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Multivariate approximation in φ-variation for nonlinear integral operators via summability methods

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Abstract: We consider convolution-type nonlinear integral operators endowed with Musielak-Orlicz φ-variation. Our aim is to get more powerful approximation results with the help of summability methods. In this study, we use φ-absolutely continuous functions for our convergence results. Moreover, we study the order of approximation using suitable Lipschitz class of continuous functions. A general characterization theorem for φ-absolutely continuous functions is also obtained. We also give some examples of kernels in order to verify our approximations. At the end, we indicate our approximations in figures together with some numerical computations.

Key words: Convolution type operators, N-dimensional nonlinear integral operators, bounded φ-variation, summability process, order of approximation.

1. Introduction

In [3], Angeloni studied the following convolution-type integral operators in BV^{φ} spaces,

$$T_w(f;\mathbf{x}) = \int\limits_{\mathbb{R}^N} K_w(\mathbf{t}, f(\mathbf{x}-\mathbf{t}))\,d\mathbf{t} \quad (w>0,\ \mathbf{x}\in\mathbb{R}^N), \tag{1.1}$$

where $K_w : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ fulfills some suitable conditions (see [4, 6] for univariate approximations in φ-variation of these operators). Now, our purpose is to improve and generalize these operators by using general summability methods (for other results of these type of operators under summability methods, we refer to [9, 11, 12]). Dealing with Tonelli-sense BV^{φ} spaces, we mainly study the convergence in φ-variation in N-dimension. Then, we investigate the rate of approximation. In addition, a general characterization theorem for AC^{φ} spaces is obtained. In the last section, certain sequence of nonlinear kernels, which satisfy all of our generalized conditions will be pointed out. Moreover, using these kernels, we visualize our approximations to better understand why we need such kind of summability methods [16, 17]. At the end, we evalute the error of these approximations.

Recall that variation of a function was first given by Jordan in the study in [25]. Later, this theory generalized to φ-variation concept, which is also known as Musielak-Orlicz φ-variation ([27, 29, 33–35]). Then, utilizing from Tonelli's idea in the study in [31], Angeloni and Vinti extended it to N-dimensional case in their study in [5]. For further studies about φ-variation, we refer to the studies in [6–8, 10, 15, 29].

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We must remind that a summability method gives us alternative ways for the convergence of divergent sequences. Here, we refer to recent papers [1, 9, 11, 12, 19, 20, 22, 23, 30, 32, 36], to show the effect of summability methods in linear and nonlinear settings of different type of operators. Throughout the paper, we consider Bell-type summability method $\mathcal{A} = \{A^{\upsilon}\}_{\upsilon\in\mathbb{N}} = \{[a^{\upsilon}_{nk}]\}_{\upsilon\in\mathbb{N}}$ $(k, n \in \mathbb{N})$ with nonnegative real entries. A sequence $x = (x_n)_{n\in\mathbb{N}}$ is called "$\mathcal{A}$-summable to L" if both $\mathcal{A}$-transform of x, that is, $\sum_{k=1}^{\infty} a^{\upsilon}_{nk} x_k$, is convergent for each $\upsilon, n \in \mathbb{N}$, and $\lim_{n\to\infty} \sum_{k=1}^{\infty} a^{\upsilon}_{nk} x_k = L$ uniformly in υ. This approximation is denoted by "$\mathcal{A}\text{-}\lim x = L$". Furthermore, we say that $\mathcal{A}$ is regular, if $\lim_{n\to\infty} x_n = L$ implies $\mathcal{A}\text{-}\lim x = L$. It is shown in [17] that $\mathcal{A}$ is regular if and only if the following expressions hold

- for each $k \in \mathbb{N}$, $a^{\upsilon}_{nk} \to 0$ (uniformly in υ) as $n \to \infty$,
- $\sum_{k=1}^{\infty} a^{\upsilon}_{nk} \to 1$ (uniformly in υ) as $n \to \infty$,
- for each $\upsilon, n \in \mathbb{N}$, $\sum_{k=1}^{\infty} |a^{\upsilon}_{nk}| < \infty$ and there exist integers N, M such that $\sup_{n\geq N, \upsilon\in\mathbb{N}} \sum_{k=1}^{\infty} |a^{\upsilon}_{nk}| \leq M < \infty$.

We denote by Φ that the class of all convex functions $\varphi : \mathbb{R}^{+}_{0} \to \mathbb{R}^{+}_{0}$ such that φ is continuous, nondecreasing, $\varphi(0) = 0$, $\varphi(x) > 0$ for all $x > 0$ and $\lim_{x\to\infty} \varphi(x) = \infty$.

From now on, we assume that $\varphi \in \Phi$ and $\lim_{x\to 0^{+}} \varphi(x)/x = 0$. Note that due to the lack of integral representation of BV^{φ}-spaces (defined below), we require this limit condition to be able to use φ-modulus of continuity (see [2]).

Since we deal with N-dimensional Tonelli-sense φ-variation, we need the following notations ([3, 5]).

- For any $\mathbf{x} := (x_1, \ldots, x_N)$, if we are considering the j-th term of $\mathbf{x}$, we write $\mathbf{x} := (x'_j, x_j)$, where $x'_j := (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \in \mathbb{R}^{N-1}$,
- $|\mathbf{x}|$ denotes the Euclidean norm of $\mathbf{x}$,
- any closed N-dimensional interval will be denoted by $I := \Pi_{i=1}^{N} [a_i, b_i]$, and the following $N-1$-dimensional interval $\Pi_{i=1, i\neq j}^{N} [a_i, b_i]$ will be denoted by $I'_j := [a'_j, b'_j]$ for $j = 1, \ldots, N$,
- by $g_j(x_j)$, we mean the j-th section of f, namely for every fixed $x'_j \in \mathbb{R}^{N-1}$, $g_j(x_j) := f(x'_j, x_j)$ for $j = 1, \ldots, N$,
- we define the following $(N-1)$-dimensional integral such that

$$\Phi^{\varphi}_{j}(f, I) := \int_{a'_j}^{b'_j} V^{\varphi}_{[a_j, b_j]}[f(x'_j, \cdot)] dx'_j \text{ for every } j = 1, \ldots, N,$$

  where $V^{\varphi}_{[a_j, b_j]}[f(x'_j, \cdot)]$ denotes the one-dimensional φ-variation of j-th section of f,
- recall that, one-dimensional φ-variation of $g : [a, b] \to \mathbb{R}$ is defined by

$$V^{\varphi}_{[a,b]}[g] := \sup_{U} \sum_{i=1}^{n} \varphi\left(|g(x_i) - g(x_{i-1})|\right),$$

where U forms a partition of $[a,b]$. Here g is called "bounded φ-variation", if there exists a $\lambda > 0$ such that $V^{\varphi}_{[a,b]}[\lambda g] < \infty$. The space of all functions of bounded φ-variation will be denoted by $BV^{\varphi}[a,b]$,

- the Euclidean norm of $(\Phi_1^{\varphi}(f,I), \ldots, \Phi_N^{\varphi}(f,I))$ is denoted by $\Phi^{\varphi}(f,I)$ and therefore $\Phi^{\varphi}(f,I) = \infty$ if $\Phi_j^{\varphi}(f,I) = \infty$ for some $j = 1, \ldots, N$.

Considering the above expressions, Tonelli-sense multidimensional φ-variation of f on $I \subset \mathbb{R}^N$ is given as

$$V_I^{\varphi}[f] := \sup_{\{J_1,\ldots,J_m\}} \sum_{k=1}^{m} \Phi^{\varphi}(f, J_k),$$

where $\{J_1, \ldots, J_m\}$ forms a partition of I. Then φ-variation of f on $\mathbb{R}^N$ is given as

$$V^{\varphi}[f] := \sup_{I \subset \mathbb{R}^N} V_I^{\varphi}[f].$$

Finally, the space of functions of bounded φ-variation on $\mathbb{R}^N$ is defined by

$$BV^{\varphi}\left(\mathbb{R}^N\right) := \left\{ f \in L^1\left(\mathbb{R}^N\right) : \ \exists \lambda > 0, \ V^{\varphi}[\lambda f] < \infty \right\}.$$

An important property of φ-variation in one dimension is that if $f_1, \ldots, f_n \in L^1(\mathbb{R})$, then

$$V^{\varphi}\left[\sum_{i=1}^{n} f_i\right] \leq \frac{1}{n} \sum_{i=1}^{n} V^{\varphi}[n f_i]. \tag{1.2}$$

In N-dimension ($N > 1$), similar expression is hold by changing $\frac{1}{n}$ with $\frac{1}{\sqrt{n}}$ ([5]).

In our approximating operator, we will use φ-absolutely continuous functions. For this reason, we also need the following definitions.

Let $g : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be given. Then g is called "φ-absolutely continuous", if one can find a $\lambda > 0$ satisfying that

$$\sum_{k=1}^{m} \varphi\left(\lambda \left| g(\beta_i) - g(\alpha_i) \right|\right) < \varepsilon$$

for every (finite) collections of nonoverlapping intervals $[\alpha_i, \beta_i] \subset [a,b]$, $(i = 1, \ldots, m)$ such that

$$\sum_{k=1}^{m} \varphi(\beta_i - \alpha_i) < \delta.$$

For a given $f : \mathbb{R}^N \to \mathbb{R}$, if for all interval $I = \Pi_{i=1}^{N}[a_i, b_i]$ and for all $j = 1, \ldots, N$, the j-th section of f, namely, $g_j : [a_j, b_j] \to \mathbb{R}$ is (uniformly) φ-absolutely continuous for almost all $x'_j \in [a'_j, b'_j]$, then f is called "locally φ-absolutely continuous" in Tonelli sense. We denote by $AC^{\varphi}_{loc}\left(\mathbb{R}^N\right)$, the space of all locally φ-absolutely continuous functions.

Now, we say that f is "φ-absolutely continuous on $\mathbb{R}^N$", if $f \in BV^{\varphi}\left(\mathbb{R}^N\right) \cap AC^{\varphi}_{loc}\left(\mathbb{R}^N\right) := AC^{\varphi}\left(\mathbb{R}^N\right)$.

In Section 2, considering the new general conditions we introduce our new operator, which is generalized and improved by the summability method. Our main approximation theorem is also given in this section. Section

3 is devoted to the rate of approximations by means of Lipschitz class of absolutely continuous functions (in particular Zygmund-type classes). In Section 4, we obtain a general characterization theorem for the space of φ-absolutely continuous on $\mathbb{R}^N$. Then, in Section 5, we will mention about the special cases of our operator. Afterwards, defining nontrivial examples of our kernels we give two illustrative examples for our approximation and characterization theorems. We also investigate the error of these approximation numerically. Finally we show that our theory can be extended to $\mathcal{F}^{\varphi}$-variation.

2. Multivariate approximation in φ-variation

Our operator, which is the $\mathcal{A}$-transform of the operator (1.1), is given as follows:

$$\mathcal{T}_{n,\upsilon}(f;\mathbf{x}) = \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int\limits_{\mathbb{R}^N} K_k(\mathbf{t}, f(\mathbf{x}-\mathbf{t}))\, d\mathbf{t} \quad (\upsilon, n \in \mathbb{N},\ \mathbf{x} \in \mathbb{R}^N). \tag{2.1}$$

Here, the sequence of nonlinear kernels are given by $K_k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R},\ K_k(\mathbf{t}, u) := L_k(\mathbf{t}) H_k(u),$ where $L_k : \mathbb{R}^N \to \mathbb{R}$ and $H_k : \mathbb{R} \to \mathbb{R}$. In addition, $\{L_k\} \subset L^1(\mathbb{R}^N)$ endowed with the L^1 norm ($\|\cdot\|_{\mathbf{1}}$) and H_k satisfies the following assumption:

- $(*)$ denoted by $G_k(u) := H_k(u) - u,\ u \in \mathbb{R},$ for all $\gamma > 0,$ there can be found a $\lambda > 0$ satisfying that $\lim\limits_{k\to\infty} \frac{V^{\varphi}[\lambda G_k, J]}{\varphi(\gamma m(J))} = 0$ (uniformly in every proper bounded interval $J \subset \mathbb{R}$), namely, for all $\gamma > 0$, there exists a $\lambda > 0$ such that for a given $\varepsilon > 0$, there exists a number k_0 satisfying that $\frac{V_{\varphi}[\lambda G_k, J]}{\varphi(\gamma m(J))} < \varepsilon$ for all $k \geq k_0$ ($m(J)$ denotes the length of J).

Assumption $(*)$ is natural in this setting and an example is given in the last section (for more examples, see [3, 4, 6, 9]).

We remark that the sum in the operator (2.1) starts from k_0, where k_0 is sufficiently large for which (2.1) is well-defined (see also Remark 3.1 in [9]).

Here are our kernel assumptions on L_k:

- (I) There exists a constant $A > 0$ such that $\|L_k\|_{\mathbf{1}} \leq A < \infty$ for all $k \in \mathbb{N}$.

- (II) for every fixed $\delta > 0$, $\mathcal{A}\text{-}\lim \left(\int\limits_{|\mathbf{t}|\geq\delta} |L_k(\mathbf{t})|\, d\mathbf{t} \right) = 0,$

- (III) $\mathcal{A}\text{-}\lim \left(\int\limits_{\mathbb{R}^N} L_k(\mathbf{t})\, d\mathbf{t} \right) = 1.$

Taking $\mathcal{A}$ as identity matrix, it is possible to see that the above conditions turn into classical approximate identities ([3–5]).

Lemma 2.1 *Assume that (I) holds. Then there can be found a $\mu > 0$ satisfying that $V^{\varphi}[\mu\mathcal{T}_{n,\upsilon}(f)] < \infty$ for all $f \in BV^{\varphi}(\mathbb{R}^N) \cap C^0(\mathbb{R}^N)$, where $C^0(\mathbb{R}^N)$ is the space of all continuous functions $f : \mathbb{R}^N \to \mathbb{R}$.*

Proof Let $I = \Pi_{i=1}^{N}[a_i, b_i] \subset \mathbb{R}^N$ and $\{J_1, \ldots, J_m\}$ be a partition of I, where $J_q = \Pi_{j=1}^{N}[{}^q a_j, {}^q b_j]$ for $q = 1, \ldots, m$. For a given $j = 1, \ldots, N$ and $q = 1, \ldots, m$, consider the partition $\left\{s_j^0 = {}^q a_j, \ldots, s_j^\eta = {}^q b_j\right\}$ of the interval $[{}^q a_j, {}^q b_j]$. Now for all $\mu > 0$,

$$\begin{aligned} S &= \sum_{\zeta=1}^{\eta} \varphi\left(\mu \left|\mathcal{T}_{n,\upsilon}(f;(s_j', s_j^\zeta)) - \mathcal{T}_{n,\upsilon}(f;(s_j', s_j^{\zeta-1}))\right|\right) \\ &= \sum_{\zeta=1}^{\eta} \varphi\left(\mu \left|\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} K_k(\mathbf{t}, f(s_j' - t_j', s_j^\zeta - t_j))d\mathbf{t} - \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} K_k(\mathbf{t}, f(s_j' - t_j', s_j^{\zeta-1} - t_j))d\mathbf{t}\right|\right) \\ &\le \sum_{\zeta=1}^{\eta} \varphi\left(\mu \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} |L_k(\mathbf{t})| \left|H_k(f(s_j' - t_j', s_j^\zeta - t_j)) - H_k(f(s_j' - t_j', s_j^{\zeta-1} - t_j))\right| d\mathbf{t}\right). \end{aligned}$$

From Jensen's inequality (discrete version), we get

$$S \le \sum_{\zeta=1}^{\eta} \sum_{k=k_0}^{\infty} \frac{a_{nk}^{\upsilon}}{a_{n,\upsilon}} \varphi\left(\mu a_{n,\upsilon} \int_{\mathbb{R}^N} |L_k(\mathbf{t})| \left|H_k(f(s_j' - t_j', s_j^\zeta - t_j)) - H_k(f(s_j' - t_j', s_j^{\zeta-1} - t_j))\right| d\mathbf{t}\right),$$

where $a_{n,\upsilon} := \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}$. Since φ is convex and $\|L_k\|_{\mathbf{1}} \le A$, by the Jensen's inequality,

$$\begin{aligned} S &\le \frac{1}{Aa_{n,\upsilon}} \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} |L_k(\mathbf{t})| \sum_{\zeta=1}^{\eta} \varphi\left(\mu A a_{n,\upsilon} \left|H_k(f(s_j' - t_j', s_j^\zeta - t_j)) - H_k(f(s_j' - t_j', s_j^{\zeta-1} - t_j))\right|\right) d\mathbf{t} \\ &\le \frac{1}{Aa_{n,\upsilon}} \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} |L_k(\mathbf{t})| V^{\varphi}_{[{}^q a_j, {}^q b_j]}\left[\mu A a_{n,\upsilon} H_k(f(s_j' - t_j', \cdot - t_j))\right] d\mathbf{t} \end{aligned}$$

holds. Now, using two times Fubini-Tonelli theorem, for all $j = 1, \ldots, N$

$$\begin{aligned} \Phi_j^{\varphi}(\mu \mathcal{T}_{n,\upsilon}(f), J_q) &= \int_{{}^q a_j'}^{{}^q b_j'} V^{\varphi}_{[{}^q a_j, {}^q b_j]}\left[\mu \mathcal{T}_{n,\upsilon}(f;(s_j', \cdot))\right] ds_j' \\ &\le \frac{1}{Aa_{n,\upsilon}} \int_{{}^q a_j'}^{{}^q b_j'} \left(\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} |L_k(\mathbf{t})| V^{\varphi}_{[{}^q a_j, {}^q b_j]}\left[\mu A a_{n,\upsilon} H_k(f(s_j' - t_j', \cdot - t_j))\right] d\mathbf{t}\right) ds_j' \\ &= \frac{1}{Aa_{n,\upsilon}} \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{{}^q a_j'}^{{}^q b_j'} \left(\int_{\mathbb{R}^N} |L_k(\mathbf{t})| V^{\varphi}_{[{}^q a_j, {}^q b_j]}\left[\mu A a_{n,\upsilon} H_k(f(s_j' - t_j', \cdot - t_j))\right] d\mathbf{t}\right) ds_j' \\ &= \frac{1}{Aa_{n,\upsilon}} \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} |L_k(\mathbf{t})| \left(\int_{{}^q a_j'}^{{}^q b_j'} V^{\varphi}_{[{}^q a_j, {}^q b_j]}\left[\mu A a_{n,\upsilon} H_k(f(s_j' - t_j', \cdot - t_j))\right] ds_j'\right) d\mathbf{t} \\ &= \frac{1}{Aa_{n,\upsilon}} \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} |L_k(\mathbf{t})| \left\{\Phi_j^{\varphi}(\mu A a_{n,\upsilon} H_k(f(\cdot - \mathbf{t})), J_q)\right\} d\mathbf{t} \end{aligned}$$

yields. Then by the generalized Minkowski inequality, one can observe that

$$\begin{aligned}\Phi^{\varphi}\left(\mu\mathcal{T}_{n,\upsilon}\left(f\right),J_q\right) &:= \left\{\sum_{j=1}^{N}\left[\Phi_j^{\varphi}\left(\mu\mathcal{T}_{n,\upsilon}\left(f\right),J_q\right)\right]^2\right\}^{\frac{1}{2}}\\
&\leq \frac{1}{Aa_{n,\upsilon}}\left\{\sum_{j=1}^{N}\left[\sum_{k=k_0}^{\infty}a_{nk}^{\upsilon}\int_{\mathbb{R}^N}\left|L_k\left(\mathbf{t}\right)\right|\left\{\Phi_j^{\varphi}\left(\mu Aa_{n,\upsilon}H_k(f(\cdot-\mathbf{t})),J_q\right)\right\}d\mathbf{t}\right]^2\right\}^{\frac{1}{2}}\\
&\leq \frac{1}{Aa_{n,\upsilon}}\sum_{k=k_0}^{\infty}a_{nk}^{\upsilon}\left\{\sum_{j=1}^{N}\left(\int_{\mathbb{R}^N}\left|L_k\left(\mathbf{t}\right)\right|\left\{\Phi_j^{\varphi}\left(\mu Aa_{n,\upsilon}H_k(f(\cdot-\mathbf{t})),J_q\right)\right\}d\mathbf{t}\right)^2\right\}^{\frac{1}{2}}\\
&\leq \frac{1}{Aa_{n,\upsilon}}\sum_{k=k_0}^{\infty}a_{nk}^{\upsilon}\int_{\mathbb{R}^N}\left|L_k\left(\mathbf{t}\right)\right|\left(\sum_{j=1}^{N}\left\{\Phi_j^{\varphi}\left(\mu Aa_{n,\upsilon}H_k(f(\cdot-\mathbf{t})),J_q\right)\right\}^2\right)^{\frac{1}{2}}d\mathbf{t}\\
&= \frac{1}{Aa_{n,\upsilon}}\sum_{k=k_0}^{\infty}a_{nk}^{\upsilon}\int_{\mathbb{R}^N}\left|L_k\left(\mathbf{t}\right)\right|\Phi^{\varphi}\left(\mu Aa_{n,\upsilon}H_k(f(\cdot-\mathbf{t})),J_q\right)d\mathbf{t}\end{aligned}$$

for every $q=1,\ldots,m$. Summing over q from 1 to m and taking supremum over $\{J_1,\ldots,J_m\}$, we immediately get

$$V_I^{\varphi}\left[\mu\mathcal{T}_{n,\upsilon}\left(f\right)\right]\leq\frac{1}{Aa_{n,\upsilon}}\sum_{k=k_0}^{\infty}a_{nk}^{\upsilon}\int_{\mathbb{R}^N}\left|L_k\left(\mathbf{t}\right)\right|V_I^{\varphi}\left[\mu Aa_{n,\upsilon}H_k(f(\cdot-\mathbf{t}))\right]d\mathbf{t}.$$

Then taking supremum over all $I\subset\mathbb{R}^N$, from (I) there holds

$$\begin{aligned}V^{\varphi}\left[\mu\mathcal{T}_{n,\upsilon}\left(f\right)\right] &\leq \frac{1}{Aa_{n,\upsilon}}\sum_{k=k_0}^{\infty}a_{nk}^{\upsilon}\int_{\mathbb{R}^N}\left|L_k\left(\mathbf{t}\right)\right|V^{\varphi}\left[\mu Aa_{n,\upsilon}H_k(f(\cdot-\mathbf{t}))\right]d\mathbf{t}\\
&= \frac{1}{Aa_{n,\upsilon}}\sum_{k=k_0}^{\infty}a_{nk}^{\upsilon}V^{\varphi}\left[\mu Aa_{n,\upsilon}H_k(f)\right]\int_{\mathbb{R}^N}\left|L_k\left(\mathbf{t}\right)\right|d\mathbf{t}\\
&\leq \frac{1}{a_{n,\upsilon}}\sum_{k=k_0}^{\infty}a_{nk}^{\upsilon}V^{\varphi}\left[\mu Aa_{n,\upsilon}H_k(f)\right].\end{aligned}$$

By Proposition 4.1 in [3], condition $(*)$ implies that, for every $\gamma>0$, there can be found a $\lambda>0$ satisfying that for every $\varepsilon>0$

$$V^{\varphi}\left[\lambda\left(H_k\circ f-f\right)\right]<\varepsilon V^{\varphi}\left[\gamma f\right]$$

holds for sufficiently large $k\in\mathbb{N}$. Then for $\varepsilon=1$, we can find a $k_0'\in\mathbb{N}$ such that

$$V^{\varphi}\left[\lambda\left(H_k\circ f-f\right)\right]<V^{\varphi}\left[\gamma f\right]$$

holds for all $k\geq k_0'$, where γ is sufficiently small for which $V^{\varphi}\left[\gamma f\right]<\infty$. Considering (1.2) and taking $k_0=k_0'$, we immediately observe

$$\begin{aligned}V^{\varphi}\left[\mu\mathcal{T}_{n,\upsilon}\left(f\right)\right] &\leq \frac{1}{\sqrt{2}a_{n,\upsilon}}\left\{\sum_{k=k_0}^{\infty}a_{nk}^{\upsilon}V^{\varphi}\left[2\mu Aa_{n,\upsilon}(H_k(f)-f)\right]+\sum_{k=k_0}^{\infty}a_{nk}^{\upsilon}V^{\varphi}\left[2\mu Aa_{n,\upsilon}f\right]\right\}\\
&\leq \frac{1}{\sqrt{2}}\left\{V^{\varphi}\left[\gamma f\right]+V^{\varphi}\left[2\mu Aa_{n,\upsilon}f\right]\right\}\end{aligned}$$

for sufficiently small $\mu > 0$. Here $\mu \leq \min\left\{\lambda/2Aa_{n,\upsilon}, \bar{\lambda}/2Aa_{n,\upsilon}\right\}$, where $V^{\varphi}\left[\bar{\lambda}f\right] < \infty$. Finally, since $f \in BV^{\varphi}\left(\mathbb{R}^N\right)$, the proof completes. □

We should state that our operator becomes well-defined for $k_0 = k_0'$, where k_0' is given in the above proof. Thus, from now on we will assume $k_0 = k_0'$.

Now, we state our main approximation theorem, which includes many summation methods such as almost convergence method and Cesàro summability as well as conventional convergence (for details, see Section 5).

Theorem 2.2 *Let $f \in AC^{\varphi}\left(\mathbb{R}^N\right)$. If $(I)-(III)$ hold, then there can be found a $\mu > 0$ satisfying that*

$$\lim_{n\to\infty} V^{\varphi}\left[\mu\left(\mathcal{T}_{n,\upsilon}\left(f\right)-f\right)\right] = 0, \text{ uniformly in } \upsilon \in \mathbb{N}.$$

Proof From the previous lemma, it is clear that $\mathcal{T}_{n,\upsilon}\left(f\right)-f \in BV^{\varphi}\left(\mathbb{R}^N\right)$. Let $I = \Pi_{i=1}^{N}\left[a_i, b_i\right] \subset \mathbb{R}^N$ and $\{J_1, \ldots, J_m\}$ be a partition of I, where $J_q = \Pi_{j=1}^{N}\left[{}^{q}a_j, {}^{q}b_j\right]$ for $q = 1, \ldots, m$. Considering the partition $\left\{s_j^0 = {}^{q}a_j, \ldots, s_j^{\eta} = {}^{q}b_j\right\}$ of the interval $\left[{}^{q}a_j, {}^{q}b_j\right]$, then for all $\mu > 0$

$$\begin{aligned}
L &:= \sum_{\zeta=1}^{\eta}\varphi\left(\mu\left|\mathcal{T}_{n,\upsilon}(f;(s_j', s_j^{\zeta})) - f(s_j', s_j^{\zeta}) - \mathcal{T}_{n,\upsilon}(f;(s_j', s_j^{\zeta-1})) + f(s_j', s_j^{\zeta-1})\right|\right) \\
&\leq \sum_{\zeta=1}^{\eta}\varphi\Bigg(\mu\Bigg|\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N} L_k\left(\mathbf{t}\right)\Big\{H_k(f(s_j'-t_j', s_j^{\zeta}-t_j)) - f(s_j'-t_j', s_j^{\zeta}-t_j) \\
&\quad -H_k(f(s_j'-t_j', s_j^{\zeta-1}-t_j)) + f(s_j'-t_j', s_j^{\zeta-1}-t_j)\Big\}\,d\mathbf{t} \\
&\quad +\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N} L_k\left(\mathbf{t}\right)\Big\{f(s_j'-t_j', s_j^{\zeta}-t_j) - f(s_j', s_j^{\zeta}) - f(s_j'-t_j', s_j^{\zeta-1}-t_j) + f(s_j', s_j^{\zeta-1})\Big\}\,d\mathbf{t} \\
&\quad +\Big\{f(s_j', s_j^{\zeta}) - f(s_j', s_j^{\zeta-1})\Big\}\left(\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N} L_k\left(\mathbf{t}\right)d\mathbf{t}-1\right)\Bigg|\Bigg) \\
&\leq \sum_{\zeta=1}^{\eta}\frac{1}{3}\varphi\Bigg(3\mu\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N}\left|L_k\left(\mathbf{t}\right)\right|\Big|H_k(f(s_j'-t_j', s_j^{\zeta}-t_j)) - f(s_j'-t_j', s_j^{\zeta}-t_j) \\
&\quad -H_k(f(s_j'-t_j', s_j^{\zeta-1}-t_j)) + f(s_j'-t_j', s_j^{\zeta-1}-t_j)\Big|\,d\mathbf{t}\Bigg) \\
&\quad +\sum_{\zeta=1}^{\eta}\frac{1}{3}\varphi\Bigg(3\mu\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N}\left|L_k\left(\mathbf{t}\right)\right|\Big|f(s_j'-t_j', s_j^{\zeta}-t_j) - f(s_j', s_j^{\zeta}) \\
&\quad -f(s_j'-t_j', s_j^{\zeta-1}-t_j) + f(s_j', s_j^{\zeta-1})\Big|\,d\mathbf{t}\Bigg) \\
&\quad +\sum_{\zeta=1}^{\eta}\frac{1}{3}\varphi\left(3\mu\left|f(s_j', s_j^{\zeta}) - f(s_j', s_j^{\zeta-1})\right|\left|\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N} L_k\left(\mathbf{t}\right)d\mathbf{t}-1\right|\right) := L_1 + L_2 + L_3
\end{aligned}$$

holds. Using (I), convexity of φ and two times Jensen's inequality in L_1 and L_2, we have the followings

$$\begin{aligned}
L_1 &\leq \frac{1}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N}\left|L_k\left(\mathbf{t}\right)\right|\sum_{\zeta=1}^{\eta}\varphi\Big(3\mu AM\Big|H_k(f(s_j'-t_j', s_j^{\zeta}-t_j)) - f(s_j'-t_j', s_j^{\zeta}-t_j) \\
&\quad -H_k(f(s_j'-t_j', s_j^{\zeta-1}-t_j)) + f(s_j'-t_j', s_j^{\zeta-1}-t_j)\Big|\Big)\,d\mathbf{t} \\
&\leq \frac{1}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N}\left|L_k\left(\mathbf{t}\right)\right| V_{\left[{}^{q}a_j, {}^{q}b_j\right]}^{\varphi}\left[3\mu AM(H_k\circ f - f)(s_j'-t_j', \cdot - t_j)\right]d\mathbf{t}
\end{aligned}$$

and

$$L_2 \leq \frac{1}{3AM} \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} |L_k(\mathbf{t})| V_{[^q a_j, ^q b_j]}^{\varphi} \left[3\mu AM(f(s_j' - t_j', \cdot - t_j) - f(s_j', \cdot))\right] d\mathbf{t}$$

for all $n \geq N$. Here M, N are the numbers coming from the regularity of $\mathcal{A}$ such that $\sup_{n \geq N, \upsilon \in \mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}^{\upsilon}| \leq M$. Now by (III) and by the convexity of φ, one can clearly observe that

$$L_3 \leq \frac{1}{3} \left| \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} L_k(\mathbf{t})\, d\mathbf{t} - 1 \right| V_{[^q a_j, ^q b_j]}^{\varphi} \left[3\mu f(s_j', \cdot)\right]$$

for sufficiently large $n \in \mathbb{N}$. Afterwards, applying Fubini-Tonelli theorem, we get

$$\begin{aligned}
\Phi_j^{\varphi}\left(\mu\left(\mathcal{T}_{n,\upsilon}(f) - f\right), J_q\right) &:= \int_{^q a_j'}^{^q b_j'} V_{[^q a_j, ^q b_j]}^{\varphi} \left[\mu\left(\mathcal{T}_{n,\upsilon}(f; (s_j', \cdot)) - f(s_j', \cdot)\right)\right] ds_j' \\
&\leq \frac{1}{3AM} \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} |L_k(\mathbf{t})| \Phi_j^{\varphi}\left(3\mu AM(H_k \circ f - f)(\cdot - \mathbf{t}), J_q\right) d\mathbf{t} \\
&+ \frac{1}{3AM} \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} |L_k(\mathbf{t})| \Phi_j^{\varphi}\left(3\mu AM(f(\cdot - \mathbf{t}) - f(\cdot)), J_q\right) d\mathbf{t} \\
&+ \frac{1}{3} \left| \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} L_k(\mathbf{t})\, d\mathbf{t} - 1 \right| \Phi_j^{\varphi}(3\mu f, J_q).
\end{aligned}$$

Now using Minkowski inequality

$$\begin{aligned}
\Phi^{\varphi}\left(\mu\left(\mathcal{T}_{n,\upsilon}(f) - f\right), J_q\right) &:= \left\{ \sum_{j=1}^{N} \left[\Phi_j^{\varphi}\left(\mu(\mathcal{T}_{n,\upsilon}(f) - f), J_q\right)\right]^2 \right\}^{\frac{1}{2}} \\
&\leq \frac{1}{3AM} \left\{ \sum_{j=1}^{N} \left[\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} |L_k(\mathbf{t})| \Phi_j^{\varphi}\left(3\mu AM(H_k \circ f - f)(\cdot - \mathbf{t}), J_q\right) d\mathbf{t} \right]^2 \right\}^{\frac{1}{2}} \\
&+ \frac{1}{3AM} \left\{ \sum_{j=1}^{N} \left[\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} |L_k(\mathbf{t})| \Phi_j^{\varphi}\left(3\mu AM(f(\cdot - \mathbf{t}) - f(\cdot)), J_q\right) d\mathbf{t} \right]^2 \right\}^{\frac{1}{2}} \\
&+ \frac{1}{3} \left| \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} L_k(\mathbf{t})\, d\mathbf{t} - 1 \right| \left\{ \sum_{j=1}^{N} \left[\Phi_j^{\varphi}(3\mu f, J_q)\right]^2 \right\}^{\frac{1}{2}} \\
&:= I_1 + I_2 + I_3
\end{aligned}$$

and considering two times generalized Minkowski inequality, we have

$$
\begin{aligned}
I_1 &\leq \frac{1}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\left\{\sum_{j=1}^{N}\left[\int_{\mathbb{R}^N}|L_k(\mathbf{t})|\,\Phi_j^{\varphi}\left(3\mu AM(H_k\circ f-f)(\cdot-\mathbf{t}),J_q\right)d\mathbf{t}\right]^2\right\}^{\frac{1}{2}}\\
&\leq \frac{1}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N}|L_k(\mathbf{t})|\left\{\sum_{j=1}^{N}\left[\Phi_j^{\varphi}\left(3\mu AM(H_k\circ f-f)(\cdot-\mathbf{t}),J_q\right)\right]^2\right\}^{\frac{1}{2}}d\mathbf{t}\\
&= \frac{1}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N}|L_k(\mathbf{t})|\,\Phi^{\varphi}\left(3\mu AM(H_k\circ f-f)(\cdot-\mathbf{t}),J_q\right)d\mathbf{t}.
\end{aligned}
$$

Similarly, one can deduce that

$$
I_2 \leq \frac{1}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N}|L_k(\mathbf{t})|\,\Phi^{\varphi}\left(3\mu AM(f(\cdot-\mathbf{t})-f(\cdot)),J_q\right)d\mathbf{t}
$$

and

$$
I_3 \leq \frac{1}{3}\left|\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N}L_k(\mathbf{t})\,d\mathbf{t}-1\right|\Phi^{\varphi}(3\mu f,J_q).
$$

Then, if we sum up over q from 1 to m and take supremum over $\{J_1,\ldots,J_m\}$, then we see that

$$
\begin{aligned}
V_I^{\varphi}\left[\mu\left(\mathcal{T}_{n,\upsilon}(f)-f\right)\right] &\leq \frac{1}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N}|L_k(\mathbf{t})|\,V_I^{\varphi}\left[3\mu AM(H_k\circ f-f)(\cdot-\mathbf{t})\right]d\mathbf{t}\\
&\quad+\frac{1}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N}|L_k(\mathbf{t})|\,V_I^{\varphi}\left[3\mu AM(f(\cdot-\mathbf{t})-f(\cdot))\right]d\mathbf{t}\\
&\quad+\frac{1}{3}\left|\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N}L_k(\mathbf{t})\,d\mathbf{t}-1\right|V_I^{\varphi}\left[3\mu f\right].
\end{aligned}
$$

Since $I\subset\mathbb{R}^N$ is arbitrary, having supremum over all $I\subset\mathbb{R}^N$, we get

$$
\begin{aligned}
V^{\varphi}\left[\mu\left(\mathcal{T}_{n,\upsilon}(f)-f\right)\right] &\leq \frac{1}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}V^{\varphi}\left[3\mu AM(H_k\circ f-f)\right]\int_{\mathbb{R}^N}|L_k(\mathbf{t})|\,d\mathbf{t}\\
&\quad+\frac{1}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N}|L_k(\mathbf{t})|\,V^{\varphi}\left[3\mu AM(f(\cdot-\mathbf{t})-f(\cdot))\right]d\mathbf{t}\\
&\quad+\frac{1}{3}\left|\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N}L_k(\mathbf{t})\,d\mathbf{t}-1\right|V^{\varphi}\left[3\mu f\right]\\
&:= P_1+P_2+P_3
\end{aligned}
$$

From Proposition 4.1. in [3], $(*)$ implies that there exists a $\xi>0$ such that

$$
\lim_{k\to\infty}V^{\varphi}\left[\xi\left(H_k\circ f-f\right)\right]=0.
$$

In this manner, for every $\gamma>0$ there exists a $\lambda>0$ such that for all $1>\varepsilon>0$ there can be found a number $k_0''\in\mathbb{N}$ such that for every $k>k_0''$

$$
V^{\varphi}\left[3\mu AM(H_k\circ f-f)\right]<\varepsilon V^{\varphi}\left[\gamma f\right],
$$

where $3\mu AM \leq \lambda$. Then dividing P_1 into two parts and considering (I) and regularity of $\mathcal{A}$ in the previous inequality, we observe that

$$P_1 = \frac{1}{3AM}\left(\sum_{k=k_0}^{k_0''} a_{nk}^{\upsilon} + \sum_{k=k_0''+1}^{\infty} a_{nk}^{\upsilon}\right) V^{\varphi}\left[3\mu AM(H_k \circ f - f)\right] \int_{\mathbb{R}^N} |L_k(\mathbf{t})|\, d\mathbf{t}$$
$$\leq \left(\frac{(k_0'' - k_0 + 1)\, D}{3M} + \frac{V^{\varphi}[\gamma f]}{3}\right)\varepsilon,$$

where $D := \max_{k\in\{k_0,\ldots,k_0''\}} V^{\varphi}\left[3\mu AM(H_k \circ f - f)\right].$

For P_2, by Theorem 5 and Remark 4 in [2], it is known that

$$\lim_{|\mathbf{t}|\to 0} V^{\varphi}\left[9\mu AM(f(\cdot - \mathbf{t}) - f(\cdot))\right] = 0$$

for some $\mu > 0$, i.e. there exists a $\mu > 0$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying that

$$V^{\varphi}\left[9\mu AM(f(\cdot - \mathbf{t}) - f(\cdot))\right] < \varepsilon \tag{2.3}$$

whenever $|\mathbf{t}| < \delta$. If we consider this in P_2 and divide it into two parts as follows, there holds

$$P_2 = \frac{1}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{|\mathbf{t}|<\delta} |L_k(\mathbf{t})|\, V^{\varphi}\left[3\mu AM(f(\cdot - \mathbf{t}) - f(\cdot))\right] d\mathbf{t}$$
$$+ \frac{1}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{|\mathbf{t}|\geq\delta} |L_k(\mathbf{t})|\, V^{\varphi}\left[3\mu AM(f(\cdot - \mathbf{t}) - f(\cdot))\right] d\mathbf{t}$$
$$:= P_2^1 + P_2^2,$$

where

$$P_2^1 \leq \frac{\varepsilon}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{|\mathbf{t}|<\delta} |L_k(\mathbf{t})|\, d\mathbf{t} < \frac{\varepsilon}{3}$$

from (2.3), regularity of $\mathcal{A}$ and (I).

On the other hand, using (1.2) and (II),

$$P_2^2 \leq \frac{\sqrt{2}}{3AM} V^{\varphi}\left[6\mu AMf\right] \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{|\mathbf{t}|\geq\delta} |L_k(\mathbf{t})|\, d\mathbf{t}$$
$$< \frac{\sqrt{2}}{3AM} V^{\varphi}\left[6\mu AMf\right]\varepsilon$$

holds for sufficiently large $n \in \mathbb{N}$ small $\mu > 0$. And finally we conclude from (III) that

$$P_3 < \frac{\varepsilon}{3} V^{\varphi}\left[3\mu f\right]$$

for sufficiently large $n \in \mathbb{N}$ and for some $\mu > 0$, which completes the proof. □

3. Rate of approximation

In this section, we investigate the order of approximation by means of the following classes of functions.

$\Gamma := \{\tau : \mathbb{R}^N \to \mathbb{R}_0^+ : \tau$ is measurable and continuous at $\mathbf{t} = \mathbf{0}$ s.t. $\tau(\mathbf{0}) = 0$ and $\tau(\mathbf{t}) > 0$ for $\mathbf{t} \neq \mathbf{0}\}$.

Taking this definition into account, for a given $\tau \in \Gamma$, we may define the following Lipschitz class:

$$V^{\varphi}Lip_N(\tau) := \left\{ f \in AC^{\varphi}\left(\mathbb{R}^N\right) : \exists \mu > 0 \text{ s.t. } V^{\varphi}\left[\mu\left(f(\cdot - \mathbf{t}) - f(\cdot)\right)\right] = O(\tau(\mathbf{t})) \text{ as } \mathbf{t} \to \mathbf{0} \right\},$$

where $f(\mathbf{t}) = O(g(\mathbf{t}))$, that is one can find $L, \delta > 0$ satisfying that $|f(\mathbf{t})| \leq L|g(\mathbf{t})|$ whenever $|\mathbf{t}| \leq \delta$.

We remark that taking $\tau(\mathbf{t}) = |\mathbf{t}|^{\alpha}$ ($\mathbf{t} \in \mathbb{R}^N$ and $\alpha > 0$) and $\xi(u) = u^{\alpha}$ ($u \in \mathbb{R}_0^+$) in the above definition, we obtain Zygmund-type class.

Let Ψ be defined by

$$\Psi := \left\{ \xi : \mathbb{R}_0^+ \to \mathbb{R}_0^+ : \xi \text{ is continuous at } u = 0,\ \xi(0) = 0 \text{ and } \xi(u) > 0 \text{ for } u > 0 \right\}.$$

Then for a fixed $\tau \in \Gamma$ and for any fixed $\delta > 0$, we may write the following rate assumptions:

$$(I') \quad \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{|\mathbf{t}| \geq \delta} |L_k(\mathbf{t})|\, d\mathbf{t} = O(\xi(1/n)) \text{ as } n \to \infty \text{ uniformly in } \upsilon,$$

$$(II') \quad \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{|\mathbf{t}| < \delta} |L_k(\mathbf{t})|\, \tau(\mathbf{t})\, d\mathbf{t} = O(\xi(1/n)) \text{ as } n \to \infty \text{ uniformly in } \upsilon,$$

$$(III') \quad \left| \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} L_k(\mathbf{t})\, d\mathbf{t} - 1 \right| = O(\xi(1/n)) \text{ as } n \to \infty \text{ uniformly in } \upsilon.$$

Note that, although they are equivalent, instead of $(*)$, for suitability we use condition (3.1) in the following theorem.

Theorem 3.1 *Let $\tau \in \Gamma$, $\xi \in \Psi$, $\delta > 0$ and (I), $(I') - (III')$ hold. Also assume that $\{\beta_k\}$ is a null sequence (that is, $\beta_k \to 0$ as $k \to \infty$) of positive real numbers such that: for all $\gamma > 0$, there can be found a $\lambda > 0$ satisfying that*

$$\frac{V^{\varphi}[\lambda G_k, J]}{\varphi(\gamma m(J))} \leq \beta_k \tag{3.1}$$

for all bounded interval $J \subset \mathbb{R}$ and $k \in \mathbb{N}$, and

$$\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \beta_k = O(\xi(1/n)) \text{ as } n \to \infty \text{ uniformly in } \upsilon. \tag{3.2}$$

If $f \in V^{\varphi}Lip_N(\tau)$, then there holds

$$V^{\varphi}\left[\mu\left(\mathcal{T}_{n,\upsilon}(f) - f\right)\right] = O(\xi(1/n)) \text{ as } n \to \infty \text{ uniformly in } \upsilon$$

for sufficiently large $n \in \mathbb{N}$.

Proof From Theorem 2.2,

$$\begin{aligned}
V^{\varphi}\left[\mu\left(\mathcal{T}_{n,\upsilon}(f)-f\right)\right] &\leq \frac{1}{3M}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} V^{\varphi}\left[3\mu AM(H_k\circ f-f)\right] \\
&+\frac{1}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{|\mathbf{t}|<\delta}\left|L_k(\mathbf{t})\right| V^{\varphi}\left[3\mu AM(f(\cdot-\mathbf{t})-f(\cdot))\right]d\mathbf{t} \\
&+\frac{V^{\varphi}\left[6\mu AMf\right]}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{|\mathbf{t}|\geq\delta}\left|L_k(\mathbf{t})\right| d\mathbf{t} \\
&+\frac{1}{3}\left|\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}^N} L_k(\mathbf{t})\,d\mathbf{t}-1\right| V^{\varphi}\left[3\mu f\right] \\
&:= A_1+A_2+A_3+A_4
\end{aligned}$$

holds. On the other hand, from Proposition 5.1 in [3], (3.1) implies that

$$V^{\varphi}\left[3\mu AM(H_k\circ f-f)\right]\leq\beta_k$$

and therefore from (3.2),

$$A_1=O\left(\xi\left(1/n\right)\right)\ \text{as}\ n\to\infty\ \text{uniformly in}\ \upsilon.$$

Since $f\in V^{\varphi}Lip_N(\tau)$, $\exists\mu>0$ and $K>0$ s.t. $V^{\varphi}\left[\mu\left(f(\cdot-\mathbf{t})-f(\cdot)\right)\right]\leq K\tau(\mathbf{t})$ as $\mathbf{t}\to\mathbf{0}$. Considering this with (II'),

$$\begin{aligned}
A_2 &\leq \frac{K}{3AM}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\int_{|\mathbf{t}|<\delta}\left|L_k(\mathbf{t})\right|\tau(\mathbf{t})\,d\mathbf{t} \\
&= O\left(\xi\left(1/n\right)\right)\ \text{as}\ n\to\infty\ \text{uniformly in}\ \upsilon.
\end{aligned}$$

holds. In addition, from (I'),

$$A_3=O\left(\xi\left(1/n\right)\right)\ \text{as}\ n\to\infty\ \text{uniformly in}\ \upsilon.$$

Finally from (III'), we conclude

$$A_4=O\left(\xi\left(1/n\right)\right)\ \text{as}\ n\to\infty\ \text{uniformly in}\ \upsilon.$$

□

4. Characterization of absolute continuity

Using φ-absolutely continuous functions, a general characterization theroem by means of the summability method will be given in this section. To this end, we require condition (IV) such that:

(IV) For any N-dimensional interval $I=\Pi_{i=1}^{N}\ [a_i,b_i]\subset\mathbb{R}^N$ and for all nonoverlapping intervals $\{[{}^{q}\alpha_j,{}^{q}\beta_j]\}_{q=1}^{r}$ of the interval $[a_j,b_j]$, there exists a $\mu>0$ such that for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$\sum_{q=1}^{r}\sum_{k=k_0}^{\infty} a_{nk}^{\upsilon}\varphi(\mu\left|L_k(s_j',{}^{q}\beta_j)-L_k(s_j',{}^{q}\alpha_j)\right|)<\varepsilon$$

holds, whenever

$$\sum_{q=1}^{r} \varphi\left({}^{q}\beta_j - {}^{q}\alpha_j\right) < \delta$$

for every $j = 1, \ldots, N$.

Note that if $\mathcal{A} = \{A^{\upsilon}\}_{\upsilon \in \mathbb{N}}$ is a row finite summability method, namely, A^{υ} contains at most a finite number of nonzero terms for all $\upsilon \in \mathbb{N}$, then condition (IV) reduces to "$L_k \in AC^{\varphi}_{loc}\left(\mathbb{R}^N\right)$".

Let $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a nondecreasing function such that $\psi(u) = 0$ if and only if $u = 0$ and $\lim_{u \to \infty} \psi(u) = \infty$. Then we get the following lemma.

Lemma 4.1 *Assume that L_k satisfies (IV) and assume further that $\exists K > 0$ such that*

$$|H_k(u)| \leq K \psi(|u|) \tag{4.1}$$

holds for every $u \in \mathbb{R}$ and $k \in \mathbb{N}$. If $f \in BV^{\varphi}\left(\mathbb{R}^N\right) \cap C^0\left(\mathbb{R}^N\right)$ and $(\psi \circ |f|) \in L^1\left(\mathbb{R}^N\right)$, then $\mathcal{T}_{n,\upsilon}(f) \in AC^{\varphi}\left(\mathbb{R}^N\right)$ for all $\upsilon \in \mathbb{N}$ and for sufficiently large $n \in \mathbb{N}$.

Proof From $\mathbf{x} - \mathbf{t} = \mathbf{z}$ substitution, one can rewrite the operator as follows

$$\mathcal{T}_{n,\upsilon}(f;\mathbf{x}) = \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} L_k(\mathbf{x} - \mathbf{z}) H_k(f(\mathbf{z})) \, d\mathbf{z}.$$

Using the same notations in (IV), let $\{[{}^{q}\alpha_j, {}^{q}\beta_j]\}_{q=1}^{r}$ be such that $\sum_{q=1}^{r} \varphi\left({}^{q}\beta_j - {}^{q}\alpha_j\right) < \delta$. Then by Jensen's inequality and Fubini-Tonelli theorem

$$\begin{aligned}
&\sum_{q=1}^{r} \varphi\left(\mu \left|\mathcal{T}_{n,\upsilon}(f; {}^{q}\beta_j) - \mathcal{T}_{n,\upsilon}(f; {}^{q}\alpha_j)\right|\right) \\
&\leq \sum_{q=1}^{r} \varphi\left(K\mu \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \int_{\mathbb{R}^N} \left|L_k\left(x_j' - z_j', {}^{q}\beta_j - z_j\right) - L_k\left(x_j' - z_j', {}^{q}\alpha_j - z_j\right)\right| \psi(|f(\mathbf{z})|) \, d\mathbf{z}\right) \\
&\leq \frac{1}{MB} \int_{\mathbb{R}^N} \psi(|f(\mathbf{z})|) \sum_{k=k_0}^{\infty} a_{nk}^{\upsilon} \sum_{q=1}^{r} \varphi\left(KMB\mu \left|L_k\left(x_j' - z_j', {}^{q}\beta_j - z_j\right) - L_k\left(x_j' - z_j', {}^{q}\alpha_j - z_j\right)\right|\right) d\mathbf{z} \\
&\leq \frac{\varepsilon}{M}
\end{aligned}$$

holds for sufficiently large n$\in \mathbb{N}$, where $B \geq \int_{\mathbb{R}^N} \psi(|f(\mathbf{z})|) \, d\mathbf{z}$ and M comes from the regularity of $\mathcal{A}$. □

Here is our characterization theorem, which is generalized and improved by the summability method.

Theorem 4.2 *Assume that $(I) - (IV)$ and (4.1) hold and assume further that $(\psi \circ |f|) \in L^1\left(\mathbb{R}^N\right)$ and $f \in BV^{\varphi}\left(\mathbb{R}^N\right) \cap C^0\left(\mathbb{R}^N\right)$. Then we have*

$$f \in AC^{\varphi}\left(\mathbb{R}^N\right) \Longleftrightarrow \exists \mu > 0 \ s.t. \ \lim_{n \to \infty} V^{\varphi}\left[\mu\left(\mathcal{T}_{n,\upsilon}(f) - f\right)\right] = 0 \ \textit{uniformly in } \upsilon \in \mathbb{N}.$$

Proof Since $f \in AC^{\varphi}\left(\mathbb{R}^N\right)$, from the Theorem 3.1 $\exists \mu > 0$ satisfying that

$$\lim_{n\to\infty} V^{\varphi}\left[\mu\left(\mathcal{T}_{n,\upsilon}\left(f\right)-f\right)\right]=0 \text{ uniformly in } \upsilon \in \mathbb{N}.$$

On the other hand, it is known from [5] that $AC^{\varphi}\left(\mathbb{R}^N\right)$ is a closed subspace of $BV^{\varphi}\left(\mathbb{R}^N\right)$ under convergence in φ-variation. By the previous lemma, since $\mathcal{T}_{n,\upsilon}\left(f\right) \in AC^{\varphi}\left(\mathbb{R}^N\right)$, we obtain $f \in AC^{\varphi}\left(\mathbb{R}^N\right)$. □

5. Conclusions and applications

5.1. Special cases

Some crucial conclusions of summability methods in our approach are as follows:

- if we consider $\mathcal{A}$ as the identity matrix, then our generalized operator reduces to operator T_k given in 1.1, and we get the classical approximation studied in [3]. In addition to classical one, we also get the following approximations, so that larger class of the kernels and operators could be included into the approximations.
- If we take $\mathcal{A}$ as the Cesàro matrix $\{[c_{nk}]\}$, where $c_{nk}=1/n$ if $1\leq k\leq n;$ 0 if $k>n$, we obtain the Cesàro approximation [24],
- if we assume $\mathcal{A}$ is equal to $\mathcal{F}=\{[d_{nk}^{\upsilon}]\}_{\upsilon\in\mathbb{N}}$ (almost convergence method), where $d_{nk}^{\upsilon}=1/n$, whenever $\upsilon\leq k<n+\upsilon;$ 0, otherwise, then we get almost approximation [26],
- by the help of summability method, it is possible to have different characterizations of the space $AC^{\varphi}\left(\mathbb{R}^N\right)$ (see also [13]).

5.2. Examples and graphical illustrations

In this part, we exemplify our kernels, which fulfill our kernel assumptions and then display our approximations for certain values of n.

Now, we deal with the followings:

$\mathcal{A}=\{C_1\}=\{[c_{nk}]\}$, the Cesàro matrix, $N=2$, $\tau\left(\mathbf{t}\right):=\left|\mathbf{t}\right|^{\frac{1}{3}}$,

$$\xi\left(u\right):=\begin{cases} u\ln\left(1+\frac{1}{u}\right); & \text{if } u>0\\ 0; & \text{if } u=0\end{cases}$$

$$H_k\left(u\right):=u+\arctan\left(\frac{u}{k}\right) \tag{5.1}$$

$$L_k\left(t_1,t_2\right):=\begin{cases}\left(\left(-1\right)^k+1\right)\frac{7k^7}{\pi}\left(\frac{1}{k}-\left(t_1^2+t_2^2\right)^{\frac{1}{6}}\right); & \text{if } \sqrt{t_1^2+t_2^2}\leq\frac{1}{k^3}\\ 0; & \text{if } \sqrt{t_1^2+t_2^2}>\frac{1}{k^3}.\end{cases} \tag{5.2}$$

Since $G_k(u) = \arctan\left(\frac{u}{k}\right)$, then it is clear that for all $\varphi \in \Phi$ and $J = [a,b]$ that

$$\begin{aligned}
\frac{V^{\varphi}[\lambda G_k, J]}{\varphi(\gamma m(J))} &= \frac{\varphi(\gamma(G_k(b) - G_k(a)))}{\varphi(\gamma(b-a))} \\
&= \frac{\varphi\left(\gamma\left(\arctan\left(\frac{b}{k}\right) - \arctan\left(\frac{a}{k}\right)\right)\right)}{\varphi(\gamma(b-a))} \\
&\leq \frac{(1/k)\,\varphi(\gamma(b-a))}{\varphi(\gamma(b-a))} \\
&= \frac{1}{k},
\end{aligned}$$

which proves $(*)$ for $\lambda = \gamma$. In addition, letting $\beta_k := 1/k$, we get

$$\begin{aligned}
\sum_{k=1}^{\infty} c_{nk}\beta_k &= \frac{1}{n}\sum_{k=1}^{n}\frac{1}{k} \\
&\leq \frac{2\ln(n+1)}{n}.
\end{aligned}$$

So that, we proved (3.1) and (3.2) together.

Now, let us concentrate on the kernel L_k. By the definition of L_k, it is not hard to see that condition (I) holds.

For (II), let $\delta > 0$ be given. If $\delta > 1$, then

$$\frac{1}{n}\sum_{k=1}^{n} \iint\limits_{\sqrt{t_1^2+t_2^2}\geq\delta} |L_k(t_1,t_2)|\, dt_2 dt_1 = 0,$$

if $0 < \delta \leq 1$, then

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} \iint\limits_{\sqrt{t_1^2+t_2^2}\geq\delta} |L_k(t_1,t_2)|\, dt_2 dt_1 &= \frac{1}{n}\sum_{k=1}^{\left[\frac{1}{\sqrt[3]{\delta}}\right]} \iint\limits_{\sqrt{t_1^2+t_2^2}\geq\delta} |L_k(t_1,t_2)|\, dt_2 dt_1 \\
&\quad + \frac{1}{n}\sum_{k=\left[\frac{1}{\sqrt[3]{\delta}}\right]+1}^{n} \iint\limits_{\sqrt{t_1^2+t_2^2}\geq\delta} |L_k(t_1,t_2)|\, dt_2 dt_1,
\end{aligned}$$

where $[\cdot]$ denotes the integer part. From the definition of L_k, second summation must be zero. Then, there holds

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{\left[\frac{1}{\sqrt[3]{\delta}}\right]}\iint_{\sqrt{t_1^2+t_2^2}\geq\delta}\left|L_k\left(t_1,t_2\right)\right|dt_2dt_1\\
&=\frac{1}{n}\sum_{k=1}^{\left[\frac{1}{\sqrt[3]{\delta}}\right]}\iint_{\frac{1}{k^3}\geq\sqrt{t_1^2+t_2^2}\geq\delta}\left|L_k\left(t_1,t_2\right)\right|dt_2dt_1\\
&=\frac{1}{n}\sum_{k=1}^{\left[\frac{1}{\sqrt[3]{\delta}}\right]}\left((-1)^k+1\right)\frac{7k^7}{\pi}\iint_{\frac{1}{k^3}\geq\sqrt{t_1^2+t_2^2}\geq\delta}\left(\frac{1}{k}-\left(t_1^2+t_2^2\right)^{\frac{1}{6}}\right)dt_2dt_1\\
&\leq\frac{B\left(\delta\right)}{n},
\end{aligned}$$

where

$$B\left(\delta\right)=\sum_{k=1}^{\left[\frac{1}{\sqrt[3]{\delta}}\right]}\left((-1)^k+1\right)\frac{7k^7}{\pi}\iint_{\frac{1}{k^3}\geq\sqrt{t_1^2+y^2}\geq\delta}\left(\frac{1}{k}-\left(t_1^2+y^2\right)^{\frac{1}{6}}\right)dydt_1.$$

Therefore, we understand

$$\frac{1}{n}\sum_{k=1}^{\left[\frac{1}{\sqrt[3]{\delta}}\right]}\iint_{\sqrt{t_1^2+t_2^2}\geq\delta}\left|L_k\left(t_1,t_2\right)\right|dt_2dt_1=\frac{B\left(\delta\right)}{n}\leq\frac{2B\left(\delta\right)\ln\left(n+1\right)}{n},$$

which gives us (II) and (I').

For (II'), one may observe the followings,

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\iint_{\sqrt{t_1^2+t_2^2}<\delta}\left|L_k\left(t_1,t_2\right)\right|\left(\sqrt{t_1^2+t_2^2}\right)^{\frac{1}{3}}dt_2dt_1\\
&\leq\frac{1}{n}\sum_{k=1}^{n}\iint_{\mathbb{R}^2}\left|L_k\left(t_1,t_2\right)\right|\left(\sqrt{t_1^2+t_2^2}\right)^{\frac{1}{3}}dt_2dt_1\\
&=\frac{1}{n}\sum_{k=1}^{n}\left((-1)^k+1\right)\frac{7k^7}{\pi}\iint_{\sqrt{t_1^2+t_2^2}<\frac{1}{k^3}}\left(\frac{1}{k}-\left(t_1^2+t_2^2\right)^{\frac{1}{6}}\right)\left(\sqrt{t_1^2+t_2^2}\right)^{\frac{1}{3}}dt_2dt_1\\
&=\frac{3}{4n}\sum_{k=1}^{n}\left((-1)^k+1\right)\frac{1}{k}\leq K\frac{\ln\left(n+1\right)}{n}.
\end{aligned}$$

Finally, for (III) and (III'), we get

$$\left|\frac{1}{n}\sum_{k=1}^{n}\iint_{\mathbb{R}^2}|L_k(t_1,t_2)|\,dt_2dt_1-1\right|$$

$$=\left|\frac{1}{n}\sum_{k=1}^{n}\left((-1)^k+1\right)\frac{7k^7}{\pi}\iint_{\sqrt{t_1^2+t_2^2}\le\frac{1}{k^3}}\left(\frac{1}{k}-\left(t_1^2+t_2^2\right)^{\frac{1}{6}}\right)dt_2dt_1-1\right|$$

$$=\left|\frac{1}{n}\sum_{k=1}^{n}(-1)^k\right|\le\frac{1}{n}\le 2\frac{\ln(n+1)}{n}.$$

Now, we will display our approximation using the above definitions together with the following function $f(x,y)=F(x)F(y)$, where

$$F(x)=\begin{cases}\cos x; & \text{if } |x|<\pi/2\\ 0; & \text{if otherwise.}\end{cases}$$

It is clear that $f\in AC^{\varphi}\left(\mathbb{R}^2\right)$. Then our operator turns into

$$T_n(f;x,y)=\frac{1}{n}\sum_{k=1}^{n}\int_{\sqrt{t_1^2+t_2^2}\le 1/k^3}\left((-1)^k+1\right)\frac{7k^7}{\pi}\left(\frac{1}{k}-\left(t_1^2+t_2^2\right)^{\frac{1}{6}}\right)\Bigg(f(x-t_1,y-t_2)+\arctan\left(\frac{f(x-t_1,y-t_2)}{k}\right)\Bigg)dt_2dt_1. \tag{5.3}$$

This approximation is indicated in Figure 1, 2 for some even and odd values of n.

As understood from above, our kernel examples fulfilled all of our assumptions. However, it can be clearly seen that they do not satisfy the classical approximate identities in [3]. Moreover, the classical approximation fails, since even and odd terms of our operator approach to different limits.

Now, we will give an application of Theorem 4.2. In this part, we assume that $\mathcal{A}=\mathcal{F}$ (almost convergence matrix). Moreover, considering the one dimensional L_k, given in [9] in Figure 3,

$$L_k(t)=\begin{cases}2\left(k-k^2|t|\right); & \text{if } |t|\le 1/k \text{ and } k=m^2\ (m\in\mathbb{N})\\ k-k^2|t|; & \text{if } |t|\le 1/k \text{ and } k\ne m^2\ (m\in\mathbb{N})\\ 0; & \text{if } |t|>1/k\end{cases}$$

we construct a two dimensional kernel,

$$L_k(t_1,t_2):=L_k(t_1)\,L_k(t_2), \tag{5.4}$$

which satisfies (IV) and all the assumptions of Theorem 4.2.

Our two dimensional kernel in (5.4) is plotted in Figure 3.

On the other hand, taking $\psi(|u|)=|u|$, it is not hard to see that H_k in (5.1) satisfies (4.1) for $K=2$.

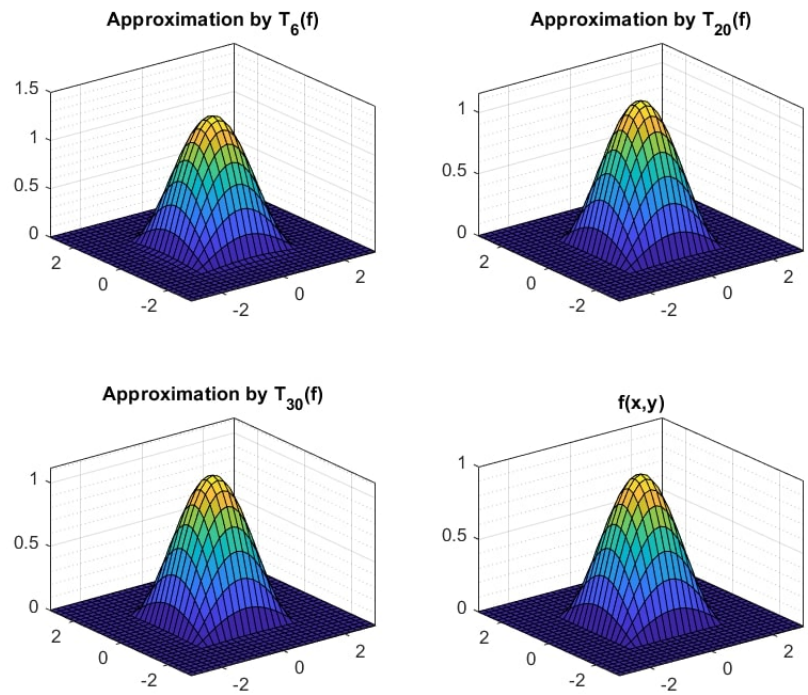

Figure 1. Approximation to f for even values of n.

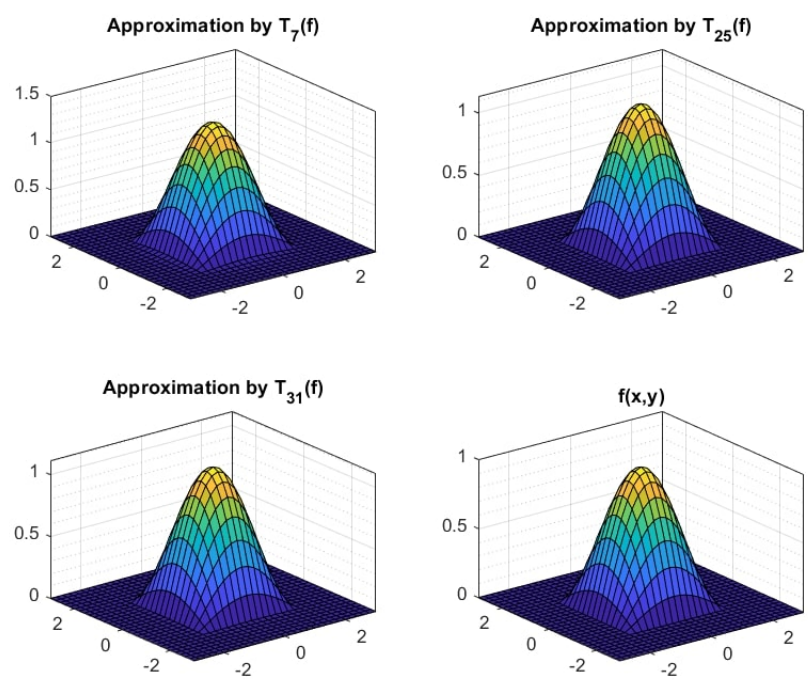

Figure 2. Approximation to f for odd values of n.

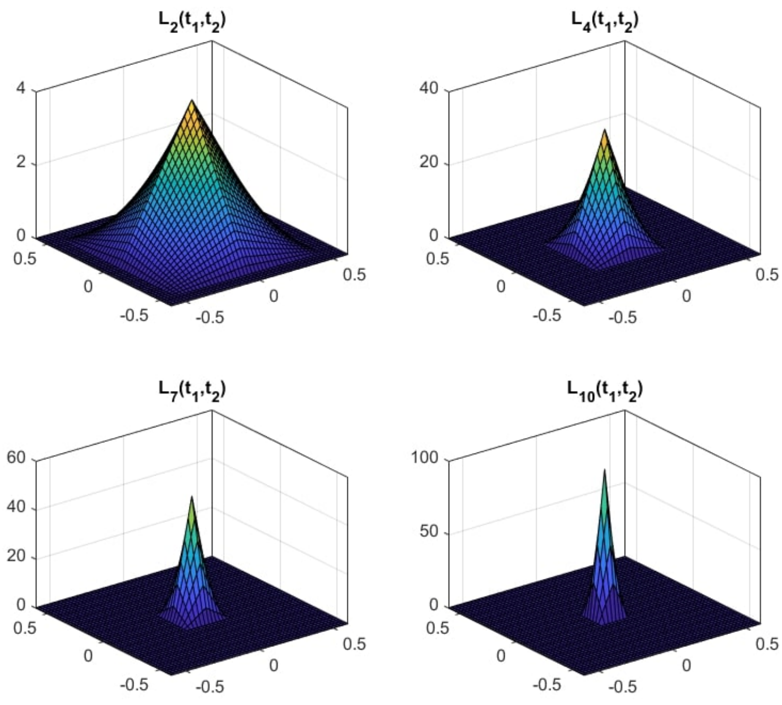

Figure 3. The kernel function $L_k(t_1, t_2)$ for $k = 2, 4, 7, 10$.

5.3. Numerical computations

We evaluated the approximation errors of the operator (5.3) with respect to φ-variation, where $\varphi(x) = x^2$. Since all the assumptions of Theorem 2.2 are satisfied for the corresponding φ, then we obtain

$$\lim_{n\to\infty} V^{\varphi}[T_n(f) - f] = 0.$$

Some φ-variation values are indicated in Table .

Table . Error of $V^{\varphi}[T_n(f) - f]$ for some odd and even values of n.

n (odd values)	$V^{\varphi}[T_n(f) - f]$	n (even values)	$V^{\varphi}[T_n(f) - f]$
5	$\approx 0.033892525826214$	4	$\approx 0.568313388764892$
23	$\approx 0.031843086450096$	18	$\approx 0.104079239418339$
97	$\approx 0.005453887845341$	90	$\approx 0.010299602502562$
195	$\approx 0.001977860928621$	168	$\approx 0.003882562638240$
275	$\approx 0.001175155118425$	290	$\approx 0.001616539490713$
323	$\approx 0.000917628363054$	358	$\approx 0.001147701794314$
501	$\approx 0.000462735016275$	586	$\approx 0.000511342504081$

5.4. Further results

As it is pointed out in [3, 5], all this theory can be extended to $\mathcal{F}^{\varphi}$-variation, where $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}_0^+$ be a functional satifying the following properties:

1. $\mathcal{F}(\mathbf{p}) = 0$ if and only if $\mathbf{p} = \mathbf{0}$, $\mathbf{p} \in \mathbb{R}^N$,

2. $\mathcal{F}(\mathbf{p}+\mathbf{q}) \leq \mathcal{F}(\mathbf{p}) + \mathcal{F}(\mathbf{q})$, $\mathbf{p}, \mathbf{q} \in \mathbb{R}^N$,

3. $\mathcal{F}(\alpha \mathbf{p}) = \alpha \mathcal{F}(\mathbf{p})$, $\mathbf{p} \in \mathbb{R}^N$, $\alpha \in \mathbb{R}_0^+$,

4. $\mathcal{F}(\mathbf{p}) \leq C \|\mathbf{p}\|$, $\mathbf{p} \in \mathbb{R}^N$, where C is the Lipschitz constant of $\mathcal{F}$.

We remark that $\mathcal{F}^{\varphi}$-variation is inspired by $\mathcal{F}$-variation, which is a generalization of classical variation ($\varphi(|u|) = |u|$) and has applications in calculus of variations [14, 18, 21, 28]. As an application, we refer to Remark 7 in [5].

Now, considering the above properties, $\mathcal{F}^{\varphi}$-variation of f on the interval $I \subset \mathbb{R}^N$ is given by

$$V_{\mathcal{F}}^{\varphi}[f, I] := \sup_{\{J_1, \ldots, J_m\}} \sum_{k=1}^{m} \mathcal{F}\left(\Phi^{\varphi}(f, J_k)\right)$$

where $\{J_i\}_{i=1,\ldots,m}$ forms a partition of I. Then, $\mathcal{F}^{\varphi}$-variation of f on $\mathbb{R}^N$ is defined as follows:

$$V_{\mathcal{F}}^{\varphi}[f] = \sup_{I \subset \mathbb{R}^N} V_{\mathcal{F}}^{\varphi}[f, I].$$

Since taking $\mathcal{F}(\mathbf{p}) = \|\mathbf{p}\|$ we obtain φ-variation, we may easily see that $\mathcal{F}^{\varphi}$-variation is a generalization of φ-variation. In addition, from [5] (see Section 7), we conclude that convergence in φ-variation is equivalent to convergence in $\mathcal{F}^{\varphi}$-variation and therefore all these studies can be generalized to $\mathcal{F}^{\varphi}$-variation.

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References

[1] Alemdar ME, Duman O. General summability methods in the approximation by Bernstein-Chlodovsky operators. Numerical Functional Analysis and Optimization 2021; 42 (5): 497-509.

[2] Angeloni L. A characterization of a modulus of smoothness in multidimensional setting. Bollettino dell'Unione Matematica Italiana 2011; 4 (1): 79-108.

[3] Angeloni L. Approximation results with respect to multidimensional φ-variation for nonlinear integral operators. Zeitschrift für Analysis und ihre Anwendungen 2013; 32 (1): 103-128.

[4] Angeloni L, Vinti G. Approximation by means of nonlinear integral operators in the space of functions with bounded φ-variation. Differential and Integral Equations 2007; 20: 339-360.

[5] Angeloni L, Vinti G. Convergence and rate of approximation for linear integral operators in BV^{φ}-spaces in multidimensional setting. Journal of Mathematical Analysis and Applications 2009; 349: 317-334.

[6] Angeloni L, Vinti G. Errata corrige to: Approximation by means of nonlinear integral operators in the space of functions with bounded φ-variation. Differential and Integral Equations 2010; 23: 795-799.

[7] Angeloni L, Vinti G. Convergence and rate of approximation in $BV_\varphi\left(\mathbb{R}_N^+\right)$ for class of Mellin integral operators. Atti della Accademia Nazionale dei Lincei, Classe di Scienze Fisiche, Matematische e Naturali, Rendiconti Lincei Matematica e Applicazioni 2014; 25 (3): 217-232.

[8] Angeloni L, Vinti G. Discrete operators of sampling type and approximation in φ-variation. Mathematische Nachrichten 2018; 291 (4): 546-555.

[9] Aslan I. Convergence in Phi-variation and rate of approximation for nonlinear integral operators using summability process. Mediterranean Journal of Mathematics 2021; 18 (1): paper no. 5.

[10] Aslan I. Approximation by sampling-type nonlinear discrete operators in φ-variation. Filomat 2021; 35 (8): 2731-2746.

[11] Aslan I, Duman O. Nonlinear approximation in N-dimension with the help of summability methods. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 2021; 115 (3): paper no. 105.

[12] Aslan I, Duman O. Approximation by nonlinear integral operators via summability process. Mathematische Nachrichten 2020; 293 (3): 430-448.

[13] Aslan I, Duman O. Characterization of absolute and uniform continuity. Hacettepe Journal of Mathematics and Statistics 2020; 49 (5): 1550-1565.

[14] Bardaro C. Alcuni teoremi di convergenza per l'integrale multiplo del Calcolo delle Variazioni (in Italian). Atti del Seminario Matematico e Fisico dell'Universita di Modena e Reggio Emilia 1982; 31: 302-324.

[15] Bardaro C, Sciamannini S, Vinti G. Convergence in BV_φ by nonlinear Mellin-type convolution operators. Functiones et Approximatio Commentarii Mathematici 2001; 29: 17-28.

[16] Bell HT. $\mathcal{A}$-summability. Thesis (PhD.)–Lehigh University 1971; 82 pp.

[17] Bell HT. Order summability and almost convergence. Proceedings of the American Mathematical Society 1973; 38: 548-552.

[18] Boni M. Sull'approssimazione dell'integrale multiplo del Calcolo delle Variazioni (in Italian). Atti del Seminario Matematico e Fisico dell'Universita di Modena e Reggio Emilia 1971; 20 (1): 187-211.

[19] Braha NL, Mansour T, Mursaleen M, Acar T. Convergence of λ Î»-Bernstein operators via power series summability method. Journal of Applied Mathematics and Computing 2021; 65 (1): 125-146.

[20] Demirci K, Karakuş S. Statistical A-summability of positive linear operators. Mathematical and Computer Modelling 2011; 53 (1-2): 189-195.

[21] Goffman C, Serrin J. Sublinear functions of measures and variational integrals. Duke Mathematical Journal 1964; 31 (1): 159-178.

[22] Gökçer TY, Duman O. Summation process by max-product operators. In: Anastassiou G., Duman O. (eds) Computational Analysis. Springer Proceedings in Mathematics & Statistics 2016; 155: 59-67.

[23] Gökçer TY, Duman O. Regular summability methods in the approximation by max-min operators. Fuzzy Sets Systems 2022; 426: 106-120.

[24] Hardy GH. Divergent series. Oxford University Press, London, 1949.

[25] Jordan C. Sur la serie de Fourier. Comptes Rendus de l'Académie des Sciences 1881; 92: 228-230.

[26] Lorentz GG. A contribution to the theory of divergent sequences. Acta Mathematica 1948; 80: 167-190.

[27] Love ER, Young LC. Sur une classe de fonctionnelles linéaires. Fundamenta Mathematicae 1937; 28 (1): 243-257 (in French).

[28] Serrin J. On the differentiability of functions of several variables. Archive for Rational Mechanics and Analysis 1961; 7: 359-372.

[29] Musielak J, Orlicz W. On generalized variations (I). Studia Mathematica 1959; 18: 11-41.

[30] Söylemez D, Ünver M. Korovkin type theorems for Cheney-Sharma operators via summability methods. Results in Mathematics 2017; 72 (3): 1601-1612.

[31] Tonelli L. Su alcuni concetti dell'analisi moderna (in Italian). Annali della Scuola Normale Superiore di Pisa-Classe di Scienze 1942; 11 (1-2): 107-118.

[32] Turkun C, Duman O. Modified neural network operators and their convergence properties with summability methods. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 2020; 114: paper no. 132.

[33] Wiener N. The quadratic variation of a function and its Fourier coefficients. Journal of Mathematics and Physics 1924; 3 (2): 72-94.

[34] Young LC. An inequality of the Hölder type, connected with Stieltjes integration. Acta Mathematica 1936; 67: 251-282.

[35] Young LC. Sur une généralisation de la notion de variation de puissance p^{ieme} bornée au sens de M. Wiener, et sur la convergence des séries de Fourier. Comptes rendus de l'Académie des Sciences 1937; 204: 470-472 (in French).

[36] Yurdakadim T, Ünver M. Some results concerning the summability of spliced sequences. Turkish Journal of Mathematics 2016; 40 (5): 1134-1143.