y_{r+1} - y_r > k$. \qed

**Remark 2.1** It is quite easy to prove that if $\xi \in \text{cl}_R - R \setminus R$ and $Y \in \xi$, then for any $k > 0$ there is a sequence $(y_r) \subseteq Y$ such that $y_r - y_{r-1} > k$ for every $r \in \mathbb{N}$, and \{ $y_r : r \in \mathbb{N}$ \} $\in \xi$.

A point of $\hat{R}$ is called a remote point if it does not belong to the closure of any discrete subspace of $R$. As a consequence of Theorem 2.2, $\hat{R}$ has no remote points, but under the continuum hypothesis the set of remote points of $\beta R$ is dense in $R^*$ (see[5]).

**Theorem 2.3** If $\xi \in \text{cl}_R - R^+ \setminus R^*$, then every neighborhood of $\xi$ contains a topological copy of $\beta \mathbb{N} \setminus \mathbb{N}$.

**Proof.** We first note that for each $Y \in \xi$ and $W \in \mathcal{B}$

\[ C_{Y+W} = \{ \eta \in \hat{R} : Y + W \in \eta \} \]

is a neighborhood of $\xi$, and \{ $C_{Y+W} : Y \in \xi$, $W \in \mathcal{B}$ \} forms a base for the neighborhoods system of $\xi$.

Now let $G$ be a neighborhood of $\xi$. Then there exists $Y \in \xi$ and $W \in \mathcal{B}$ satisfying $C_{Y+W} \subseteq G$. There is a sequence $(x_n) \subseteq Y$ with $x_{n+1} - x_n \geq 1$ by Theorem 2.2. Hence, $C_{(x_n)+W}$ is a neighborhood of $\xi$ which is contained in $G$. Let $H = C_{(x_n)+W}$. It is easy to see that $H \supseteq \text{cl}_R \{ (x_n) \} \setminus \{ x_n \}$.

Now we define a mapping $h$ from $\mathbb{R}$ to $\hat{R}$ such that $h(n) = x_n$. Clearly, the mapping $\beta$ is continuous and it extends to a continuous mapping $h^\beta$ from $\beta \mathbb{N}$ onto $\text{cl}_R(x_n)$. Let $\xi$ and $\eta$ be two distinct points in $\beta \mathbb{N}$. Then there will be $U \subseteq \eta$ and $V \in \xi$ satisfying $U \cap V = \emptyset$ and so $h^\beta(U) \cap h^\beta(V) = \emptyset$ because of the fact that $h$ is one to one on $\mathbb{N}$. Let $W$ be the interval $(-\frac{1}{3}, \frac{1}{3})$. Clearly, $(h^\beta(U) + W) \cap (h^\beta(V) + W) = \emptyset$ since $|h^\beta(n) - h^\beta(m)| \geq 1$ for every $n, m \in \mathbb{N}$ with $n \in U, m \in V$. Also,

\[ \text{cl}_R h^\beta(U) \cap \text{cl}_R h^\beta(V) = \emptyset. \]

Since $\xi \in \text{cl}_R - U$, $h^\beta(\xi) \in \text{cl}_R h^\beta(U)$ and since $\eta \in \text{cl}_R - V$, $h^\beta(\eta) \in \text{cl}_R h^\beta(V)$. Therefore, $h^\beta(\xi) \neq h^\beta(\eta)$. Hence, $h$ is one to one on $\beta \mathbb{N}$.

It is a well-known fact that a one to one continuous mapping of a compact space onto a Hausdorff space is a homeomorphism. Therefore, $h^\beta$ is a homeomorphism between $[\text{cl}_R(x_n)] \setminus \{ x_n \}$ and $\mathbb{N}^*$. \qed
 Remark 2.2 It can be easily proved that every neighborhood of $\xi$ in $\text{cl}_{R^+}R^+\setminus R$ contains a topological copy of $\beta\mathbb{N}\setminus\mathbb{N}$.

 Corollary 2.1 No point in $\gamma R$ has a countable base of neighborhoods in $\gamma R$.

 Proof. If $\xi \in \gamma R$, by Theorem 2.3 there is a subset $X$ of $R$ such that $\xi \in X$ and $X$ is homeomorphic to $\mathbb{N}^\ast$.

 We may suppose that $\xi \in \text{cl}_{R^+}R^+\setminus R^+$ and that it has a countable base of neighborhoods of $(U_n)$ in $\gamma R$. Then $(\cap U_n) \cap X$ is a singleton. But this is a contradiction, since $(\cap U_n) \cap X$ is homeomorphic to a nonempty $G_\delta$-set of $\mathbb{N}^\ast$, and it is a well-known fact that in $\mathbb{N}^\ast$ every nonempty $G_\delta$-set has nonempty interior (see[5]).

 It is immediate from Corollary 2.1 that $\bar{R}\setminus R$ is not metrizable and has not have a countable base.

 Theorem 2.4 If $\eta \in \gamma R$, there is no sequence $(x_n)$ in $R$ converging to $\eta$.

 Proof. We may suppose that $\eta \in \text{cl}_{R^+}R^+\setminus R$ and that there is such a sequence $(x_n)$ in $R$. By Theorem 2.2, we may suppose that $x_{n+1} - x_n \geq 1$ for all $n \in \mathbb{N}$. Clearly, the sequence $(x_n)$ can not be bounded, otherwise it would have a subsequence $(x_{n_r})$ which converges to a real number $k$ which is a contradiction since $x_{n_r+1} - x_{n_r} \geq 1$ and $(x_{n_r})$ also converges to $\eta$.

 We define a function $f$ from $R$ to $R$ as follows:

$$f(x_n) = \begin{cases} 0 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd} \end{cases}$$

and complete the definition by piecewise linearity. It is easy to see that $f$ is uniformly continuous and hence that it extends to a continuous function $\tilde{f}$ from $\bar{R}$ to $\bar{R}$ (cf.[4]). Therefore, $\tilde{f}(\eta) = \lim_n f(x_{2n}) = 0$ and $\tilde{f}(\eta) = \lim_n f(x_{2n+1}) = 1$ which is a contradiction since $(x_{2n})$ and $(x_{2n+1})$ both converges to $\eta$.

 We state the following lemma that will be used later on, and its proof is straightforward.

 Lemma 2.2 Let $\xi \in \text{cl}_{R^+}R^+\setminus R$ and $(x_n)$ be a sequence in $R$ such that $x_{n+1} - x_n \geq 1$ and $\xi \in \text{cl}_{R}(x_n)$. Then $U = \{ A \subseteq \mathbb{N} : \xi \in \text{cl}_{R}(x_n)_{n \in A} \}$ is an ultrafilter on $\mathbb{N}$.

 Remark 2.3 We can easily proved that if $\xi \in \text{cl}_{R^+}R^+\setminus R$ and if $(x_n)$ is a sequence in $R$ such that $x_n - x_{n+1} \geq 1$ and $\xi \in \text{cl}_{R^+}(x_n)$, then $U = \{ A \subseteq \mathbb{N} : \xi \in \text{cl}_{R^+}(x_n)_{n \in A} \}$ is an ultrafilter on $\mathbb{N}$. 457
Lemma 2.3 For each $m \in \mathbb{N}$, there is a unique point $\xi_m \in \text{cl}_{\mathbb{R}} \mathbb{R}^+ \setminus \mathbb{R}$ satisfying

$$\xi_m \in \bigcap_{A \in U} \text{cl}_{\mathbb{R}} \{ x_n + \frac{1}{m} \}_{n \in A}.$$  

Proof. Clearly, for each $m \in \mathbb{N}$, $\{ x_n + \frac{1}{m} \}_{n \in A}, A \in U$ has near finite intersection property and so such near ultrafilter exists. Now we will show that such near ultrafilter is unique. To see this, suppose that $\eta, \zeta \in \bigcap_{A \in U} \text{cl}_{\mathbb{R}} \{ x_n + \frac{1}{m} \}_{n \in A}$. If $\eta \neq \zeta$, there exists $Y \in \eta$ and $Z \in \zeta$ and $W \in B$ such that $(Y + W) \cap (Z + W) = \emptyset$. Now, for any $A \in U$, $\{ x_n + \frac{1}{m} \}_{n \in A} \in \eta$ and $\{ x_n + \frac{1}{m} \}_{n \in A} \in \zeta$. We claim that $A = \{ n \in \mathbb{N} : x_n + \frac{1}{m} \in Y + W \} \subset U$ for all $W \in B$. To see this suppose that $A \notin U$, then there is a set $B \in U$ such that $A \cap B = \emptyset$. Since $\{ x_n + \frac{1}{m} \}_{n \in B} \in \eta$ and $Y \in \eta$, for all $W \in B$, $\{ x_n + \frac{1}{m} \}_{n \in B} \cap (Y + W) \neq \emptyset$ which implies that $x_{n_0} + \frac{1}{m} \in Y + W$ for some $n_0 \in B$. Hence, $n_0 \in A$ and so $A \cap B \neq \emptyset$ and it is a contradiction. Hence, $A \in U$. Similarly, $C = \{ n \in \mathbb{N} : x_n + \frac{1}{m} \in Z + W \} \subset U$. Therefore, there exists $n \in A \cap C$, and so $x_n + \frac{1}{m} \in Y + W$ and $x_n + \frac{1}{m} \in Z + W$ which implies that $(Y + W) \cap (Z + W) \neq \emptyset$, it is a contradiction. Therefore, for all $Y \in \eta$ and $Z \in \zeta$ and $W \in B$, $(Y + W) \cap (Z + W) \neq \emptyset$ and so $\xi = \zeta$. 

Remark 2.4 We can easily prove that for each $m \in \mathbb{N}$, there is a unique point $\xi_m \in \text{cl}_{\mathbb{R}} \mathbb{R}^+ \setminus \mathbb{R}$ satisfying $\xi_m \in \bigcap_{A \in U} \text{cl}_{\mathbb{R}} \{ x_n - \frac{1}{m} \}_{n \in A}$. 

Theorem 2.5 Every point $\xi$ in $\mathbb{R} \setminus \mathbb{R}$ is a limit point of a countable subset of $\mathbb{R}$ which does not contain $\xi$.

Proof. We may suppose that $\xi \in \text{cl}_{\mathbb{R}} \mathbb{R}^+ \setminus \mathbb{R}$, the case $\xi \in \text{cl}_{\mathbb{R}} \mathbb{R}^+ \setminus \mathbb{R}$ can be proved similarly. To see this, we will show that $\xi \in \text{cl}_{\mathbb{R}} \{ \xi_m \}$. Let $C_{Y+W_1}$ be a basic neighborhood of $\xi$. We will show that there exists $m_0 \in \mathbb{N}$ such that for every $W \in B$ and $A \in U$, $\{ x_n + \frac{1}{m_0} \}_{n \in A} \cap (Y + W_1) = \emptyset$. Then $\{ x_n + \frac{1}{m_0} \}_{n \in A} \cap (Y + W_1) = \emptyset$. Let $W_2$ be symmetric such that $W_2 + W_2 \subset W_1$. Then $\{ x_n + \frac{1}{m_0} \}_{n \in A} \cap (Y + W_2) = \emptyset$ which implies that $\{ x_n \}_{n \in A} \cap (Y + W_2 - \frac{1}{m_0}) = \emptyset$. Let $m_0$ be the smallest integer such that $\frac{1}{m_0} \in W_2$, then $W_2 - \frac{1}{m_0} = W_3$ is in $B$ and $\{ x_n \}_{n \in A} \cap (Y + W_3) = \emptyset$ which is a contradiction since $Y \in \xi$ and $\{ x_n \}_{n \in A} \notin \xi$ for all $A \in U$. 

We remind the reader that a point of a topological space is called a P-point if every $G_\delta$-set containing the point is a neighborhood of the point. Since no P-point can be a non-trivial limit of any sequence (see [5]), we have the following Corollary.
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Corollary 2.2 \( \mathbb{R} \setminus \mathbb{R} \) has no P-point.

References


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