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Classification of some geometric structures on 4-dimensional Riemannian Lie group

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Abstract: In this paper we study the spectral geometry of a 4-dimensional Lie group. The main focus of this paper is to study the 2-Stein and 2-Osserman structures on a 4-dimensional Riemannian Lie group. In this paper, we study the spectrum and trace of Jacobi operator and also we study the characteristic polynomial of generalized Jacobi operator on the non-abelian 4-dimensional Lie group G, whenever G is equipped with an orthonormal left invariant Riemannian metric g. The Lie algebra structures in dimension four have key role in this paper. It is known that in the classification of 4-dimensional non-abelian Lie algebras there are nineteen classes of Lie algebras up to isomorphism [12]. We consider these classes and study all of them. Finally, we study the space form problem and spectral properties of Szabo operator on G.

Key words: Lie group, pointwise Osserman, sectional curvature, 2-Stein.

1. Introduction

Spectral geometry is an area of differential geometry that studies the spectrum of operators and it has intersection with analysis, partial differential equations and differential geometry. Originally, spectral geometry investigates the dependence and properties of eigenvalues and eigenfunctions of the Laplacian, Jacobi operator and other operators. Recently, many applications of spectral geometry are given in the field of computer science, shape recognition, machine learning, heat propagation and vibration (see [13, 15] for instance). Furthermore, Osserman spaces, k-Osserman spaces, 1 and 2-Stein spaces are samples of the study of eigenvalues in the spectral geometry [4–6, 8, 11, 14].

In recent years, there are a lot of papers and results appeared about Osserman spaces in both Riemannian and Lorentzian setting such as [1–3, 7, 9, 10], but as mentioned in [9], [10], spectral geometry is a big branch and it has a lot of unsolved problems. Also there are a few papers and results in the spectral geometry of Lie groups, so in this paper we focus on the spectral geometry of a 4-dimensional Lie group G in the Riemannian setting.

The Jacobi operator $J_X(Y) = R(Y,X)X$ is a very important tool for understanding the relation between the curvature and the geometry of Riemannian manifold (M,g). The Jacobi operator is a self adjoint operator and it plays an important role in the curvature theory. Let $spec(J_X)$ be the set of eigenvalues of Jacobi operator J_X and $S(M,g)$ be the sphere bundle of unit tangent vectors. One says that (M,g) is Osserman at $p \in M$, if

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for every $X, Y \in S_p(M,g)$ the characteristic polynomial of J_X is equal to characteristic polynomial of J_Y. In fact the eigenvalues of J_X are independent of $X \in S_p(M)$. Furthermore, (M,g) is called pointwise Osserman, if it is Osserman at each $p \in M$. Also (M,g) is called globally Osserman manifold if for any point $p \in M$ and any unit tangent vector $X \in T_pM$, the eigenvalues of Jacobi operator depend neither on X nor on p. In fact the eigenvalues of J_X are constant on $S(M,g)$. We recall that globally Osserman manifolds are clearly pointwise Osserman.

Let (M,g) be a Riemannian manifold, $p \in M$, $Z \in S_p(M)$, associated to the Jacobi operators and natural number t, there are functions f_t defined by $f_t(p,Z) = trace(J_Z^{(t)})$, where $J_Z^{(t)}$ is the t^{th} power of the Jacobi operator J_Z. We say that the Riemannian manifold (M,g) is k-Stein at $p \in M$, if $f_t(p,Z)$ is independent of $Z \in S_p(M)$ for every $1 \leqslant t \leqslant k$. Also (M,g) is called k-Stein, if it is k-Stein at each $p \in M$.

Let $Gr_k(T_pM)$ be the Grasmannian of k-planes in T_pM of a Riemannian manifold (M,g). For each $E \in Gr_k(T_pM)$, we define the generalized Jacobi operator as follows

$$J_E(.) = R(.,x_1)x_1 + \cdots + R(.,x_k)x_k,$$

where $\{x_1, \cdots, x_k\}$ is an orthonormal basis for E and R is the Riemannian curvature tensor (note that J_E is independent of the choice of orthonormal basis).

A Riemannian manifold (M,g) is called k-Osserman at $p \in M$, if the characteristic polynomial of $J_E(.)$ is independent of $E \in Gr_k(T_pM)$, that is, the eigenvalues of linear operator $J_E(.)$ counted with multiplicities are constant for every $E \in Gr_k(T_pM)$. Also (M,g) is called globally k-Osserman if the characteristic polynomial of $J_E(.)$ is independent of $E \in \cup_{p\in M} Gr_k(T_pM)$ [10]. It is known that any Riemannian Osserman manifold is 1-Osserman.

This paper is concerned with the spectral properties of Jacobi operator and generalized Jacobi operator of 4-dimensional Lie group G, whenever G is equipped with orthonormal left invariant Riemannian metric g. It is remarkable that if we replace Riemannian metric with an orthonormal left invariant Lorentzian or orthonormal left invariant neutral metrics then we have gap and different results. We do this work by the study of characteristic polynomials, trace and spectrum of the Jacobi operator and the generalized Jacobi operator. Our paper has two parts and two geometric consequences. At the first part, we are looking for Lie algebras such that G equipped with them be a Osserman manifold at identity, and in the second part, we introduce the concept of Szabo manifold at a point, then we find Lie algebras that can provide Szabo manifold at identity on the Lie group G. It is known that there exist nineteen classes of non-abelian 4-dimensional Lie algebras as mentioned in [12], and this classification has key role in our investigation. Also according to this classification we study 2-Stein, 1-Stein and 2-Osserman structures at identity element of the Lie group G, also we study on the pointwise Osserman property. Furthermore, we check the constancy of sectional curvature at identity in the all of options for the Lie algebras of the Lie group G.

2. 2-Stein, 2-Osserman and space form structures

Throughout the paper we consider a non-abelian 4-dimensional Lie group G equipped with a left invariant Riemannian metric g. The goal of this section is to examine the 2-stein and 2-Osserman property and constancy of the sectional curvature of the Lie group G. Our aim is to show which of 4-dimensional Lie algebra structures on a Lie group G can provide pointwise Osserman structure on it.

Let $\mathfrak{g}_n$ be an n-dimensional Lie algebra over the field of real numbers with generator $e_1, \cdots, e_n$, $n \leq 4$.

It is known that there exists only one non-abelian Lie algebra of dimension two, that is the solvable $\mathfrak{af}(1)$ with Lie bracket $[e_1, e_2] = e_1$. This Lie algebra is denoted by $\mathfrak{g}_{2,1}$. In [12], Mubarakzyanov has proved that there exist eight classes of non-abelian Lie algebra of dimension 3. These Lie algebras are denoted by $\mathfrak{g}_{2,1} \oplus \mathfrak{g}_1$, $\mathfrak{g}_{3,k}$, $k = 1, \cdots, 7$. Also, he classified the Lie algebras of dimensional 4 in nineteen classes.

Lemma 2.1 *[7] Let (M, g) be a 4-dimensional Riemannian manifold. Then (M, g) is pointwise Osserman if and only if (M, g) is 2-Stein.*

We divide our study in all of cases as the following;

$\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$: This is a decomposable nilpotent Lie algebra with $[e_2, e_3] = e_1$.

It can easily check that the only nonzero components of the left invariant Levi-Civita connection for Lie groups are as follows

$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = -\frac{1}{2} e_3, \qquad \nabla_{e_1} e_3 = \nabla_{e_3} e_1 = \frac{1}{2} e_2, \qquad \nabla_{e_2} e_3 = -\nabla_{e_3} e_2 = \frac{1}{2} e_1.$$

We consider $e_3, e_4 \in S_e(G, g)$. Using the Levi-Civita connection and the Riemannian curvature tensor we get the Jacobi operator as follows

$$J_{e_3}(e_1) = \frac{1}{4} e_1, \qquad J_{e_3}(e_2) = -\frac{3}{4} e_2, \qquad J_{e_3}(e_j) = 0, \qquad J_{e_4}(e_k) = 0, \quad j = 3, 4, \ k = 1, 2, 3, 4.$$

Thus, the spectrums of Jacobi operators are as follows

$$spec(J_{e_3}) = \{0, \frac{1}{4}, -\frac{3}{4}\}, \qquad spec(J_{e_4}) = \{0\}.$$

Since $spec(J_{e_3}) \neq spec(J_{e_4})$, hence the spectrum of the Jacobi operator depends on the unit tangent vectors. Therefore G is not Osserman at identity, especially G is not pointwise Osserman. Also we have

$$f_1(e, e_3) = trace(J_{e_3}) = -\frac{1}{2}, \qquad f_1(e, e_4) = trace(J_{e_4}) = 0.$$

Thus, G is not 1-stein. Also, according to Lemma 2.1, G is not 2-Stein.

Here, we consider 2-planes $E = span\{e_1, e_2\}$ and $F = span\{e_2, e_3\}$ in T_eG. Using the Levi-Civita connection and the Riemannian curvature tensor we get the generalized Jacobi operator as follows

$$J_E(e_1) = \frac{1}{2} J_F(e_1) = \frac{1}{4} e_1, \ J_E(e_2) = -\frac{1}{3} J_F(e_2) = \frac{1}{4} e_2, \ J_E(e_3) = \frac{2}{3} J_F(e_3) = -\frac{1}{2} e_3, \ J_E(e_4) = J_F(e_4) = 0.$$

Thus, the characteristic polynomial of the generalized Jacobi operators are as follows

$$p_{J_E}(x) = (x - \frac{1}{4})^2 (x + \frac{1}{2}) x, \qquad p_{J_F}(x) = (x - \frac{1}{2})(x + \frac{3}{4})^2 x.$$

Since we have different characteristic polynomials, hence the characteristic polynomials of the generalized Jacobi operators depend on 2-planes E and F. Therefore G is not 2-Osserman at identity. Also, if we take 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then we have $\kappa_\pi = \frac{1}{4}$ and $\kappa_\sigma = 0$, so G is not a space form.

$\mathfrak{g}_{4,1}$**:** This is an indecomposable nilpotent with Lie brackets $[e_2,e_4]=e_1$, $[e_3,e_4]=e_2$.
From the Koszul's formula we obtain all of nonzero components of left invariant Levi-Civita connection ∇ as follows

$$\nabla_{e_1}e_2=\nabla_{e_2}e_1=-\frac{1}{2}e_4,\qquad \nabla_{e_1}e_4=\nabla_{e_4}e_1=\frac{1}{2}e_2,\qquad \nabla_{e_2}e_3=\nabla_{e_3}e_2=-\frac{1}{2}e_4,$$

$$\nabla_{e_2}e_4=\frac{1}{2}e_1+\frac{1}{2}e_3,\qquad \nabla_{e_3}e_4=-\nabla_{e_4}e_3=\frac{1}{2}e_2,\qquad \nabla_{e_4}e_2=-\frac{1}{2}e_1+\frac{1}{2}e_3.$$

Considering $e_3,e_4\in S_e(G,g)$ and using the Levi-Civita connection and the Riemannian curvature tensor we get

$$J_{e_3}(e_1)=J_{e_3}(e_3)=J_{e_4}(e_4)=0,\qquad J_{e_3}(e_2)=-\frac{1}{2}J_{e_4}(e_2)=\frac{1}{4}e_2,\qquad J_{e_3}(e_4)=-\frac{3}{4}e_4,$$
$$J_{e_4}(e_1)=\frac{1}{4}e_1-\frac{1}{4}e_3,\qquad J_{e_4}(e_3)=-\frac{1}{4}e_1-\frac{3}{4}e_3.$$

So $\{0,\frac{1}{4},-\frac{3}{4}\}=spec(J_{e_3})\neq spec(J_{e_4})=\{0,-\frac{1}{2},\frac{-1\pm\sqrt{5}}{4}\}$ and consequently G is not Osserman at identity. Also we have

$$f_1(e,e_3)=trace(J_{e_3})=-\frac{1}{2},\qquad f_1(e,e_4)=trace(J_{e_4})=-1.$$

Thus, G is not 1-stein. Also, according to Lemma 2.1, G is not 2-Stein.

We consider $E=span\{e_1,e_2\}$ and $F=span\{e_1,e_3\}$ in T_eG, so we can get again the generalized Jacobi operators as follows

$$J_E(e_1)=\frac{1}{4}e_1+\frac{1}{4}e_3,\quad J_E(e_2)=\frac{1}{2}J_F(e_2)=\frac{1}{4}e_2,\quad J_E(e_3)=\frac{1}{4}e_1+\frac{1}{4}e_3,\quad J_E(e_4)=\frac{1}{2}J_F(e_4)=-\frac{1}{4}e_4,$$
$$J_F(e_1)=J_F(e_3)=0,\quad J_F(e_4)=-\frac{1}{2}e_4.$$

Also we have $p_{_{J_E}}(x)=(x+\frac{1}{4})(x-\frac{1}{4})x(x-\frac{1}{2})$ and $p_{_{J_F}}(x)=x^2(x-\frac{1}{2})(x+\frac{1}{2})$, therefore G is not 2-Osserman at identity. If we take 2-planes $\pi=\{e_1,e_2\}$ and $\sigma=\{e_1,e_3\}$, we have $\kappa_\pi=\frac{1}{4}$ and $\kappa_\sigma=0$. Hence G is not a space form.

$\mathfrak{g}_{3,7}\oplus\mathfrak{g}_1$**:** This is an unsolvable Lie algebra with the Lie brackets $[e_1,e_2]=e_3$, $[e_2,e_3]=e_1$ and $[e_3,e_1]=e_2$.

By virtue of the Koszul's formula the left invariant Levi-Civita connection is as follows

$$\nabla_{e_1}e_2=-\nabla_{e_2}e_1=\frac{1}{2}e_3,\qquad \nabla_{e_1}e_3=-\nabla_{e_3}e_1=-\frac{1}{2}e_2,\qquad \nabla_{e_2}e_3=-\nabla_{e_3}e_2=\frac{1}{2}e_1.$$

Also we have

$$J_{e_3}(e_1)=\frac{1}{4}e_1,\qquad J_{e_3}(e_2)=\frac{1}{4}e_2,\qquad J_{e_3}(e_i)=0,\qquad i=3,4.$$

Also $J_{e_4}(e_i)=0$, for $i=1,2,3,4$. Since $\{0\}=spec(J_{e_4})\neq spec(J_{e_3})=\{0,\frac{1}{4}\}$, as the same proof in the last cases we deduce that G is not pointwise Osserman. Also we obtain

$$f_1(e,e_3)=trace(J_{e_3})=\frac{1}{2},\qquad f_1(e,e_4)=trace(J_{e_4})=0.$$

Thus, G is not 1-stein, and so according to Lemma 2.1, G is not 2-Stein.

Here, we get the generalized Jacobi operators associated to 2-planes $E=span\{e_1,e_2\}$ and $F=span\{e_1,e_4\}$ in T_eG as follows

$$J_E(e_1)=\frac{1}{4}e_1,\quad J_E(e_2)=J_F(e_2)=\frac{1}{4}e_2,\quad J_E(e_3)=2J_F(e_3)=\frac{1}{2}e_3,\quad J_F(e_1)=J_F(e_4)=J_E(e_4)=0.$$

Since $p_{_{J_E}}(x)=(x-\frac{1}{4})^2(x-\frac{1}{2})x$ and $p_{_{J_F}}(x)=(x-\frac{1}{4})^2x^2$, as the same proof in the last cases this implies that G is not 2-Osserman. If we take 2-planes $\pi=\{e_1,e_2\}$ and $\sigma=\{e_1,e_4\}$, then we have $\kappa_\pi=\frac{1}{4}$ and $\kappa_\sigma=0$, thus, G is not a space form.

$\mathfrak{g}_{3,6}\oplus\mathfrak{g}_1$: This is an unsolvable Lie algebra with the Lie brackets $[e_1,e_2]=e_1$, $[e_1,e_3]=2e_2$ and $[e_2,e_3]=e_3$.

From the Koszul's formula we get

$$\nabla_{e_1}e_1=-e_2,\qquad \nabla_{e_1}e_2=e_1-e_3,\qquad \nabla_{e_1}e_3=\nabla_{e_3}e_3=-\nabla_{e_3}e_1=e_2,\qquad \nabla_{e_2}e_1=-e_3,$$
$$\nabla_{e_2}e_3=e_1,\qquad \nabla_{e_3}e_2=e_1-e_3.$$

We get also

$$J_{e_3}(e_1)=-2e_1,\qquad J_{e_3}(e_i)=0,\qquad i=2,3,4,$$

and $J_{e_4}(e_i)=0$, for $i=1,2,3,4$. In this case we have $\{0,-2\}=spec(J_{e_3})\neq spec(J_{e_4})=\{0\}$, thus, G is not pointwise Osserman. We have also

$$f_1(e,e_3)=trace(J_{e_3})=-2,\qquad f_1(e,e_4)=trace(J_{e_4})=0.$$

Hence, G is not 1-stein, and so from Lemma 2.1, we have (G,g) is not 2-Stein at identity.

Considering 2-planes $E=span\{e_1,e_2\}$ and $F=span\{e_1,e_4\}$ in T_eG, we get

$$J_E(e_1)=2e_3,\qquad J_E(e_2)=0,\qquad J_E(e_3)=2e_1-2e_3,\qquad J_E(e_4)=0,$$

$$J_F(e_1)=0,\qquad J_F(e_2)=0,\qquad J_F(e_3)=-2e_3,\qquad J_F(e_4)=0.$$

In this case we have $p_{_{J_E}}(x)=x^2(x^2+2x-4)$ and $p_{_{J_F}}(x)=x^3(x+2)$, so G is not 2-Osserman at identity. If we take 2-planes $\pi=\{e_1,e_3\}$ and $\sigma=\{e_1,e_4\}$, then we have $\kappa_\pi=-2$ and $\kappa_\sigma=0$, thus, G is not a space form.

$\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1$: The decomposable solvable Lie algebra with the Lie bracket $[e_1,e_2]=e_1$.

The Koszul's formula implies that the only nonzero components of the left invariant Levi-Civita connection ∇ are as follows

$$\nabla_{e_1}e_1=-e_2,\qquad \nabla_{e_1}e_2=e_1.$$

We get also

$$J_{e_1}(e_i) = 0, \qquad i = 1, 3, 4, \qquad J_{e_1}(e_2) = -e_2,$$

and $J_{e_4}(e_i) = 0$, for $i = 1, 2, 3, 4$. So we have $\{0, -1\} = spec(J_{e_1}) \neq spec(J_{e_4}) = \{0\}$, thus, G is not pointwise Osserman. We get also

$$f_1(e, e_1) = trace(J_{e_1}) = -1, \qquad f_1(e, e_4) = trace(J_{e_4}) = 0.$$

Hence, G is not 1-stein, and so (G, g) is not 2-Stein.

Consider 2-planes $E = span\{e_1, e_2\}$ and $F = span\{e_3, e_4\}$ in T_eG. Therefore we get

$$J_E(e_1) = -e_1, \qquad J_E(e_2) = -e_2, \qquad J_E(e_3) = 0, \qquad J_E(e_4) = 0,$$

$$J_F(e_1) = 0, \qquad J_F(e_2) = 0, \qquad J_F(e_3) = 0, \qquad J_F(e_4) = 0.$$

In this case we have $p_{J_E}(x) = (x+1)^2x^2$ and $p_{J_F}(x) = x^4$. Thus, G is not 2-Osserman at identity. If we take 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then we have $\kappa_\pi = -1$ and $\kappa_\sigma = 0$, therefore G is not a space form.

The following is the first case of Lie algebra structure $\mathfrak{g}$ such that $(G, \mathfrak{g})$ has 1-Stein property at identity element in Riemannian setting. Also we emphasize that in Lorentzian setting there exists the same result, if G equipped with the following Lie algebra structure.

$2\mathfrak{g}_{2,1}$: The decomposable solvable Lie algebra with the Lie brackets $[e_1, e_2] = e_1$ and $[e_3, e_4] = e_3$.

In this case, the only nonzero components of the left invariant Levi-Civita connection ∇ are as follows

$$\nabla_{e_1}e_1 = -e_2, \qquad \nabla_{e_1}e_2 = e_1, \qquad \nabla_{e_3}e_3 = -e_4, \qquad \nabla_{e_3}e_4 = e_3.$$

Now we consider the unit vector $v = re_1 + se_2 + te_3 + ke_4 \in S_e(G, g)$. Using the Levi-Civita connection and the Riemannian curvature tensor, we get

$$J_v(e_1) = -s^2e_1 + sre_2, \qquad J_v(e_2) = rse_1 - r^2e_2, \qquad J_v(e_3) = -k^2e_3 + kte_4, \qquad J_v(e_4) = tke_3 - t^2e_4.$$

Finally we obtain $trace(J_v) = -(r^2 + s^2 + t^2 + k^2) = -1$. Since trace is independent of v, thus, G is 1-Stein at identity. We get also $spec(J_v) = \{0, -(s^2 + r^2), -(t^2 + k^2)\}$. Therefore the spectrum depends to the unit vector v. Hence, G is not pointwise Osserman. Using Lemma 2.1 , we deduce that (G, g) is not 2-Stein at identity.

Now we consider the 2-plane $A = span\{v_1, v_2\}$, where $v_1 = re_1+se_2+te_3+ke_4$, $v_2 = xe_1+ye_2+ze_3+we_4$ and $\{v_1, v_2\}$ is a orthonormal basis for A. Direct computations give us

$$J_A(e_1) = (-s^2 - y^2)e_1 + (sr + xy)e_2, \qquad J_A(e_2) = (rs + xy)e_1 + (-r^2 - x^2)e_2,$$

$$J_A(e_3) = (-k^2 - w^2)e_3 + (kt + wz)e_4, \qquad J_A(e_4) = (tk + zw)e_3 + (-t^2 - z^2)e_4.$$

We obtain also

$$p_{J_A}(X) = (X^2 + (x^2 + y^2 + s^2 + r^2)X + (sx - yr)^2)(X^2 + (z^2 + w^2 + t^2 + k^2)X + (kz - wt)^2). \qquad (2.1)$$

Here we consider the 2-palnes $E = span\{e_1, e_2\}$ and $F = span\{\frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{\sqrt{2}}e_3, e_4\}$. According to (2.1), we have $p_{J_E}(x) = x^2(x+1)^2$ and $p_{J_F}(x) = x(x+\frac{1}{2})^2(x+1)$. So we conclude that G is not 2-Osserman at identity. If we take the 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then we have $\kappa_\pi = -1$ and $\kappa_\sigma = 0$, thus, G is not a space form.

$\mathfrak{g}_{3,2} \oplus \mathfrak{g}_1$: The decomposable solvable Lie algebra with the Lie brackets $[e_1, e_3] = e_1$ and $[e_2, e_3] = e_1 + e_2$.

In this case, we conclude that all components of the left invariant Levi-Civita connection ∇ are zero, except

$$\nabla_{e_1}e_1 = -e_3, \qquad \nabla_{e_1}e_2 = \nabla_{e_2}e_1 = -\frac{1}{2}e_3, \qquad \nabla_{e_1}e_3 = e_1 + \frac{1}{2}e_2,$$

$$\nabla_{e_2}e_2 = -e_3, \qquad \nabla_{e_2}e_3 = \frac{1}{2}e_1 + e_2, \qquad \nabla_{e_3}e_1 = \frac{1}{2}e_2, \qquad \nabla_{e_3}e_2 = -\frac{1}{2}e_1.$$

We get also

$$J_{e_3}(e_1) = -\frac{3}{4}e_1 - e_2, \qquad J_{e_3}(e_2) = -e_1 - \frac{7}{4}e_2, \qquad J_{e_3}(e_i) = 0, \qquad i = 3, 4,$$

and $J_{e_4}(e_i) = 0$, for $i = 1, 2, 3, 4$. So we obtain $spec(J_{e_3}) = \{0, \frac{-5\pm 2\sqrt{5}}{4}\}$ and $spec(J_{e_4}) = \{0\}$. Thus, G is not pointwise Osserman. Furthermore we have

$$f(e, e_3) = trace(J_{e_3}) = -\frac{5}{2}, \qquad f(e, e_4) = trace(J_{e_4}) = 0.$$

Hence, G is not 1-Stein. Also according to Lemma 2.1, we imply that (G, g) is not 2-Stein. Considering the 2-planes $E = span\{e_1, e_2\}$ and $F = span\{e_3, e_4\}$, we obtain

$$J_E(e_1) = -\frac{3}{4}e_1, \qquad J_E(e_2) = -\frac{3}{4}e_2, \qquad J_E(e_3) = -\frac{5}{2}e_3, \qquad J_E(e_4) = 0.$$

$$J_F(e_1) = -\frac{3}{4}e_1 - e_2, \qquad J_F(e_2) = -e_1 - \frac{7}{4}e_2, \qquad J_F(e_3) = 0, \qquad J_F(e_4) = 0.$$

We get also

$$p_{J_E}(x) = (x + \frac{3}{4})^2(x + \frac{5}{2})x, \qquad p_{J_F}(x) = x^2(x^2 + \frac{5}{2}x + \frac{5}{16}).$$

Thus, G is not 2-Osserman. If we take the 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then we have $\kappa_\pi = -\frac{3}{4}$ and $\kappa_\sigma = 0$, thus, G is not a space form.

$\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1$: The decomposable solvable Lie algebra with Lie brackets $[e_1, e_3] = e_1$ and $[e_2, e_3] = e_2$.

Here we study the left invariant Levi-Civita connection ∇. From the Koszul's formula we deduce that the following are the nonzero components of ∇:

$$\nabla_{e_1}e_1 = -e_3, \qquad \nabla_{e_1}e_3 = e_1, \qquad \nabla_{e_2}e_2 = -e_3, \qquad \nabla_{e_2}e_3 = e_2,$$

We have also

$$J_{e_3}(e_1) = -e_1, \qquad J_{e_3}(e_2) = -e_2, \qquad J_{e_3}(e_i) = 0, \qquad i = 3, 4,$$

and $J_{e_4}(e_i) = 0$, for $i = 1, 2, 3, 4$. Therefore $spec(J_{e_3}) = \{0, -1\}$ and $spec(J_{e_4}) = \{0\}$. Thus, G is not pointwise Osserman. Furthermore we have

$$f(e, e_3) = trace(J_{e_3}) = -2, \qquad f(e, e_4) = trace(J_{e_4}) = 0.$$

Hence, G is not 1-Stein, and consequently (G, g) is not 2-Stein.

Consider the 2-planes $E = span\{e_1, e_2\}$ and $F = span\{e_3, e_4\}$. So we get the following

$$J_E(e_1) = -e_1, \qquad J_E(e_2) = -e_2, \qquad J_E(e_3) = -2e_3, \qquad J_E(e_4) = 0.$$

$$J_F(e_1) = -e_1, \qquad J_F(e_2) = -e_2, \qquad J_F(e_3) = 0, \qquad J_F(e_4) = 0.$$

So

$$p_{J_E}(x) = (x + 1)^2(x + 2)x, \qquad p_{J_F}(x) = (x + 1)^2x^2.$$

Thus, G is not 2-Osserman. If we take the 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then we have $\kappa_\pi = -1$ and $\kappa_\sigma = 0$, so G is not a space form.

$\mathfrak{g}_{3,4} \oplus \mathfrak{g}_1$: The decomposable solvable Lie algebra with $[e_1, e_3] = e_1$ and $[e_2, e_3] = \alpha e_2$, $-1 \leq \alpha < 1, \alpha \neq 0$.

In this case we have

$$\nabla_{e_1}e_1 = -e_3, \qquad \nabla_{e_1}e_3 = e_1, \qquad \nabla_{e_2}e_2 = -\alpha e_3, \qquad \nabla_{e_2}e_3 = \alpha e_2.$$

We obtain also

$$J_{e_3}(e_1) = -e_1, \qquad J_{e_3}(e_2) = -\alpha^2 e_2, \qquad J_{e_3}(e_i) = 0, \qquad i = 3, 4,$$

and $J_{e_4}(e_i) = 0$, for $i = 1, 2, 3, 4$. So $spec(J_{e_3}) = \{0, -1, -\alpha^2\}$ and $spec(J_{e_4}) = \{0\}$. Thus, G is not Osserman at identity. Furthermore we have

$$f(e, e_3) = trace(J_{e_3}) = -1 - \alpha^2, \qquad f(e, e_4) = trace(J_{e_4}) = 0.$$

Hence, G is not 1-Stein. Also according to Lemma 2.1, we derive that (G, g) is not 2-Stein. Considering $E = span\{e_1, e_2\}$ and $F = span\{e_3, e_4\}$, we get

$$J_E(e_1) = -\alpha e_1, \qquad J_E(e_2) = -\alpha e_2, \qquad J_E(e_3) = (-\alpha^2 - 1)e_3, \qquad J_E(e_4) = 0.$$

$$J_F(e_1) = -e_1, \qquad J_F(e_2) = -\alpha^2 e_2, \qquad J_F(e_3) = 0, \qquad J_F(e_4) = 0,$$

and

$$p_{J_E}(x) = (x + \alpha)^2(x + \alpha^2 + 1)x, \qquad p_{J_F}(x) = (x + 1)(x + \alpha^2)x^2.$$

Hence, G is not 2-Osserman. If we take the 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then we have $\kappa_\pi = -\alpha$ and $\kappa_\sigma = 0$, so G is not a space form.

The following is the first case of Lie algebra structure $\mathfrak{g}$, in Riemannian setting such that $(G, \mathfrak{g})$ has 2-Stein and 2-Osserman property at the identity element and this achievement will gain by little effort. Also we emphasize that about 2-Stein property there exists a gap and different result in Lorentzian setting whenever G eqipped with the following Lie algebra structure.

$\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1$**:** The decomposable Lie algebra with $[e_1, e_3] = \beta e_1 - e_2$, $[e_2, e_3] = e_1 + \beta e_2$, $\beta \geq 0$.

Koszul's formula concludes that the nonzero coefficients of the left invariant Levi-Civita connection are as follows

$$\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = -\beta e_3, \qquad \nabla_{e_1} e_3 = -\beta \nabla_{e_3} e_2 = \beta e_1, \qquad \nabla_{e_2} e_3 = \beta \nabla_{e_3} e_1 = \beta e_2.$$

If $\beta > 0$, we get

$$J_{e_3}(e_1) = -\beta^2 e_1, \qquad J_{e_3}(e_2) = -\beta^2 e_2, \qquad J_{e_3}(e_i) = 0, \qquad i = 3, 4,$$

and $J_{e_4}(e_i) = 0$, for $i = 1, 2, 3, 4$. So $spec(J_{e_3}) = \{0, -\beta^2\}$ and $spec(J_{e_4}) = \{0\}$. Thus, G is not Osserman at identity. Furthermore we have

$$f(e, e_3) = trace(J_{e_3}) = -2\beta^2, \qquad f(e, e_4) = trace(J_{e_4}) = 0.$$

Hence, G is not 1-Stein. Now we consider $\beta = 0$ and assume that $v = re_1 + se_2 + te_3 + ke_4 \in S_e(G, g)$. The Riemann curvature tensor and direct computations give us

$$J_v(e_1) = J_v(e_2) = J_v(e_3) = J_v(e_4) = 0.$$

Thus, matrix representation of J_v is zero. Therefore, $spec(J_v) = \{0\}$, i.e. the spectrum is independent of unit vector v. Hence, G is Osserman at identity. Also according to the Lemma 2.1, we have G is 2-Stein at identity.

For $\beta > 0$, we get $E = span\{e_1, e_2\}$ and $F = span\{e_3, e_4\}$. So we have

$$J_E(e_1) = J_F(e_1) = -\beta^2 e_1, \qquad J_E(e_2) = J_F(e_2) = -\beta^2 e_2, \qquad J_E(e_3) = -2\beta^2 e_3,$$
$$J_E(e_4) = J_F(e_4) = J_F(e_3) = 0.$$

Also we get

$$p_{J_E}(x) = (x + \beta^2)^2 (x + 2\beta^2) x, \qquad p_{J_F}(x) = (x + \beta^2)^2 x^2.$$

Thus, in this case G is not 2-Osserman at identity. If $\beta = 0$ and $E = span\{v_1, v_2\}$ is an arbitrary 2-plane, where $v_1 = re_1 + se_2 + te_3 + ke_4$ and $v_2 = xe_1 + ye_2 + ze_3 + we_4$, using the Riemann curvature tensor and the Levi-Civita connection we obtain

$$J_E(e_1) = J_E(e_2) = J_E(e_3) = J_E(e_4) = 0.$$

Thus matrix representation of J_E is zero. Therefore, $p_{J_E}(x) = x^4$, i.e. the characteristic polynomial of generalized Jacobi operator is independent of 2-plane E. Hence, G is 2-Osserman at identity. Now we take 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then we have $\kappa_\pi = -\beta^2$ and $\kappa_\sigma = 0$, if $\beta \neq 0$. Therefore, $\kappa_\sigma \neq \kappa_\pi$ and consequently, G is not a space form. If $\beta = 0$, we take arbitrary orthonormal basis $\pi = \{v_1, v_2\}$, where

$v_1 = re_1 + se_2 + te_3 + ke_4$ and $v_2 = xe_1 + ye_2 + ze_3 + we_4$. Then direct computations give us $\kappa_\pi = 0$, so in this case G has constant sectional curvature at identity.

$\mathfrak{g}_{4,3}$: The indecomposable solvable Lie algebra with $[e_1, e_4] = e_1$, $[e_3, e_4] = e_2$.

Koszul's formula implies that the nonzero coefficients of the left invariant Levi-Civita connection ∇ are as follows

$$\nabla_{e_1} e_4 = e_1, \qquad \nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \frac{1}{2}\nabla_{e_1} e_1 = -\frac{1}{2}e_4,$$

$$\nabla_{e_3} e_4 = -\nabla_{e_4} e_3 = \frac{1}{2}e_2, \qquad \nabla_{e_4} e_2 = \nabla_{e_2} e_4 = \frac{1}{2}e_3.$$

Considering $e_1, e_4 \in S_e(G, g)$, we get

$$J_{e_1}(e_2) = \frac{2}{3}J_{e_4}(e_3) = -\frac{1}{2}e_3, \qquad J_{e_1}(e_3) = -2J_{e_4}(e_2) = -\frac{1}{2}e_2, \qquad J_{e_1}(e_4) = -e_4,$$
$$J_{e_4}(e_1) = -e_1, \qquad J_{e_i}(e_i) = 0, \qquad i = 1, 4.$$

Therefore

$$spec(J_{e_1}) = \{0, -1, \pm\frac{1}{2}\}, \qquad spec(J_{e_4}) = \{0, -1, \frac{1}{4}, -\frac{3}{4}\}.$$

Since $spec(J_{e_1}) \neq spec(J_{e_4})$, so G is not Osserman at identity, especially G is not pointwise Osserman. We have also

$$f_1(e, e_1) = trace(J_{e_1}) = -1, \qquad f_1(e, e_4) = trace(J_{e_4}) = -\frac{3}{2}.$$

Thus, G is not 1-Stein, and consequently G is not 2-Stein.

Considering $E = span\{e_1, e_2\}$ and $F = span\{e_2, e_3\}$, we get

$$J_E(e_1) = J_F(e_1) = 0, \qquad J_E(e_2) = -2J_F(e_3) = -\frac{1}{2}e_3, \qquad J_E(e_3) = -\frac{1}{2}e_2 + \frac{1}{4}e_3,$$
$$J_F(e_2) = \frac{1}{4}e_2, \qquad J_E(e_4) = \frac{3}{2}J_F(e_4) = -\frac{3}{4}e_4.$$

Therefore,

$$p_{J_E}(x) = x(x + \frac{3}{4})(x^2 - \frac{x}{4} - \frac{1}{4}), \qquad p_{J_F}(x) = x(x - \frac{1}{4})^2(x + \frac{1}{2}).$$

Hence, G is not 2-Osserman at identity. If we take 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then we have $\kappa_\pi = 0$ and $\kappa_\sigma = -1$. This implies that G is not a space form.

$\mathfrak{g}_{4,2}$: The indecomposable solvable Lie algebra with $[e_1, e_4] = \beta e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_2 + e_3$, $\beta \neq 0$.

In this case, we get the following nonzero coefficients for the left invariant Levi-Civita connection ∇:

$$\nabla_{e_1} e_1 = -\beta e_4, \qquad \nabla_{e_1} e_4 = \beta e_1, \qquad \nabla_{e_2} e_2 = -e_4, \qquad \nabla_{e_2} e_3 = \nabla_{e_3} e_2 = -\frac{1}{2}e_4, \qquad \nabla_{e_2} e_4 = e_2 + \frac{1}{2}e_3,$$

$$\nabla_{e_3}e_3=-e_4,\qquad \nabla_{e_3}e_4=\frac{1}{2}e_2+e_3,\qquad \nabla_{e_4}e_2=\frac{1}{2}e_3,\qquad \nabla_{e_4}e_3=-\frac{1}{2}e_2.$$

Considering $e_2,e_3\in S_e(G,g)$, we get

$$J_{e_3}(e_1)=-\beta e_1,\qquad J_{e_3}(e_2)=-\frac{3}{4}e_2,\qquad J_{e_3}(e_4)=-\frac{7}{4}e_4,$$

$$J_{e_2}(e_1)=-\beta e_1,\qquad J_{e_2}(e_3)=-\frac{3}{4}e_3,\qquad J_{e_2}(e_4)=-\frac{3}{4}e_4,\qquad J_{e_i}(e_i)=0\qquad i=2,3.$$

Therefore, $\{0,-\beta,-\frac{3}{4}\}=spec(J_{e_2})\neq spec(J_{e_3})=\{0,-\beta,-\frac{3}{4},-\frac{7}{4}\}$, i.e. G is not Osserman at identity. We have also

$$f_1(e,e_3)=trace(J_{e_3})=-\beta-\frac{5}{2},\qquad f_1(e,e_2)=trace(J_{e_2})=-\beta-\frac{3}{2}.$$

Thus, G is not 1-Stein and G is not 2-Stein.

Consider $E=span\{e_2,e_4\}$ and $F=span\{e_1,e_2\}$. By direct computations we obtain

$$J_E(e_1)=-(\beta+\beta^2)e_1,\qquad J_E(e_2)=-\frac{3}{4}e_2-e_3,\qquad J_E(e_3)=-e_2-\frac{5}{2}e_3,\qquad J_E(e_4)=-\frac{3}{4}e_4,$$

$$J_F(e_1)=-\beta e_1,\qquad J_F(e_2)=-\beta e_2-\frac{\beta}{2}e_3,\qquad J_F(e_3)=-\frac{\beta}{2}e_2-(\beta+\frac{3}{4})e_3,\qquad J_F(e_4)=-(\beta^2+\frac{3}{4})e_4.$$

Also we have

$$p_{_{J_E}}(x)=(x+\beta+\beta^2)(x+\frac{3}{4})(x^2+\frac{13}{4}x+\frac{7}{8}),\quad p_{_{J_F}}(x)=(x+\beta^2+\frac{3}{4})(x+\beta)(x^2+(2\beta+\frac{3}{4})x+(\frac{3}{4}\beta^2+\frac{3}{4}\beta)).$$

Since we have different characteristic polynomials, G is not 2-Osserman at identity. If we take 2-planes $\pi=\{e_2,e_4\}$ and $\sigma=\{e_3,e_4\}$, then we have $\kappa_\pi=-\frac{3}{4}$ and $\kappa_\sigma=-\frac{7}{4}$, so G is not a space form.

$\mathfrak{g}_{4,4}$: The indecomposable solvable Lie algebra with $[e_1,e_4]=e_1$, $[e_2,e_4]=e_1+e_2$, $[e_3,e_4]=e_2+e_3$. Koszul's formula implies that

$$\nabla_{e_1}e_1=-e_4,\qquad \nabla_{e_1}e_2=\nabla_{e_2}e_1=-\frac{1}{2}e_4,\qquad \nabla_{e_1}e_4=e_1+\frac{1}{2}e_2,\qquad \nabla_{e_2}e_2=-e_4,$$

$$\nabla_{e_2}e_3=\nabla_{e_3}e_2=-\frac{1}{2}e_4,\qquad \nabla_{e_2}e_4=\frac{1}{2}e_1+e_2+\frac{1}{2}e_3,\qquad \nabla_{e_3}e_3=-e_4,$$

$$\nabla_{e_3}e_4=\frac{1}{2}e_2+e_3,\qquad \nabla_{e_4}e_1=\frac{1}{2}e_2,\qquad \nabla_{e_4}e_2=-\frac{1}{2}e_1+\frac{1}{2}e_3,\qquad \nabla_{e_4}e_3=-\frac{1}{2}e_2.$$

Considering $e_1,e_3\in S_e(G,g)$, we have

$$J_{e_3}(e_1)=-e_1-\frac{1}{2}e_2,\qquad J_{e_3}(e_2)=-\frac{1}{2}e_1-\frac{3}{4}e_2,\qquad J_{e_3}(e_4)=-\frac{7}{4}e_4,$$

$$J_{e_i}(e_i)=0, \qquad i=1,3 \qquad J_{e_1}(e_2)=-\frac{3}{4}e_2-\frac{1}{2}e_3, \qquad J_{e_1}(e_3)=-\frac{1}{2}e_2-e_3, \qquad J_{e_1}(e_4)=-\frac{3}{4}e_4.$$

Since $\{0,-\frac{7}{4},\frac{-7\pm\sqrt{17}}{8}\}=spec(J_{e_3})\neq spec(J_{e_1})=\{0,-\frac{3}{4},\frac{-7\pm\sqrt{17}}{8}\}$, then we deduce that G is not pointwise Osserman. Also we obtain

$$f_1(e,e_3)=trace(J_{e_3})=-\frac{7}{2}, \qquad f_1(e,e_1)=trace(J_{e_1})=-\frac{5}{2}.$$

Thus, G is not 1-Stein and G is not 2-Stein.

Considering $E=span\{e_1,e_2\}$ and $F=span\{e_1,e_3\}$, we obtain

$$J_E(e_1)=-\frac{3}{4}e_1+\frac{1}{4}e_3, \qquad J_E(e_2)=-\frac{3}{4}e_2-\frac{1}{2}e_3, \qquad J_E(e_3)=\frac{1}{4}e_1-\frac{1}{2}e_2-\frac{7}{4}e_3, \qquad J_E(e_4)=-\frac{9}{4}e_4,$$

$$J_F(e_1)=-e_1-\frac{1}{2}e_2, \qquad J_F(e_2)=-\frac{1}{2}e_1-\frac{3}{2}e_2-\frac{1}{2}e_3, \qquad J_F(e_3)=-\frac{1}{2}e_1-e_3, \qquad J_F(e_4)=-\frac{5}{2}e_4.$$

We get also

$$p_{J_E}(x)=(x+\frac{9}{4})(x+\frac{3}{4})(x^2+\frac{5}{2}x+1), \qquad p_{J_F}(x)=(x+\frac{5}{2})(x+1)(x^2+\frac{5}{2}x+1).$$

This implies that G is not 2-Osserman at identity. If we take the 2-planes $\pi=\{e_1,e_2\}$ and $\sigma=\{e_1,e_3\}$, we have $\kappa_\pi=-\frac{3}{4}$ and $\kappa_\sigma=-1$. This implies that G is not a space form.

The following is the first case of the indecomposable solvable Lie algebra structure $\mathfrak{g}$ such that $(G,\mathfrak{g})$ has 1-Stein property at identity. In this case $\mathfrak{g}$ is not 2-Stein.

$\mathfrak{g}_{4,5}$: The indecomposable solvable Lie algebra with $[e_1,e_4]=\alpha e_1$, $[e_2,e_4]=\beta e_2$, $[e_3,e_4]=\gamma e_3$, $\alpha\beta\gamma\neq 0$.

From the Koszul's formula we obtain the nonzero components of the left invariant Levi-Civita connection ∇ are as follows

$$\nabla_{e_1}e_1=-\alpha e_4, \qquad \nabla_{e_1}e_4=\alpha e_1, \qquad \nabla_{e_2}e_2=-\beta e_4,$$

$$\nabla_{e_2}e_4=\beta e_2, \qquad \nabla_{e_3}e_3=-\gamma e_4, \qquad \nabla_{e_3}e_4=\gamma e_3.$$

We get also

$$J_{e_3}(e_1)=-\gamma\alpha e_1, \qquad J_{e_3}(e_2)=-\gamma\beta e_2, \qquad J_{e_3}(e_4)=-\gamma^2 e_4,$$

$$J_{e_4}(e_1)=-\alpha^2 e_1, \qquad J_{e_4}(e_2)=-\beta^2 e_2, \qquad J_{e_4}(e_3)=-\gamma^2 e_3, \qquad J_{e_i}(e_i)=0, \qquad i=3,4.$$

There are two cases. If α,β,γ are distinct, then in this case we have $\{0,-\gamma\alpha,-\gamma\beta,-\gamma^2\}=spec(J_{e_3})\neq spec(J_{e_4})=\{0,-\alpha^2,-\beta^2,-\gamma^2\}$, thus, G is not pointwise Osserman. We have also

$$f_1(e,e_3)=trace(J_{e_3})=-\gamma\alpha-\gamma\beta-\gamma^2, \qquad f_1(e,e_4)=trace(J_{e_4})=-\alpha^2-\beta^2-\gamma^2.$$

Hence, G is not 1-stein. If $\alpha=\beta=\gamma$ and $v=re_1+se_2+te_3+ke_4\in S_e(G,g)$, then in this case we have

$$J_v(e_1)=\alpha^2\{-(s^2+t^2+k^2)e_1+(sr)e_2+(tr)e_3+(kr)e_4\},$$

$$J_v(e_2)=\alpha^2\{(rs)e_1-(r^2+t^2+k^2)e_2+(ts)e_3+(ks)e_4\},$$

$$J_v(e_3)=\alpha^2\{(rt)e_1+(st)e_2-(r^2+s^2+k^2)e_3+(kt)e_4\},$$

$$J_v(e_4)=\alpha^2\{(rk)e_1+(sk)e_2+(tk)e_3-(r^2+s^2+t^2)e_4\}.$$

Therefore, we have $trace(J_v)=-3\alpha^2(r^2+s^2+t^2+k^2)=-3\alpha^2$. Since trace is independent of v, hence, G is 1-Stein at identity. Now a direct computation gives us $p_{J_v}(x)=x(x+(r^2+s^2+t^2+k^2)\alpha^2)^3$. But $(r^2+s^2+t^2+k^2)=1$, since v is unit. Therefore $spec(p_{J_v})=\{0,-\alpha^2\}$. Thus, in this case G is Osserman.

There are three states. If α,β,γ are distinct pairwise, considering $E=span\{e_1,e_2\}$ and $F=span\{e_1,e_3\}$, we obtain

$$J_E(e_1)=-\beta\alpha e_1,\qquad J_E(e_2)=-\beta\alpha e_2,\qquad J_E(e_3)=-(\alpha\gamma+\beta\gamma)e_3,\qquad J_E(e_4)=-(\alpha^2+\beta^2)e_4,$$

$$J_F(e_1)=-\alpha\gamma e_1,\qquad J_F(e_2)=-(\alpha\beta+\gamma\beta)e_2,\qquad J_F(e_3)=-\alpha\gamma e_3,\qquad J_F(e_4)=-(\alpha^2+\gamma^2)e_4.$$

We get also

$$p_{J_E}(x)=(x+\alpha\beta)^2(x+(\alpha+\beta)\gamma)(x+(\alpha^2+\beta^2)),\qquad p_{J_F}(x)=(x+\alpha\gamma)^2(x+(\alpha+\gamma)\beta)(x+(\alpha^2+\gamma^2)).$$

Thus, in this case G is not 2-Osserman.
If $\alpha\neq\beta=\gamma$, considering $H=spec\{e_3,e_4\}$, and in comparison with two plane E we get

$$p_{J_H}(x)=(x+\alpha(\alpha+\beta))(x+2\beta^2)(x+\beta^2)^2.$$

Thus, in this case, it implies that G is not 2-Osserman.
If $\alpha=\beta=\gamma$, considering $E=span\{v_1,v_2\}$, where $v_1=re_1+se_2+te_3+ke_4$ and $v_2=xe_1+ye_2+ze_3+we_4$, we have

$$J_E(e_1)=-(s^2+t^2+k^2+y^2+z^2+w^2)\alpha^2e_1+(sr+xy)\alpha^2e_2+(tr+zx)\alpha^2e_3+(kr+wx)\alpha^2e_4,$$
$$J_E(e_2)=(rs+xy)\alpha^2e_1-(r^2+t^2+k^2+x^2+z^2+w^2)\alpha^2e_2+(ts+zy)\alpha^2e_3+(ks+wy)\alpha^2e_4,$$
$$J_E(e_3)=(rt+xz)\alpha^2e_1+(st+yz)\alpha^2e_2-(r^2+s^2+k^2+x^2+y^2+w^2)\alpha^2e_3+(kt+wz)\alpha^2e_4,$$
$$J_E(e_4)=(rk+xw)\alpha^2e_1+(sk+yw)\alpha^2e_2+(tk+zw)\alpha^2e_3-(r^2+s^2+t^2+x^2+y^2+z^2)\alpha^2e_4.$$

Now if $E=span\{e_1,e_2\}$ and $F=span\{e_1,\frac{1}{\sqrt{2}}e_3+\frac{1}{\sqrt{2}}e_4\}$, we get

$$p_{J_E}(x)=(x+\alpha^2)^2(x+2\alpha^2)^2,\qquad p_{J_F}(x)=(x+\alpha^2)(x+2\alpha^2)(x^2+3\alpha^2x+2\alpha^4).$$

Thus, in this case G is not 2-Osserman. Now we take $\pi=\{e_1,e_2\}$ and $\sigma=\{e_1,e_3\}$. Then we get $\kappa_\pi=-\alpha\beta$ and $\kappa_\sigma=-\alpha\gamma$, so if $\beta\neq\gamma$, G is not a space form. If $\beta=\gamma\neq\alpha$, we take $\pi=\{e_1,e_2\}$ and $\sigma=\{e_2,e_3\}$, then

we obtain $\kappa_\pi = -\alpha\beta$ and $\kappa_\sigma = -\beta\gamma$, which implies that G is not a space form. If $\alpha = \beta = \gamma$ we consider arbitrary orthonormal basis $\pi = \{v_1, v_2\}$, where $v_1 = re_1 + se_2 + te_3 + ke_4$ and $v_2 = xe_1 + ye_2 + ze_3 + we_4$, then we have the following

$$r^2 + s^2 + t^2 + k^2 = 1, \qquad x^2 + y^2 + z^2 + w^2 = 1, \qquad rx + sy + zt + kw = 0. \tag{2.2}$$

Now direct computations give us $\kappa_\pi = p\alpha^2$, where

$$\begin{aligned} p = &-r^2(y^2 + z^2 + w^2) - s^2(x^2 + z^2 + w^2) - t^2(x^2 + y^2 + w^2) - k^2(x^2 + y^2 + z^2) \\ &+ 2yxrs + 2zxtr + 2szty + 2rkxw + 2sykw + 2tkwz. \end{aligned}$$

From (2.2), we deduce that $p = -1$, which implies $\kappa_\pi = -\alpha^2$. Therefore, the sectional curvature is independent of orthonormal 2-plane π. Thus, G has constant sectional curvature at identity.

The following is the second case of indecomposable solvable Lie algebra structure $\mathfrak{g}$ such that $(G, \mathfrak{g})$ has 1-Stein property at identity. Note that in this case $\mathfrak{g}$ is not 2-Stein.

$\mathfrak{g}_{4,6}$: The indecomposable solvable Lie algebra with $[e_1, e_4] = \alpha e_1$, $[e_2, e_4] = \beta e_2 - e_3$, $[e_3, e_4] = e_2 + \beta e_3$, $\alpha > 0$.

In this case we have

$$\nabla_{e_1} e_1 = -\alpha e_4, \qquad \nabla_{e_1} e_4 = \alpha e_1, \qquad \nabla_{e_2} e_2 = -\beta e_4, \qquad \nabla_{e_2} e_4 = \beta e_2,$$

$$\nabla_{e_3} e_3 = -\beta e_4, \qquad \nabla_{e_3} e_4 = \beta e_3, \qquad \nabla_{e_4} e_2 = e_3, \qquad \nabla_{e_4} e_3 = -e_2,$$

and

$$J_{e_3}(e_1) = -\alpha\beta e_1, \qquad J_{e_3}(e_2) = -\beta^2 e_2, \qquad J_{e_3}(e_4) = -\beta^2 e_4,$$

$$J_{e_4}(e_1) = -\alpha^2 e_1, \qquad J_{e_4}(e_2) = -\beta^2 e_2, \qquad J_{e_4}(e_3) = -\beta^2 e_3, \qquad J_{e_i}(e_i) = 0, \qquad i = 3, 4.$$

Now we consider two possible cases. If $\alpha \neq \beta$, then we get $\{0, -\beta^2, -\alpha\beta\} = spec(J_{e_3}) \neq spec(J_{e_4}) = \{0, -\beta^2, -\alpha^2\}$, so G is not pointwise Osserman. We have also

$$f_1(e, e_3) = trace(J_{e_3}) = -\alpha\beta - 2\beta^2, \qquad f_1(e, e_4) = trace(J_{e_4}) = -\alpha^2 - 2\beta^2.$$

Hence G is not 1-Stein. If $\alpha = \beta$, then considering $v = re_1 + se_2 + te_3 + ke_4 \in S_e(G, g)$ we conclude that the matrix representation of J_v and its trace is the same as in the last case, i.e. $trace(J_v) = -3\alpha^2$. Therefore, G is 1-Stein at identity. As the same proof in last case $spec(J_v) = \{0, -\alpha^2\}$. Therefore, G is Osserman space at identity.

Now, we study the 2-Osserman property. If $\alpha \neq \beta$, and $\beta \neq 0$, considering $E = span\{e_1, e_2\}$ and $F = span\{e_3, e_4\}$, we get

$$p_{J_E}(x) = (x + \alpha\beta)^2(x + \alpha\beta + \beta^2)(x + \alpha^2 + \beta^2), \qquad p_{J_F}(x) = (x + \alpha\beta + \alpha^2)(x + 2\beta^2)(x + \beta^2)^2.$$

Therefore, G is not 2-Osserman. If $\alpha = \beta$, considering $E = span\{v_1, v_2\}$, where $v_1 = re_1 + se_2 + te_3 + ke_4$ and $v_2 = xe_1 + ye_2 + ze_3 + we_4$, the generalized Jacobi operator is exactly as the same as the generalized Jacobi

operator in last case.Considering $E = span\{e_1, e_2\}$ and $F = span\{e_1, \frac{1}{\sqrt{2}}e_3 + \frac{1}{\sqrt{2}}e_4\}$, as the same as in last case, we conclude that G is not 2-Osserman. If $\beta = 0$, then $J_E(e_2) = J_E(e_3) = 0$ and

$$J_E(e_1) = -(k^2 + w^2)\alpha^2 e_1 + (kr + wx)\alpha^2 e_4, \qquad J_E(e_4) = (rk + xw)\alpha^2 e_1 - (r^2 + x^2)\alpha^2 e_4,$$

so we obtain J_E as follows

$$p_{J_E}(x) = X^2(X^2 + (k^2 + w^2 + r^2 + x^2)\alpha^2 X + (kx - wr)^2\alpha^4) = 0.$$

It is clear that by different two planes we gain different characteristic polynomials, so in this case G is not 2-Osserman. Now we take $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then we get $\kappa_\pi = -\alpha\beta$ and $\kappa_\sigma = -\alpha^2$, if $\alpha \neq \beta$. Therefore, G is not a space form. If $\alpha = \beta$, then the proof is exactly as the same as in last case, i.e. $\mathfrak{g}_{4,5}$, so in this case $\kappa_\pi = -\alpha^2$, where π is arbitrary orthonormal basis. Therefore, G has constant sectional curvature.

$\mathfrak{g}_{4,7}$: The indecomposable solvable Lie algebra with $[e_2, e_3] = e_1$, $[e_1, e_4] = 2e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_2 + e_3$.

The Koszul's formula gives us the following

$$\nabla_{e_1}e_1 = -2e_4, \qquad \nabla_{e_1}e_2 = \nabla_{e_2}e_1 = -\frac{1}{2}e_3, \qquad \nabla_{e_1}e_3 = \nabla_{e_3}e_1 = \frac{1}{2}e_2, \qquad \nabla_{e_1}e_4 = 2e_1,$$

$$\nabla_{e_2}e_2 = -e_4, \qquad \nabla_{e_2}e_3 = \frac{1}{2}e_1 - \frac{1}{2}e_4, \qquad \nabla_{e_2}e_4 = e_2 + \frac{1}{2}e_3,$$

$$\nabla_{e_3}e_2 = -\frac{1}{2}e_1 - \frac{1}{2}e_4, \qquad \nabla_{e_3}e_3 = -e_4, \qquad \nabla_{e_3}e_4 = \frac{1}{2}e_2 + e_3,$$

$$\nabla_{e_4}e_2 = \frac{1}{2}e_3, \qquad \nabla_{e_4}e_3 = -\frac{1}{2}e_2.$$

Considering unit vector $e_1, e_2 \in S_e(G, g)$, we get

$$J_{e_1}(e_2) = -\frac{7}{4}e_2 - e_3, \qquad J_{e_1}(e_3) = -e_2 - \frac{7}{4}e_3, \qquad J_{e_1}(e_4) = -4e_4,$$

$$J_{e_2}(e_1) = -\frac{7}{4}e_1 - \frac{1}{4}e_4, \qquad J_{e_i}(e_i) = 0, \qquad i = 1, 2, \qquad J_{e_2}(e_3) = -\frac{3}{2}e_3, \qquad J_{e_2}(e_4) = -\frac{1}{4}e_1 - \frac{3}{4}e_4.$$

We obtain the following

$$p_{J_{e_1}}(x) = x(x + 4)(x^2 + \frac{7}{2}x + \frac{33}{16}), \qquad p_{J_{e_2}}(x) = x^4 + 4x^3 + \frac{41}{8}x^2 + \frac{33}{16}x.$$

Since $spec(J_{e_1}) = \{0, -4, -\frac{11}{4}, -\frac{3}{4}\}$ and (-4) do not belong to $spec(J_{e_2})$, therefore, G is not Osserman at identity. We have also $trace(J_{e_1}) = -\frac{15}{2}$ and $trace(J_{e_2}) = -4$, hence, G is not 1-Stein. Lemma 2.1 implies that (G, g) is not 2-Stein. Considering $E = span\{e_1, e_2\}$ and $F = span\{e_1, e_3\}$, we get

$$J_E(e_1) = -\frac{7}{4}e_1 - \frac{1}{4}e_4, \qquad J_E(e_2) = -\frac{7}{4}e_2 - e_3, \qquad J_E(e_3) = -e_2 - \frac{13}{4}e_3, \qquad J_E(e_4) = -\frac{1}{4}e_1 - \frac{19}{4}e_4,$$

$$J_F(e_1)=-\frac{7}{4}e_1+\frac{1}{4}e_4,\qquad J_F(e_2)=-\frac{13}{4}e_2-e_3,\qquad J_F(e_3)=-e_2-\frac{7}{4}e_3,\qquad J_F(e_4)=\frac{1}{4}e_1-\frac{23}{4}e_4.$$

Thus, little effort gives us the following

$$p_{J_E}(x)=(x^2+\frac{13}{2}x+\frac{33}{4})(x^2+5x+\frac{75}{16}),\qquad p_{J_F}(x)=(x^2+\frac{15}{2}x+10)(x^2+5x+\frac{87}{4}).$$

Thus, we conclude that G is not 2-Osserman. If we take 2-planes $\pi=\{e_1,e_2\}$ and $\sigma=\{e_2,e_3\}$, we obtain $\kappa_\pi=-\frac{7}{4}$ and $\kappa_\sigma=-\frac{3}{2}$. Therefore, G is not a space form.

$\mathfrak{g}_{4,8}$: The indecomposable solvable Lie algebra with $[e_2,e_3]=e_1$, $[e_1,e_4]=(1+\beta)e_1$, $[e_2,e_4]=e_2$, $[e_3,e_4]=\beta e_3$, $-1\leq\beta\leq 1$.

Little effort gives us the following

$$\nabla_{e_1}e_1=-(1+\beta)e_4,\qquad \nabla_{e_1}e_2=\nabla_{e_2}e_1=-\frac{1}{2}e_3,\qquad \nabla_{e_1}e_3=\nabla_{e_3}e_1=\frac{1}{2}e_2,\qquad \nabla_{e_1}e_4=(1+\beta)e_1,$$

$$\nabla_{e_2}e_2=-e_4,\qquad \nabla_{e_2}e_3=-\nabla_{e_3}e_2=\frac{1}{2}e_1,\qquad \nabla_{e_2}e_4=e_2,$$

$$\nabla_{e_3}e_3=-\beta e_4,\qquad \nabla_{e_3}e_4=\beta e_3.$$

So we get

$$J_{e_1}(e_2)=(-\frac{3}{4}-\beta)e_2,\qquad J_{e_1}(e_3)=(-\beta^2-\beta+\frac{1}{4})e_3,\qquad J_{e_1}(e_4)=-(1+\beta)^2e_4.$$

$$J_{e_2}(e_1)=(-\frac{3}{4}-\beta)e_1,\qquad J_{e_i}(e_i)=0,\qquad i=1,2,\qquad J_{e_2}(e_3)=(-\frac{3}{4}-\beta)e_3,\qquad J_{e_2}(e_4)=-e_4.$$

Also we obtain $spec(J_{e_2})=\{0,-(\frac{3}{4}+\beta),-1\}$ and $spec(J_{e_1})=\{0,-(\frac{3}{4}+\beta),(-\beta^2-\beta+\frac{1}{4}),-(1+\beta)^2\}$. Thus, G is not pointwise Osserman. Now we study the 1-Stein property at identity. Considering $v=re_1+se_2+te_3+ke_4\in S_e(G,g)$, by direct computation we obtain

$$trace(J_v)=r^2(-2\beta^2-4\beta-\frac{3}{2})+s^2(-2\beta-\frac{5}{2})+t^2(-2\beta^2-2\beta-\frac{1}{2})+k^2(-2\beta^2-2\beta-2).$$

Therefore, G is not 1-Stein. Also according to Lemma 2.1, we get (G,g) is not 2-Stein. Considering $E=span\{e_1,e_2\}$ and $F=span\{e_3,e_4\}$, direct computations give us

$$J_E(e_1)=-(\frac{3}{4}+\beta)e_1,\qquad J_E(e_2)=-(\frac{3}{4}+\beta)e_2,\qquad J_E(e_3)=-(\beta^2+2\beta+\frac{1}{2})e_3,$$

$$J_E(e_4)=-(\beta^2+2\beta+2)e_4,\qquad J_F(e_1)=-(2\beta^2+3\beta+\frac{3}{4})e_1,\qquad J_F(e_2)=-(\beta+\frac{7}{4})e_2,$$

$$J_F(e_3)=-\beta^2e_3,\qquad J_F(e_4)=-\beta^2e_4.$$

So we obtain the following

$$p_{J_E}(x) = (x + \frac{3}{4} + \beta)^2(x + \beta^2 + 2\beta + \frac{1}{2})(x + \beta^2 + 2\beta + 2),$$
$$p_{J_F}(x) = (x + 2\beta^2 + 3\beta + \frac{3}{4})(x + \beta + \frac{7}{4})(x + \beta^2)^2,$$

i.e. G is not 2-Osserman. Now we consider 2-planes $\pi = \{e_3, e_4\}$ and $\sigma = \{e_2, e_4\}$, then we obtain $\kappa_\pi = -\beta^2$ and $\kappa_\sigma = -1$. If $\beta \neq \pm 1$ then G is not a space form. If $\beta = \pm 1$, then we take $\pi = \{e_1, e_4\}$ and $\sigma = \{e_1, e_2\}$, and we get $\kappa_\pi = -(1 + \beta)^2$ and $\kappa_\sigma = -(\beta + \frac{3}{4})$. This shows that G is not a space form.

$\mathfrak{g}_{4,9}$**:** The indecomposable solvable Lie algebra with $[e_2, e_3] = e_1$, $[e_1, e_4] = 2\alpha e_1$, $[e_2, e_4] = \alpha e_2 - e_3$, $[e_3, e_4] = e_2 + \alpha e_3$, $\alpha \geq 0$.
The Koszul's formula gives us

$$\nabla_{e_1}e_1 = -2\alpha e_4, \qquad \nabla_{e_1}e_2 = \nabla_{e_2}e_1 = -\frac{1}{2}e_3, \qquad \nabla_{e_1}e_3 = \nabla_{e_3}e_1 = \frac{1}{2}e_2, \qquad \nabla_{e_1}e_4 = 2\alpha e_1,$$

$$\nabla_{e_2}e_2 = -\alpha e_4, \qquad \nabla_{e_2}e_3 = -\nabla_{e_3}e_2 = \frac{1}{2}e_1, \qquad \nabla_{e_2}e_4 = \alpha e_2,$$

$$\nabla_{e_3}e_3 = -\alpha e_4, \qquad \nabla_{e_3}e_4 = \alpha e_3, \qquad \nabla_{e_4}e_2 = e_3, \qquad \nabla_{e_4}e_3 = -e_2.$$

We get also the following

$$J_{e_1}(e_2) = (-2\alpha^2 + \frac{1}{4})e_2, \qquad J_{e_1}(e_3) = (-2\alpha^2 + \frac{1}{4})e_3, \qquad J_{e_1}(e_4) = -4\alpha^2 e_4.$$

$$J_{e_4}(e_1) = -4\alpha^2 e_1, \qquad J_{e_4}(e_2) = -\alpha^2 e_2, \qquad J_{e_4}(e_3) = -\alpha^2 e_3, \qquad J_{e_i}(e_i) = 0, \qquad i = 1, 4.$$

Therefore, we have $spec(J_{e_1}) = \{0, (-2\alpha^2 + \frac{1}{4}), -4\alpha^2\}$ and $spec(J_{e_4}) = \{0, -\alpha^2, -4\alpha^2\}$. Thus, G is not pointwise Osserman. Furthermore we have

$$f(e, e_1) = trace(J_{e_1}) = -8\alpha^2 + \frac{1}{2}, \qquad f(e, e_4) = trace(J_{e_4}) = -6\alpha^2.$$

Hence, G is not 1-Stein, and so (G, g) is not 2-Stein. Considering $E = span\{e_1, e_2\}$ and $F = span\{e_3, e_4\}$, we get

$$J_E(e_i) = (-2\alpha^2 + \frac{1}{4})e_i, \qquad i = 1, 2, \qquad J_E(e_3) = (-3\alpha^2 - \frac{1}{2})e_3, \qquad J_E(e_4) = -5\alpha^2 e_4,$$

$$J_F(e_1) = (-6\alpha^2 + \frac{1}{4})e_1, \qquad J_F(e_2) = (-2\alpha^2 - \frac{3}{4})e_2, \qquad J_F(e_i) = -\alpha^2 e_i, \qquad i = 3, 4.$$

Therefore, we conclude that

$$p_{J_E}(x) = (x + 2\alpha^2 - \frac{1}{4})^2(x + 3\alpha^2 + \frac{1}{2})(x + 5\alpha^2), \qquad p_{J_F}(x) = (x + 6\alpha^2 - \frac{1}{4})(x + 2\alpha^2 + \frac{3}{4})(x + \alpha^2)^2,$$

i.e. G is not 2-Osserman at identity. Now we consider 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$. Then we obtain $\kappa_\pi = -2\alpha^2 + \frac{1}{4}$ and $\kappa_\sigma = -4\alpha^2$, so G is not a space form.

$\mathfrak{g}_{4,10}$: The indecomposable solvable Lie algebra with $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$, $[e_1, e_4] = -e_2$, $[e_2, e_4] = e_1$.

As the same proof in last cases we get

$$\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = -e_3, \qquad \nabla_{e_1} e_3 = -\nabla_{e_4} e_2 = e_1, \qquad \nabla_{e_2} e_3 = \nabla_{e_4} e_1 = e_2.$$

We recall that other components of the left invariant Levi-Civita connection are zero. Considering $v = re_1 + se_2 + te_3 + ke_4 \in S_e(G, g)$, we get

$$J_v(e_1) = -(s^2 + t^2)e_1 + (rs)e_2 + (tr)e_3, \qquad J_v(e_2) = (rs)e_1 - (r^2 + t^2)e_2 + (ts)e_3,$$

$$J_v(e_3) = (rt)e_1 + (st)e_2 - (r^2 + s^2)e_3, \qquad J_v(e_4) = 0.$$

Now we consider $v_1 = e_1$ and $v_2 = \frac{1}{2}e_1 + \frac{1}{2}e_3 + \frac{1}{\sqrt{2}}e_4 \in S_e(G, g)$. Therefore, we have

$$p_{J_{v_1}}(x) = x^2(x+1)^2, \qquad p_{J_{v_2}}(x) = x^2(x + \frac{1}{2})^2.$$

Thus, G is not Osserman at identity. Furthermore we have

$$trace(J_v) = -2(r^2 + s^2 + t^2).$$

Since trace depends to the coordinate of v, G is not 1-Stein. Also according to Lemma 2.1, we conclude that (G, g) is not 2-Stein. Considering $E = span\{e_1, e_2\}$ and $F = span\{e_3, e_4\}$, as the same as in last cases, we obtain

$$J_E(e_i) = -e_i, \qquad i = 1, 2, \qquad J_E(e_3) = -2e_3, \qquad J_E(e_4) = 0,$$

$$J_F(e_i) = -e_i, \qquad i = 1, 2, \qquad J_F(e_i) = 0, \qquad i = 3, 4.$$

Thus, we have $p_{J_E}(x) = (x+1)^2(x+2)x$ and $p(J_F) = (x+1)^2x^2$, and we conclude that G is not 2-Osserman. Now we consider 2-planes $\pi = \{e_1, e_2\}$ and $\sigma = \{e_1, e_4\}$, then we obtain $\kappa_\pi = -1$ and $\kappa_\sigma = 0$, so G is not a space form.

We summarize the above discussion in the following that is the main result of this section.

Theorem 2.2 *If G is a non-abelian 4-dimensional Lie group with an orthonormal left invariant Riemannian metric, then the following assertions hold*

(1) The only structures that can provide 1-Stein property are $2\mathfrak{g}_{2,1}$, $\mathfrak{g}_{4,5}$, (whenever $\alpha = \beta = \gamma$), $\mathfrak{g}_{4,6}$, (whenever $\alpha = \beta$) and $\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1$ (whenever $\beta = 0$).

(2) The only structures that can provide Osserman (2-Stein) property are $\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1$, (whenever $\beta = 0$), $\mathfrak{g}_{4,5}$, (whenever $\alpha = \beta = \gamma$), and $\mathfrak{g}_{4,6}$, (whenever $\alpha = \beta$).

(3) The only structure that can provide 2-Osserman property is $\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1$, whenever $\beta = 0$.

(4) $\mathfrak{g}_{4,5}$, (whenever $\alpha = \beta = \gamma$), $\mathfrak{g}_{4,6}$, (whenever $\alpha = \beta$), and $\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1$ (whenever $\beta = 0$) are the only Lie algebras of constant sectional curvature.

3. Szabo structure at identity element

As mentioned in ([9], [10]) spectral geometry is a big branch in differential geometry and there are a lot of unsolved problems in it. Thus, in this section we study the spectrum of Szabo operator $S_X(Y) = (\nabla_X R)(Y,X)X$ at identity element. In fact we are looking for Lie algebra structures such that G equipped with them be a Szabo manifold at identity, whenever G equipped with an orthonormal left invariant metric g that introduced in last section. We do this work by Lie algebra classification in dimension four.

Definition 3.1 *Let (M,g) be a Riemannian manifold, (M,g) is said to be Szabo manifold if $spec(S_X)$ is constant on the unit sphere bundle $S(M,g)$. Also (M,g) is called Szabo manifold at $p \in M$, if $spec(S_X)$ is independent of any $X \in S_p(M,g) = \{X \in T_pM \mid g_p(X,X) = 1\}$. Moreover, we say that (M,g) is pointwise Szabo if (M,g) is Szabo at any $p \in M$.*

The main result of this section is the following:

Theorem 3.2 *If G is a non-abelian 4-dimensional Lie group with an orthonormal left invariant Riemannian metric , then the only Lie algebra structures that can provide Szabo structure at identity of G are $\mathfrak{g}_{3,7} \oplus \mathfrak{g}_1$, $\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1$, $2\mathfrak{g}_{2,1}$, $\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1$, $\mathfrak{g}_{3,4} \oplus \mathfrak{g}_1$, $\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1$, $\mathfrak{g}_{4,5}$ whenever $\alpha = \beta = \gamma$, and $\mathfrak{g}_{4,6}$ whenever $\beta = 0$.*

Proof Considering unit vectors e_1, e_2, we obtain $spec(S_{e_1}) = \{0\}$ and $spec(S_{e_2}) = \{0, \pm\frac{1}{2}\}$ for $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$; $spec(S_{e_1}) = \{0\}$ and $spec(S_{e_2}) = \{0, \frac{1}{32} \pm \frac{\sqrt{17}}{16}\}$ for $\mathfrak{g}_{4,1}$; $spec(S_{e_1}) = \{0, \pm\frac{\sqrt{5}}{2}\}$ and $spec(S_{e_2}) = \{0, \pm\frac{\sqrt{2}}{2}\}$ for $\mathfrak{g}_{4,4}$; $spec(S_{e_1}) = \{0\}$ and $spec(S_{e_2}) = \{0, \pm(2\alpha^2 - \frac{1}{2})\}$ for $\mathfrak{g}_{4,9}$; $spec(S_{e_1}) = \{0, -2, \frac{3}{2}\}$ and $spec(S_{e_2}) = \{0, \pm\frac{3\sqrt{2}}{2}\}$ for $\mathfrak{g}_{4,7}$; $spec(S_{e_1}) = \{0, \pm(1-\beta^2)\}$ and $spec(S_{e_2}) = \{0, \pm(\beta + \frac{1}{2})\}$ for $\mathfrak{g}_{4,8}$. Therefore, from Definition 3.1 it implies that G is not Szabo manifold at identity. Considering unit vectors e_1, e_3, also we obtain $spec(S_{e_1}) = \{0\}$ and $spec(S_{e_3}) = \{0, \pm\frac{\sqrt{5}}{2}\}$ for $\mathfrak{g}_{3,2} \oplus \mathfrak{g}_1$; $spec(S_{e_1}) = \{0\}$ and $spec(S_{e_3}) = \{0, \pm\frac{1}{2}\}$ for $\mathfrak{g}_{4,3}$; $spec(S_{e_1}) = \{0\}$ and $spec(S_{e_3}) = \{0, \pm\frac{1}{2}\}$ for $\mathfrak{g}_{4,2}$, thus, in this case G is not Szabo manifold at identity. For $\mathfrak{g}_{3,6} \oplus \mathfrak{g}_1$, if we consider unit vectors e_1, e_4, we get $spec(S_{e_1}) = \{0, \pm 4\}$ and $spec(S_{e_4}) = \{0\}$, and consequently from Definition 3.1 we deduce that G is not Szabo at identity. For $\mathfrak{g}_{4,10}$ if we consider $v_1 = e_1$ and $v_2 = \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2$, then we get $spec(S_{v_1}) = \{0\}$ and $spec(S_{v_2}) = \{0, \frac{\sqrt{2}}{2}\}$, thus, in this case G is not Szabo manifold.

Let $v = re_1 + se_2 + te_3 + ke_4$ be a unit tangent vector on $\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1$ (or $2\mathfrak{g}_{2,1}$). Direct computations imply that the Szabo operator is zero map, so $spec(S_v) = \{0\}$. Hence, the spectrum of Szabo operator is independent of unit tangent vector, and consequently G is Szabo manifold at identity.

For $\mathfrak{g}_{3,7} \oplus \mathfrak{g}_1$, $\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1$, $\mathfrak{g}_{3,4} \oplus \mathfrak{g}_1$ and $\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1$, if we consider $v = re_1 + se_2 + te_3 + ke_4$ as a unit tangent vector, then direct computations give us $spec(S_v) = \{0\}$, i.e. Szabo operator is zero map, so the spectrum of Szabo operator is independent of unit tangent vector. Thus, G is Szabo manifold at identity.

For $\mathfrak{g}_{4,5}$ there are three cases. If $\alpha = \beta = \gamma$, by direct computations we deduce that the Szabo operator is zero, so in this case G is Szabo manifold at identity. If $\alpha = \beta \neq \gamma$, then G is not Szabo at identity, because if $\alpha = \beta = 1$ and $\gamma = 2$, we can consider tangent unit vectors $v_1 = e_1$ and $v_2 = \frac{1}{\sqrt{2}}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3$ to obtain $spec(S_{v_1}) = \{0\}$ and $spec(S_{v_2}) = \{0, \pm\frac{\sqrt{3}}{2}\}$. If α, β, γ are pairwise distinct, G is not Szabo manifold, because if $\alpha = 1, \beta = 2, \gamma = 3$, putting $v_1 = e_1$ and $v_2 = \frac{1}{\sqrt{2}}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3$, we get $spec(S_{v_1}) = \{0\}$ and

$spec(S_{v_2}) = \{0, \pm\sqrt{\frac{19}{2}}\}$.

For $\mathfrak{g}_{4,6}$ there are two cases. If $\beta = 0$ then for every unit tangent vector v, Szabo operator is the zero map, so in this case G is Szabo manifold at identity. If $\beta \neq 0$ then in this case G is not Szabo manifold, because if we consider $\alpha = \beta = 1$ and $v_1 = e_1$ and $v_2 = \frac{1}{\sqrt{2}}e_2 + \frac{1}{2}e_3 + \frac{1}{2}e_4$ we obtain $spec(S_{v_1}) = \{0\}$ and $spec(S_{v_2}) = \{0, \frac{1}{8}\}$.

□

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