Tr. J. of Mathematics
21 (1997) , 437 – 441.

ON EXISTENCE OF A MEROMORPHIC FUNCTION WITH INFINITELY MANY OF DEFICIENT FUNCTIONS

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Abstract

Let $\{a_n(z) : \{a_n(z)\}_{n=1}^{w}, \quad w \leq \infty\}$ be a finite or denumerable set of entire functions satisfying

$$T(r, a_n) \leq d_0 r^{\alpha} + d_n, \quad n = 1, 2, \ldots,$$

where $d_0, d_n > 0$ do not depend on $r, 0 < \alpha < 1$.

In this paper we show that there is a meromorphic function of order 1 and normal type having $a_1(z), a_2(z), \ldots, a_n(z), \ldots,$ as deficient functions.

1. Introduction

Goldberg [1, 2] constructed the first examples of meromorphic functions $f(z)$ with infinitely many deficient values in 1954. Modifications of these examples were presented by W.K. Hayman in his monograph [3].

These meromorphic functions $f(z)$ could have any positive order and a prescribed finite or denumerable set of deficient values.

We shall prove the following theorem by using Hayman's idea.

Theorem. *Let $\{a_n(z) : \{a_n(z)\}_{n=1}^{w}, \quad w \leq \infty\}$ be a finite or denumerable set of entire functions satisfying the conditions*

$$T(r, a_n) \leq d_0 \; r^{\alpha} + d_n, \quad n = 1, 2, \ldots, \tag{1.1}$$

where $d_0, d_n > 0$ do not depend on r and $0 < \alpha < 1$. Then there exists a meromorphic function $f(z)$ of order 1 and normal type such that

$$\delta(a_n, f(z)) > 0, \quad n = 1, 2, \ldots$$

This theorem seems to be of interest in connection with recent investigations related to deficient functions (see e.g. [4], p.34-41).

2. Proof of Theorem

Without loss of generality we can assume $w = \infty$, otherwise we can consider the sequence $a_1(z), a_2(z), \dots, a_w(z), a_w(z), a_w(z), \dots$.

Let $\{\eta_v\}_{v=1}^{\infty}$ be a decreasing sequence of positive numbers such that $\sum_1^{\infty} \eta_v = 1$. We set $\eta_1 = \eta_0$ and for all $1 < n < \infty$ define $\theta_0 = 0$, $\theta_n = \pi \sum_{v=0}^{n-1} \eta_v$, $(n = 1, \text{ to } \infty)$. We see that $\{\theta_n\}_{n=1}^{\infty}$ is an increasing sequence and $0 \le \theta_n < \pi$. Note that the well-known inequality,

$$\log^+ M(r, f) \le 3T(2r, f)$$

is valid for any entire function $f(z)$. Therefore using (1.1), we have

$$\log^+ M(r, a_n(z)) \le 3(d_0 2^\alpha r^\alpha + d_n), \quad n = 1, 2, \dots \tag{1.2}$$

whence

$$|a_n(z)| \le e^{3d_n} \exp(3d_0 2^\alpha r^\alpha).$$

We now choose a sequence $\{c_n\}_{n=1}^{\infty}$ of positive numbers such that

$$S_1 = \sum_{n=1}^{\infty} c_n < \infty, \qquad S_2 = \sum_{n=1}^{\infty} c_n e^{3d_n} < \infty$$

and define

$$F_1(z) = \sum_{n=1}^{\infty} c_n a_n(z) \exp(z e^{-i\theta_n}), \qquad F_2(z) = \sum_{n=1}^{\infty} c_n \exp(z e^{-i\theta_n}),$$

$$f(z) = F_1(z)/F_2(z).$$

We see that for $|z| = r$,

$$\begin{aligned} |F_1(z)| &= \left| \sum_{n=1}^{\infty} c_n a_n(z) \exp(z e^{-i\theta_n}) \right| \le \sum_{n=1}^{\infty} c_n |a_n(z)| e^r \\ &\le \sum_{n=1}^{\infty} c_n e^r e^{3d_n} \exp(3d_0 2^\alpha r^\alpha) \\ &\le S_2 \exp(3d_0 2^\alpha r^\alpha + r), \\ |F_2(z)| &\le \sum_{n=1}^{\infty} c_n e^r < S_1 \exp(r), \end{aligned}$$

and so

$$\begin{aligned} T(r, f) &< T(r, F_1) + T(r, F_2) + O(1) \le \log^+ M(r, F_1) + \log^+ M(r, F_2) + O(1) \\ &\le 2r + o(r), \qquad \text{as} \qquad r \to \infty. \end{aligned} \tag{1.3}$$

Suppose that $n \geq 1$ and that

$$\theta_n - \frac{1}{3}\pi\eta_n \leq \theta \leq \theta_n + \frac{1}{3}\pi\eta_n. \tag{1.4}$$

For $v < n$ we have the following inequalities

$$\theta \geq \theta_n - \frac{1}{3}\pi\eta_n, \quad \theta_v \leq \theta_{n-1}, \quad \eta_{n-1} > \eta_n$$

$$\theta - \theta_v \geq \theta_n - \theta_{n-1} - \frac{1}{3}\pi\eta_n = \pi\eta_{n-1} - \frac{1}{3}\pi\eta_n > \frac{2}{3}\pi\eta_n,$$

and for $v > n$, we have

$$\theta_v - \theta \geq \theta_{n+1} - \theta_n - \frac{1}{3}\pi\eta_n = \pi\eta_n - \frac{1}{3}\pi\eta_n = \frac{2}{3}\pi\eta_n.$$

Hence, for all $v \neq n$, we have

$$\pi \geq |\theta - \theta_v| \geq \frac{2}{3}\pi\eta_n.$$

$$\cos(\theta - \theta_v) \leq \cos(\frac{2}{3}\pi\eta_n).$$

We now define

$$\begin{aligned} F_{1,n}(z) &= F_1(z) - c_n a_n \exp(ze^{-i\theta_n}), \\ F_{2,n}(z) &= F_2(z) - c_n \exp(ze^{-i\theta_n}). \end{aligned}$$

Thus if θ lies in the range (1.4) and $z = r\exp(i\theta)$, then we have

$$\begin{aligned} F_1(z) &= c_n a_n(z)\exp(ze^{-i\theta_n}) + F_{1,n}(z), \\ |F_{1,n}(z)| &\leq \sum_{v=1, v\neq n}^{\infty} c_v |a_v(z)| \exp(r\cos(\theta - \theta_v)) \\ &\leq S_2 \exp(3d_0 2^\alpha r^\alpha)\exp(r\cos\frac{2}{3}\pi\eta_n). \end{aligned} \tag{1.5}$$

In the similar way, for θ lying in the range (1.4), we have

$$|F_{2,n}(z)| \leq S_1 \exp(r\cos\frac{2}{3}\pi\eta_n) \tag{1.6}$$

and so

$$\begin{aligned} |F_2(z)| &= |c_n \exp(ze^{-i\theta_n}) + F_{2,n}(z)| \geq |c_n \exp(ze^{-i\theta_n})| - |F_{2,n}(z)| \\ &\geq c_n \exp(r\cos\frac{\pi}{3}\eta_n) - S_1\exp(r\cos\frac{2}{3}\pi\eta_n) \\ &> \frac{c_n}{2}\exp(r\cos\frac{\pi}{3}\eta_n) \end{aligned} \tag{1.7}$$

for all sufficiently large r. Tuhs, we have

$$\begin{aligned} F_1(z) - a_n(z)F_2(z) &= c_n a_n(z)\exp(ze^{-i\theta_n}) + \sum_{v\neq n}^{\infty} c_v a_v(z)\exp(ze^{-i\theta_v}) \\ &\quad -c_n a_n(z)\exp(ze^{-i\theta_n}) - a_n(z)\sum_{v\neq n}^{\infty} c_v \exp(ze^{-i\theta_v}) \\ &= F_{1,n}(z) - a_n(z)F_{2,n}(z). \end{aligned}$$

Using (1.5), (1.6), (1.2) and (1.7), we obtain

$$\begin{aligned} |f(z) - a_n(z)| &\leq \frac{|F_1(z) - a_n(z)F_2(z)|}{|F_2(z)|} \leq \frac{|F_{1,n}(z)| + |a_n(z)|\,|F_{2,n}(z)|}{|F_2(z)|} \\ &= \frac{2\exp(3d_0 2^\alpha r^\alpha)}{c_n}(S_2 + S_1 e^{3d_n})\exp(r(\cos\frac{2}{3}\pi\eta_n - \cos\frac{\pi}{3}\eta_n)) \\ &= \frac{2\exp(3d_0 2^\alpha r^\alpha)}{c_n}(S_2 + S_1 e^{3d_n})\exp(-2r\sin(\frac{\pi}{2}\eta_n)\sin(\frac{\pi}{6}\eta_n)). \end{aligned}$$

From this, we deduce that

$$\begin{aligned} \log^+\left|\frac{1}{f(z) - a_n(z)}\right| &\geq 3d_0 2^\alpha r^\alpha + 2r\sin\frac{\pi}{2}\eta_n \sin\frac{\pi}{6}\eta_n + O(1) \\ &\geq 3d_0 2^\alpha r^\alpha + \frac{2r}{3}\eta_n^2 + O(1) \\ &= \frac{2}{3}\eta_n^2 r + o(r), \quad \text{as} \quad r\to\infty, \end{aligned}$$

and so

$$\begin{aligned} m(r, a_n, f) &\geq \frac{1}{2\pi}\int_{\theta_n - \frac{1}{3}\pi\eta_n}^{\theta_n + \frac{1}{3}\pi\eta_n} \log^+\left|\frac{1}{f(r_e^{i\theta}) - a_n(r_e^{i\theta})}\right| d\theta \\ &\geq \frac{2r}{9}\eta_n^3 + o(r), \quad \text{as} \quad r\to\infty. \end{aligned}$$

Thus in view of (1.3), we have

$$\delta(a_n(z), f(z) = \lim\inf_{r\to\infty}\frac{m(r, a_n, f)}{T(r,f)} \geq \lim\inf_{r\to\infty}\frac{\frac{2r}{9}\eta_n^3 + O(r)}{2r + O(r)} = \frac{1}{9}\eta_n^3 > 0.$$

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Received 28.06.1996