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On isolated gaps in numerical semigroups

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Abstract: A numerical semigroup is said to be perfect if it does not contain any isolated gaps. In this paper, we will look at some basic properties of isolated gaps in numerical semigroups. In particular, we will see how they are related to elements of the Apéry set. We will use these properties to find all of the isolated gaps in a numerical semigroup of embedding dimension two and demonstrate a simple method of generating some examples of perfect numerical semigroups of embedding dimension three.

Key words: Numerical semigroup, frobenius number, gap, chinese remainder theorem

1. Introduction

Let $\mathbb{N}$ denote the set of all nonnegative integers. A *numerical semigroup* is an additive submonoid S of $\mathbb{N}$ such that $\mathbb{N}\backslash S$ is finite. If S is a numerical semigroup and $A \subseteq S$, we say that A is a *generating set* of S if for every $s \in S$ there exist $a_1, \ldots, a_k \in A$ and $n_1, \ldots, n_k \in \mathbb{N}$ such that $s = n_1a_1 + \cdots + n_ka_k$. In this case, we use the notation $S = \langle A \rangle$. It is known that every numerical semigroup has a unique finite minimal set of generators (see, for example, Theorem 2.7 of [5]). The cardinality of this minimal set of generators is called the *embedding dimension* of S.

For any numerical semigroup S, the elements of $G(S) := \mathbb{N}\backslash S$ are called the *gaps* of S. The largest element of $G(S)$ is called the *Frobenius number* of S and denoted by $F(S)$. Determining the Frobenius number of a numerical semigroup is an old and celebrated problem (see [4]).

A gap g of a numerical semigroup S is an *isolated gap* if $g-1, g+1 \in S$. We will let $I(S)$ denote the set of all isolated gaps of S. A *perfect numerical semigroup* is one in which $I(S) = \emptyset$. Perfect numerical semigroups are first explored in [1] and [2]. In [1], the authors introduce the concept and show how to define an order on all perfect numerical semigroups, allowing them to establish an algorithm which constructs all of them. In [2], the authors characterize all perfect numerical semigroups with embedding dimension three and establish formulas for some of their key elements.

In this paper, we will investigate some properties of isolated gaps in numerical semigroups. In particular, we will see how they can be located by using an important and well-known set of elements of the numerical semigroup called the Apéry set. We will then demonstrate a very simple method of constructing some perfect numerical semigroups of embedding dimension three. Specifically, start with a numerical semigroup of embedding dimension two, identify the smallest isolated gap, and include it as a third generator. This is given

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in proposition 3.1 and differs from the method described in [2]. While the method described here does not produce all numerical semigroups of embedding dimension three, it is very simple, does not involve complicated formulas, and can be used to generate examples quickly.

2. Basic properties of isolated gaps

If S is a numerical semigroup, then an important property of isolated gaps that we will use repeatedly is that they conduct isolated gaps of S into S and they conduct elements of S either into $I(S)$ or into S (implying that $S \cup I(S)$ is always a numerical semigroup). For $x \in \mathbb{N}$ and $A \subseteq \mathbb{N}$, we use the notation $x + A = \{x + a : a \in A\}$.

Proposition 2.1 *Let S be a numerical semigroup that is not perfect. If $x \in I(S)$, then $x + I(S) \subseteq S$ and $x + S \subseteq I(S) \cup S$.*

Proof Let $x \in I(S)$. Then $x-1, x+1 \in S$, and since $y+1 \in S$ for any $y \in I(S)$, we have $x + y = (x-1) + (y+1) \in S$. Hence, $x + I(S) \subseteq S$. Moreover, for any $s \in S$, we have $x - 1 + s, x + 1 + s \in S$ and so $x + s \in S$ or $x + s \in I(S)$. Therefore, $x + S \subseteq I(S) \cup S$, as required. □

Let S be a numerical semigroup such that $I(S) \neq \emptyset$ and let a be a nonzero element of S. For each $i \in \{1, \ldots, a-1\}$, let $I_i(S) := \{x \in I(S) : x \equiv i \pmod{a}\}$. For each i such that $I_i(S) \neq \emptyset$, define $h_i := \min I_i(S)$ and $k_i := \max I_i(S)$. We will call the elements h_i the *minimal isolated gaps modulo a* and the elements k_i the *maximal isolated gaps modulo a*. Let $h := \min I(S)$ and call it the *minimum isolated gap* of S. Clearly h is the smallest of all the minimal isolated gaps modulo a.

Example 2.2 *Let $S = \langle 5, 8 \rangle$. Then $S = \{0, 5, 8, 10, 13, 15, 16, 18, 20, 21, 23, 24, 25, 26, 28 \rightarrow\}$ where the notation "$28 \rightarrow$" indicates that every integer greater than 28 is an element of S. It follows that $I(S) = \{9, 14, 17, 19, 22, 27\}$. Note that each isolated gap is congruent to either 2 or 4 modulo 5, so the only minimal isolated gaps modulo 5 are $h = h_4 = 9$ and $h_2 = 17$. The maximal isolated gaps modulo 5 are $k_4 = 19$ and $k_2 = 27$.*

If S is a numerical semigroup and a is a nonzero element of S, the *Apéry set* of a in S is defined to be the set $Ap(S, a) = \{s \in S : s - a \notin S\}$. The Apéry set is considered one of the most useful tools for studying numerical semigroups. Recall that if a is a positive integer, then a complete system of residues modulo a is a set of integers such that every integer from 0 to $a - 1$ is congruent modulo a to exactly one element of the set. It is known that $Ap(S, a)$ is a complete system of residues modulo a. Moreover, $Ap(S, a) = \{0 = w_0, w_1, \ldots, w_{a-1}\}$ where, for each $i \in \{0, 1, \ldots, a-1\}$, w_i is the smallest element of S that is congruent to i modulo a (see, for example, Lemma 2.4 of [5]). Note that Example 2.2 demonstrates that the elements h_i and k_i generally do not comprise a complete system of residues modulo a.

The next two results will illustrate how the Apéry set may be used to locate the maximal and minimal isolated gaps modulo a.

Proposition 2.3 *Let S be a numerical semigroup such that $I(S) \neq \emptyset$ and let a be a nonzero element of S. If k_i is a maximal isolated gap modulo a, then $k_i + a \in Ap(S, a)$.*

Proof Since $k_i \in I(S)$, $a \in S$, and $k_i + a \equiv i \pmod{a}$, $k_i + a$ cannot be a gap of S (otherwise, by Proposition 2.1 it would be an isolated gap congruent to i modulo a, contradicting the maximality of k_i). Hence, $k_i + a \in S$. Moreover, since $k_i + a \equiv i \pmod{a}$ and $k_i + a > k_i$, it follows that $k_i + a \in Ap(S, a)$. □

Proposition 2.4 *Let S be a numerical semigroup such that $I(S) \neq \emptyset$ and let a be a nonzero element of S. If h_i is a minimal isolated gap modulo a, then $h_i + 1 \in Ap(S, a)$ or $h_i - 1 \in Ap(S, a)$.*

Proof Since $h_i \in G(S)$ and $a \in S$, we have $h_i - a \in G(S)$. Moreover, since $h_i \in I(S)$, we have $h_i + 1, h_i - 1 \in S$. Note that if $h_i - a - 1$ and $h_i - a + 1$ are both elements of S, then by the definition of isolated gap, $h_i - a$ is an isolated gap congruent to i modulo a, contradicting the minimality of h_i. Hence, at least one of $h_i - a - 1$ and $h_i - a + 1$ is a gap, implying that $h_i - 1 \in Ap(S, a)$ or $h_i + 1 \in Ap(S, a)$. □

Let S be a numerical semigroup of embedding dimension two. That is, $S = \langle a, b \rangle$ where a and b are relatively prime positive integers. Numerical semigroups of embedding dimension two have been the subject of much study due to their relatively simple and predictable structure. It is known that every numerical semigroup of embedding dimension two is *symmetric*, meaning that for each $x \in \mathbb{Z}$, $x \in S$ if and only if $F(S) - x \notin S$ (see, for example, Corollary 4.7 of [5]). Furthermore, it is known that if $S = \langle a, b \rangle$, then $Ap(S, a) = \{0, b, 2b, \ldots, (a-1)b\}$ and $F(S) = ab - a - b$ (see Propositions 2.12 and 2.13 of [5]). It should come as no surprise that the isolated gaps of these numerical semigroups are easy to locate.

If $S = \langle a, b \rangle$ and $S \neq \mathbb{N}$, then S can never be a perfect numerical semigroup. To see this note that $-1, 1 \notin S$. Thus, since S is symmetric, $F(S) + 1, F(S) - 1 \in S$. It follows that $F(S) \in I(S)$, so if $\langle a, b \rangle \neq \mathbb{N}$ then $I(S) \neq \emptyset$. In order to insist that $I(S) \neq \emptyset$ when working with numerical semigroups of embedding dimension two in the results that follow, we will frequently require that $1 < a < b$. In this case, we get the following useful consequence of Proposition 2.3.

Proposition 2.5 *Let $S = \langle a, b \rangle$ be a numerical semigroup where $1 < a < b$. If k_i and k_j are any two maximal isolated gaps modulo a, then $k_i \equiv k_j \pmod{b}$.*

Proof By Proposition 2.3, $k_i + a, k_j + a \in Ap(S, a)$. Therefore, since S has embedding dimension two, $k_i + a, k_j + a \in \{0, b, 2b, \ldots, (a-1)b\}$. This implies that b divides $k_i + a - (k_j + a)$ and so $k_i \equiv k_j \pmod{b}$. □

An obvious question is whether the previous result is also true for the minimal isolated gaps modulo a. It is, but a bit more can be shown.

Proposition 2.6 *Let $S = \langle a, b \rangle$ be a numerical semigroup where $1 < a < b$. Let k_i and k_j be two maximal isolated gaps modulo a such that $k_i < k_j$. If $m \in \mathbb{N}$ and $k_i - ma \in I(S)$, then $k_j - ma \in I(S)$.*

Proof Suppose that $k_i - ma \in I(S)$. Clearly $k_j - ma \in G(S)$ (or else $k_j = k_j - ma + ma \in S$) so we just need to show that $k_j - ma + 1, k_j - ma - 1 \in S$. Since $k_i - ma \in I(S)$ note that $k_i - ma + 1, k_i - ma - 1 \in S$. By Proposition 2.5 there exists $n \in \mathbb{N}$ such that $k_i + nb = k_j$. Since $nb \in S$, $k_j - ma + 1 = k_i - ma + 1 + nb \in S$ and $k_j - ma - 1 = k_i - ma - 1 + nb \in S$. □

Theorem 2.7 *Let $S = \langle a, b \rangle$ be a numerical semigroup where $1 < a < b$. Let k_i and k_j be two maximal isolated gaps modulo a such that $k_i < k_j$. Then $|I_i(S)| = |I_j(S)|$.*

Proof Note that Proposition 2.6 asserts that $|I_i(S)| \leq |I_j(S)|$, so we just need to prove that $|I_i(S)| \geq |I_j(S)|$. We do this by contradiction.

Suppose that $|I_i(S)| < |I_j(S)|$. Then there exists $m \in \mathbb{N}$ such that $k_i - ma, k_j - ma, k_j - (m+1)a \in I(S)$ and $k_i - (m+1)a \notin I(S)$. Hence, $k_i - ma = h_i$ and $k_j - ma > h_j$. By Proposition 2.5, there exists $n \in \mathbb{N}$

such that $k_i + nb = k_j$. Consequently, $h_i + nb \equiv j \pmod{a}$. Since $h_i \in I(S)$ and $nb \in S$, by Proposition 2.1, either $h_i + nb \in S$ or $h_i + nb \in I(S)$. We claim that $h_i + nb \in I(S)$. To see why this is true, suppose that $h_i + nb \in S$. Since $h_i \le k_i$, it must then follow that $h_i + nb \le k_j$. Thus, since $h_i + nb \equiv k_j \pmod{a}$, it follows that $k_j \in S$ (a contradiction). Therefore, we have that $h_i + nb \in I(S)$.

By Proposition 2.4, $h_i + 1 \in Ap(S,a)$ or $h_i - 1 \in Ap(S,a)$. Suppose that $h_i + 1 \in Ap(S,a)$ (a similar argument works in the case when $h_i - 1 \in Ap(S,a)$). Then $h_i + 1 = rb$ where $0 \le r < a$ and so $h_i + 1 + nb = (r+n)b$.

We claim that $h_i + 1 + nb \notin Ap(S,a)$. To see why, note that $h_i + nb = k_i - ma + nb = k_j - ma > h_j$. Hence, $h_i + 1 + nb > h_j + 1$. Since $h_i + 1 + nb \equiv h_j + 1 \pmod{a}$ and $h_j + 1 \in S$ (since $h_j \in I(S)$), it follows that $h_i + 1 + nb \notin Ap(S,a)$. Hence, we have that $r + n \ge a$. This now implies that $h_i + nb = h_i + nb + 1 - 1 = (r+n)b - 1 \ge ab - 1 > ab - a - b = F(S)$. Therefore $h_i + nb \in S$, contradicting the assertion that $h_i + nb \in I(S)$. $\square$

Note that the preceding result is not always true if S has more than two generators.

Example 2.8 *Let* $S = \langle 6, 10, 15 \rangle$. *Then* $S = \{0, 6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30 \rightarrow\}$. *Note that* $I_1(S) = \{19\}$ *and* $I_5(S) = \{11, 17, 23, 29\}$.

The next three results will complete the picture of how and where the isolated gaps in a numerical semigroup of embedding dimension two arise.

Proposition 2.9 *Let* $S = \langle a, b \rangle$ *be a numerical semigroup with* $1 < a < b$. *Let* h_i *be a minimal isolated gap modulo* a. *Then, for each positive integer* n, *either* $h_i + nb \in S$ *or* $h_i + nb$, *is another minimal isolated gap modulo* a.

Proof Note that $h_i \in I(S)$ and $nb \in S$, so by Proposition 2.1, if $h_i + nb \notin S$ then $h_i + nb \in I(S)$. Thus, we just need to show that $h_i + nb$ is a minimal isolated gap modulo a.

Let $h_i + nb \equiv j \pmod{a}$. Since $h_i + nb \in I(S)$, h_j exists and $h_i + nb \ge h_j$. Suppose that $h_i + nb > h_j$. Since $h_j \in I(S)$, $h_j + 1, h_j - 1 \in S$. Hence, $h_i + nb + 1, h_i + nb - 1 \notin Ap(S,a)$.

By Proposition 2.4, $h_i + 1 \in Ap(S,a)$ or $h_i - 1 \in Ap(S,a)$. We will show that each leads to a contradiction. If $h_i + 1 \in Ap(S,a)$ then $h_i + 1 = kb$ for some $k \in \{0, 1, \ldots, a-1\}$. It follows that $h_i + nb + 1 = (k+n)b$. Since $h_i + nb + 1 \notin Ap(S,a)$, we have that $k + n \ge a$. Hence, $h_i + nb = (k+n)b - 1 \ge ab - 1 > ab - a - b = F(S)$. This implies that $h_i + nb \in S$, a contradiction. Similarly, if $h_i - 1 \in Ap(S,a)$, then $h_i - 1 = kb$ for some $k \in \{0, 1, \ldots, a-1\}$. It follows that $h_i + nb - 1 = (k+n)b$. Since $h_i + nb - 1 \notin Ap(S,a)$, we have that $k + n \ge a$. Hence, $h_i + nb = (k+n)b + 1 \ge ab + 1 > ab - a - b = F(S)$. This implies that $h_i + nb \in S$ which is another contradiction. $\square$

Proposition 2.10 *Let* $S = \langle a, b \rangle$ *be a numerical semigroup with* $1 < a < b$. *Let* n *be the smallest nonnegative integer such that* $h + nb \in S$. *Then* $h + nb \in a\mathbb{N}$.

Proof Since $h + nb \in S$, there exist $u, v \in \mathbb{N}$ such that $h + nb = ua + vb$. Note that if $v \ge n$ then $h = ua + (v-n)b \in S$ (a contradiction). Hence, $v < n$. This implies that $h + (n-v)b = ua \in S$. Since n is minimal, we have that $v = 0$. $\square$

Theorem 2.11 *Let $S = \langle a, b\rangle$ be a numerical semigroup with $1 < a < b$. Let n be the smallest nonnegative integer such that $h + nb \in a\mathbb{N}$. Then $h, h+b, ..., h+(n-1)b$ are all the minimal isolated gaps modulo a.*

Proof By Proposition 2.10, we have that $h, h+b, ..., h+(n-1)b$ are all minimal isolated gaps modulo a. We just need to prove that this list is complete.

Suppose that h_j is any minimal isolated gap modulo a. Let $h = h_i$ and suppose that $m \in \mathbb{N}$ such that $k_j = h_i + ma$. Then $I_i(S) = \{h_i, h_i + a, \ldots, h_i + ma\}$ and by Theorem 2.7, $I_j(S) = \{h_j, h_j + a, \ldots h_j + ma\}$. Hence, $k_j > k_i$ and by Proposition 2.5, $k_j = k_i + ub$ for some $u \in \mathbb{N}$. This implies that $h_j = h_i + ub$. Since $h_j \notin S$, by hypothesis $u < n$ and the proof is complete. □

Theorem 2.11 suggests that if $S \neq \mathbb{N}$ and S has embedding dimension two, then every element of $I(S)$ has the form $h + s$ for some $s \in S$. The following example will help to illustrate what we have proven.

Example 2.12 *Consider $S = \langle 7, 12\rangle$. Then $Ap(S, 7) = \{0, 12, 24, 36, 48, 60, 72\}$. With the Apéry set, it is easy to organize all the elements (and gaps) of S into an array with one row for each residue class modulo 7. This makes it easy to locate all the isolated gaps.*

0	0	7	14	21	28	35	42	49	56	63	70 →
1						36	43	50	57	64	71 →
2											72 →
3				24	31	38	45	52	59	66	73 →
4									60	67	74 →
5		12	19	26	33	40	47	54	61	68	75 →
6							48	55	62	69	76 →

The minimal isolated gap is $h = 13$. All of the isolated gaps of S are as follows: $I_2(S) = \{37, 44, 51, 58, 65\}$, $I_4(S) = \{25, 32, 39, 46, 53\}$, and $I_6(S) = \{13, 20, 27, 34, 41\}$. Note that $|I_2(S)| = |I_4(S)| = |I_6(S)|$ and that $h = h_6 = 13$, $h + 12 = 25 = h_4$, $h + 2(12) = 37 = h_2$, and $h + 3(12) = 49 \in 7\mathbb{N}$. Since each isolated gap must be of the form $h_i + 7n$ for some $i \in \{0, \ldots, 6\}$ and some $n \in \mathbb{N}$, each isolated gap is therefore of the form $h + s$ for some $s \in S$. Observe how adding 12 to an isolated gap moves you from one "row" of isolated gaps to the next (until you ultimately land in the top row), while adding 7 to an isolated gap moves you along the row you are already in.

3. Constructing examples of perfect numerical semigroups of embedding dimension three

The results of the previous section suggest a way of easily constructing a perfect numerical semigroup of embedding dimension three. Start with a numerical semigroup of embedding dimension two, identify the smallest isolated gap, and then include it as a third generator.

Theorem 3.1 *Let $S = \langle a, b\rangle$ be a numerical semigroup where $1 < a < b$. Then $\langle a, b, h\rangle$ is a perfect numerical semigroup.*

Proof Suppose that there exists $x \in I(\langle a, b, h\rangle)$. Since $S \subseteq \langle a, b, h\rangle$ we have that $x \in G(S)$. Note that x cannot be an isolated gap of S, for then, by Theorem 2.11, $x = h + s$ for some $s \in S$, implying that $x \in \langle a, b, h\rangle$. Hence, $x - 1 \in G(S)$ or $x + 1 \in G(S)$.

Suppose that $x-1 \in G(S)$. Then since $x-1 \in \langle a,b,h\rangle$, it follows that $x-1 = nh+s$ for some $n \geq 1$ and $s \in S$. By Proposition 2.1, if $n \geq 2$ then $nh \in S$ and so $x-1 \in S$ (a contradiction). Thus, we have that $n=1$. But then, by Proposition 2.1, $x-1 \in I(S) \cup S$. Since $x-1 \in G(S)$, we have that $x-1 \in I(S)$. This implies that $x = x-1+1 \in S$ and since $S \subseteq \langle a,b,h\rangle$, we also have that $x \in \langle a,b,h\rangle$ (a contradiction).

A similar argument shows that if $x+1 \in G(S)$ then once again it follows that $x \in \langle a,b,h\rangle$. □

Example 3.2 *As in Example 2.2, let* $S = \langle 5,8\rangle$. *Recall that* $I(S) = \{9,14,17,19,22,27\}$ *so* $h=9$. *Note that* $h+5=14$ *and* $h+10=19$ *are the other isolated gaps congruent to 4 modulo 5 while* $h+8=17, h+8+5=22$, *and* $h+8+10=27$ *are the isolated gaps congruent to 2 modulo 5. The numerical semigroup* $\langle 5,8,9\rangle = \{0,5,8,9,10,13\rightarrow\}$ *is perfect.*

Identifying the value of h is made easier using the following observation.

Proposition 3.3 *Let* $S = \langle a,b\rangle$ *be a numerical semigroup where* $1 < a < b$. *Then* $h=1$ *or the ordered pair* $(h \ mod \ a, \ h \ mod \ b) \in \{(1,-1),(-1,1)\}$.

Proof By Proposition 2.4, $h-1 \in Ap(S,a)$ or $h+1 \in Ap(S,a)$. Since $Ap(S,a) = \{0,b,2b,\ldots,(a-1)b\}$, it follows that $h \equiv 1 \pmod{b}$ or $h \equiv -1 \pmod{b}$.

Since h is the smallest isolated gap modulo b, we may repeat this argument to see that $h-1 \in Ap(S,b)$ or $h+1 \in Ap(S,b)$ where $Ap(S,b) = \{0,a,\ldots,(b-1)a\}$. Thus, we also have that $h \equiv 1 \pmod{a}$ or $h \equiv -1 \pmod{a}$.

We now show that if $h \neq 1$, then $h \pmod{a} = 1$ if and only if $h \pmod{b} = -1$. Suppose that $h \pmod{a} = h \pmod{b} = 1$. Then $h-1 \in Ap(S,a) \cap Ap(S,b)$, implying that $h-1 \in \{0,b,\ldots,(a-1)b\} \cap \{0,a,\ldots,(b-1)a\}$. Since a and b are relatively prime, $h-1=0$ (a contradiction). □

Proposition 3.3 suggests that identifying the smallest isolated gap in a numerical semigroup $\langle a,b\rangle$ of embedding dimension two can be accomplished using techniques from an introductory course in Number Theory (see for example, Section 2.3 of [3]). Recall that if a and b are relatively prime positive integers and m and n are arbitrary integers, then by the Chinese Remainder Theorem the system

$$x \equiv m \pmod{a}$$
$$x \equiv n \pmod{b}$$

will have a unique solution $x \in \{0,1,\ldots,ab-1\}$. Moreover, if $u,v \in \mathbb{Z}$ and $1 = ua+vb$, then $x \equiv nua+mvb \pmod{ab}$. Proposition 3.3 says that h will be congruent modulo ab to the solution x to one of the following:

$$x \equiv 1 \pmod{a}$$
$$x \equiv -1 \pmod{b}$$

or

$$x \equiv -1 \pmod{a}$$
$$x \equiv 1 \pmod{b}$$

Since a and b are relatively prime, the Euclidean algorithm can be used to find integers u and v such that $1 = au+bv$ (and additional solutions $u_0 = u+kb$ and $v_0 = v-ka$ for any integer k). It then follows that either $h \equiv (1)au + (-1)bv \pmod{ab}$ or $h \equiv (-1)au + (1)bv \pmod{ab}$. Since one of u and v will be positive and the other negative, one of these solutions will turn out to be a nonnegative integer combination of a and b. If the value of this nonnegative integer combination is less than ab, then it is clearly an element of S. Hence, the other solution will be the gap that we are looking for.

Example 3.4 *Let $S = \langle 13, 21 \rangle$. Using the Euclidean algorithm, we see that $1 = 13(-8) + 21(5)$. Hence, either $h \equiv (1)13(-8) + (-1)21(5) \pmod{273}$ or $h \equiv (-1)13(-8) + (1)21(5) \pmod{273}$. Note that if the latter statement is true, then $h \equiv 13 \cdot 8 + 21 \cdot 5 \pmod{273}$. But $13 \cdot 8 + 21 \cdot 5 = 209$ and $209 < 13 \cdot 21$. Hence, $h \in S$. Since this is a contradiction, the former statement must be true. Hence, $h \equiv -209 \pmod{273}$, which implies that $h = 64$. It follows that $\langle 13, 21, 64 \rangle$ is perfect.*

From Proposition 3.3, we get the following corollary.

Corollary 3.5 *A numerical semigroup of embedding dimension two generated by consecutive integers has exactly one isolated gap. Furthermore, this isolated gap is the Frobenius number.*

Proof It is enough to show that the smallest isolated gap is the Frobenius number. Let $S = \langle a, a+1 \rangle$ for any $a \geq 2$. Note that $1 = (-1)(a) + (1)(a+1)$. Hence, by Proposition 3.3 we have that $h \equiv (1)(-1)(a) + (-1)(1)(a+1) \pmod{(a)(a+1)}$. Thus, $h = -2a - 1 + a^2 + a = a^2 - a - 1$. Moreover, $F(S) = (a)(a+1) - a - (a+1) = a^2 - a - 1$. Therefore, $I(S) = \{F(S)\}$, as required. $\square$

Note that the method posed by Theorem 3.1 does not always work if S has an embedding dimension greater than two, since Theorem 2.7 may not be true.

Example 3.6 *Consider $S = \langle 5, 8, 11 \rangle = \{0, 5, 8, 10, 11, 13, 15, 16, 18 \rightarrow\}$. Note that $I(S) = \{9, 12, 14, 17\}$ and so $h = 9$. It is not true that $9 + s = 12$ for some $s \in S$.*

Finally, we note also that unlike the method proposed in [2], this method cannot be used to construct all perfect numerical semigroups of embedding dimension three.

Example 3.7 *It is shown in [2] (Example 19) that $\langle 19, 32, 33 \rangle$ is perfect. If $S = \langle 19, 32 \rangle$, then $h = 417$. If $S = \langle 19, 33 \rangle$, then $h = 362$. Finally, if $S = \langle 32, 33 \rangle$, then $h = 991$. Note $\langle 19, 32, 417 \rangle$, $\langle 19, 33, 362 \rangle$, and $\langle 32, 33, 991 \rangle$ are all perfect numerical semigroups, not equal to $\langle 19, 32, 33 \rangle$.*

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