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Upper and lower bounds of the A-Berezin number of operators

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Abstract: Let A be a positive bounded linear operator acting on a complex Hilbert space $\mathcal{H}$. Any positive operator A induces a semiinner product on $\mathcal{H}$ defined by $\langle x,y\rangle_A := \langle Ax,y\rangle_{\mathcal{H}}$, $\forall x,y\in\mathcal{H}$. For any $T\in\mathcal{B}(\mathcal{H}(\Omega))$, its A-Berezin symbol $\widetilde{T}^A$ is defined on Ω by $\widetilde{T}^A := \left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A$, $\lambda\in\Omega$, where $\widehat{K}_\lambda$ is the normalized reproducing kernel of $\mathcal{H}$. In this paper, we introduce the notions (A,r)-adjoint of operators and A-Berezin number of operators on the reproducing kernel Hilbert space and prove some upper and lower bounds of the A-Berezin numbers of operators. In particular, we show that

$$\frac{1}{2}\|T\|_{A-\mathrm{Ber}} \le \max\left\{|\sin|_A T, \frac{\sqrt{2}}{2}\right\}\mathrm{ber}_A(T) \le \mathrm{ber}_A(T),$$

where $|\sin|_A T$ denotes the A-sinus of angle of T.

Key words: Reproducing kernel Hilbert space, Berezin symbol, Berezin number, A-Berezin number, positive operator, semiinner product

1. Introduction

Now, we give some necessary definitions for the main results and we give some proof using inequalities.

Let $\mathcal{B}(\mathcal{H})$ stand for the Banach algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle .,.\rangle$. An operator $A\in\mathcal{B}(\mathcal{H})$ is called self-adjoint if $A^*=A$, where A^* denotes the adjoint of A. It is easy to see that an operator A is self-adjoint if and only if $\langle Ax,x\rangle\in\mathbb{R}:=(-\infty,\infty)$ for all $x\in\mathcal{H}$. An operator $A\in\mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax,x\rangle\ge 0$ for all $x\in\mathcal{H}$.

Recall that a reproducing kernel Hilbert space (shortly, RKHS) is a Hilbert space $\mathcal{H}=\mathcal{H}(\Omega)$ consisting of complex-valued functions on some set Ω such that the evaluation functionals $\varphi_\lambda(f)=f(\lambda)$, $\lambda\in\Omega$, are continuous on $\mathcal{H}$ and for any $\lambda\in\Omega$ there exists f_λ such that $f_\lambda(\lambda)\ne 0$. Then by the Riesz representation theorem for each $\lambda\in\Omega$ there exists a unique function $K_\lambda\in\mathcal{H}$ such that

$$f(\lambda)=\langle f,K_\lambda\rangle \tag{1.1}$$

for all $f\in\mathcal{H}$. The collection $\{K_\lambda:\lambda\in\Omega\}$ is called the reproducing kernel of the space $\mathcal{H}$. We say that the reproducing kernel Hilbert space admits the Ber-property (in this case we will write $\mathcal{H}\in(\mathrm{Ber})$, if for any bounded linear operator A on $\mathcal{H}(\Omega)$, $\widetilde{A}(\lambda)=0$, $\forall\lambda\in\Omega$, implies that $A=0$, i.e. for the Berezin symbols

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of operators on $\mathcal{H}(\Omega)$ the uniqueness theorem holds, (i.e. the corresponding Berezin transform is injective). In particular, the Hardy space $H^2(\mathbb{D})$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc in $\mathbb{C}$, the Bergman space $L^2_a(\mathbb{D})$, the Dirichlet space $\mathcal{D}^2(\mathbb{D})$ and the Fock space $F(\mathbb{D})$ are RKHSs with the property (Ber). A detailed presentation of the theory of RKHSs is given, for instance, in Aronszajn [4], Saitoh [25] and Halmos [18].

For any $A \in \mathcal{B}(\mathcal{H})$, its Berezin symbol $\widetilde{A}$ is defined on Ω by (see Berezin [8])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle, \ \lambda \in \Omega, \tag{1.2}$$

where $\widehat{K}_\lambda = \frac{K_\lambda}{\|K_\lambda\|_{\mathcal{H}}}$ is the normalized reproducing kernel of $\mathcal{H}$ and the inner product $\langle .,. \rangle$ is taken with respect to the the RKHS $\mathcal{H}$. The Berezin norm, Berezin set and Berezin number of the operator A are defined respectively by

$$\|A\|_{\text{Ber}} := \sup_{\lambda \in \Omega} \left\| A\widehat{K}_\lambda \right\| \tag{1.3}$$

$$\text{Ber}(A) := \text{Range}(\widetilde{A}) = \left\{ \widetilde{A}(\lambda) : \lambda \in \Omega \right\} \tag{1.4}$$

$$\text{ber}(A) := \sup_{\lambda \in \Omega} \left| \widetilde{A}(\lambda) \right|. \tag{1.5}$$

It is clear that $\text{ber}(A) \le \|A\|_{\text{Ber}} \le \|A\|$, $\text{Ber}(A) \subset W(A)$ and $\text{ber}(A) \le w(A)$, where

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\} \tag{1.6}$$

is the numerical range of the operator A and

$$w(A) := \sup_{\|x\|=1} |\langle Ax, x\rangle| \tag{1.7}$$

is its numerical radius (for more facts about reproducing kernel Hilbert spaces and Berezin symbol, see, Aronszajn [4], Berezin [8] and Karaev [21]).

For more material about the numerical radius and other results on numerical radius inequality, see, e.g., [1, 10, 17, 22, 23, 29], and the references therein.

The null space of every operator T is denoted by $\mathcal{N}(T)$, its range by $\mathcal{R}(T)$, and adjoint of T by T^*. If S is a linear subspace of $\mathcal{H}$, then $\overline{S}$ stands for its closure in the norm topology of $\mathcal{H}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called positive, denoted by $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, the absolute value of T, denoted by $|T|$, is defined as $|T| = (T^*T)^{1/2}$. Throughout the article, A denotes a nonzero positive operator on $\mathcal{H}$. Notice that any positive operator A induces a semiinner product on $\mathcal{H}$ defined by

$$\langle x, y\rangle_A := \langle Ax, y\rangle_{\mathcal{H}}, \ \forall x, y \in \mathcal{H}.$$

The seminorm induced by $\langle .,. \rangle_A$ is given by $\|x\|_A = \sqrt{\langle x, x\rangle_A} = \left\|A^{1/2}x\right\|$ for all $x \in \mathcal{H}$.

It is easy to check that $\|.\|_A$ is norm if and only if A is injective and that the seminormed space $(\mathcal{H}, \|.\|_A)$ is complete if and only if $\overline{\mathcal{R}(A)} = \mathcal{R}(A)$.

Definition 1.1 *For $T \in \mathcal{B}(\mathcal{H})$, the A-Berezin set of $\left\langle T\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A$ is defined by*

$$\operatorname{Ber}_A(T) := \left\{ \left\langle T\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A : \lambda \in \Omega \right\}.$$

It should be mentioned that $\operatorname{Ber}_A(T)$ is a nonempty subset of $\mathbb{C}$ and it is in general not closed even if $\mathcal{H}$ is finite dimensional.

Definition 1.2 *(i) A-Berezin symbol (also called A-Berezin transform) $\widetilde{T}^A$ is defined on Ω by*

$$\widetilde{T}^A(\lambda) := \left\langle T\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A \ (\lambda \in \Omega),$$

(ii) The supremum modulus of $\operatorname{Ber}_A(T)$, denoted by $\operatorname{ber}_A(T)$, is called the A-Berezin number of T, i.e.

$$\operatorname{ber}_A(T) := \sup_{\lambda \in \Omega} \left| \left\langle T\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A \right|,$$

(iii) A-Berezin norm of operators $T \in \mathcal{B}(\mathcal{H}(\Omega))$ is defined by

$$\|A\|_{A-\mathrm{Ber}} := \sup_{\lambda \in \Omega} \left\| AT\widehat{K}_\lambda \right\|_{\mathcal{H}}.$$

If $A = I$, we get the Berezin number. So, this new concept generalizes the Berezin number of reproducing kernel Hilbert space operators and the Berezin norm of operators which have recently attracted the attention of many authors (see, for instance [5, 7, 13–16, 26–28]).

Definition 1.3 *([11]) Let $T \in \mathcal{B}(\mathcal{H})$. An operator $S \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint of T if for every $\lambda, \mu \in \Omega$, identity $\left\langle T\widehat{K}_\lambda, \widehat{K}_\mu \right\rangle_A = \left\langle \widehat{K}_\lambda, S\widehat{K}_\mu \right\rangle_A$ holds.*

Definition 1.4 *Let $T \in \mathcal{B}(\mathcal{H}(\Omega))$. An operator $S \in \mathcal{B}(\mathcal{H}(\Omega))$ is called (A, r)-adjoint of T if for every $\lambda, \mu \in \Omega$, the identity $\left\langle T\widehat{K}_{\mathcal{H},\lambda}, \widehat{K}_{\mathcal{H},\mu} \right\rangle_A = \left\langle \widehat{K}_{\mathcal{H},\lambda}, S\widehat{K}_{\mathcal{H},\mu} \right\rangle_A$ holds.*

Following [11, 12], observe that the existence of an A-adjoint of T is equivalent to the existence of a solution of the equation $AX = T^*A$. This kind of equation can be studied by using a well-known theorem due to Douglas [9] (for a survey of the recent results on this theorem, the readers can consult in Moslehian, Kian and Xu [24]). Briefly, Douglas theorem says that the operator equation $TX = S$ has a bounded linear solution X if and only if $\mathcal{R}(S) \subseteq \mathcal{R}(T)$; moreover, among its many solutions it has only one, denoted by Q, which satisfies $\mathcal{R}(Q) \subseteq \overline{\mathcal{R}(T^*)}$. Such Q is called the reduced solution or Douglas solution of $TX = S$. The set of all operators in $\mathcal{B}(\mathcal{H})$ admitting A-adjoint is denoted by $\mathcal{B}_A(\mathcal{H})$. By Douglas theorem, it holds that

$$\mathcal{B}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\}.$$

Further, the set of all operators admitting $A^{1/2}$-adjoints is denoted by $\mathcal{B}_{A^{1/2}}(\mathcal{H})$. Again, by applying Douglas theorem, we obtain

$$\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \exists \lambda > 0, \ \|Tx\|_A \leq \lambda \|x\|_A, \forall x \in \mathcal{H}\}.$$

Operators in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ are called A-bounded.

If $T \in \mathcal{B}_A(\mathcal{H})$, the reduced solution (or Douglas solution) of the equation $AX = T^*A$ is a distinguished A-adjoint operator of T, which is denoted by T^{*_A}. We observe that

$$T^{*_A} = A^{\dagger}T^*A,$$

where $A^{\dagger}$ is the Moore-Penrose inverse of A (see [2, 3]). It is well-known that the operator T^{*_A} satisfies

$$AT^{*_A} = T^*A,\ \mathcal{R}(T^{*_A}) \subseteq \overline{\mathcal{R}(A)} \text{ and } \mathcal{N}(T^{*_A}) = \mathcal{N}(T^*A).$$

Also, note that if $T \in \mathcal{B}_A(\mathcal{H})$, then $T^{*_A} \in \mathcal{B}_A(\mathcal{H})$ and $(T^{*_A})^{*_A} = P_A T P_A$, where P_A denotes the ortogonal projection onto $\overline{\mathcal{R}(A)}$. Moreover, if $T \in \mathcal{B}_A(\mathcal{H})$, then $\|T^{*_A}\| = \|T\|_A$. For more results and proofs related to this class of operators, the reader can consult in [2, 3] and their references.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be A-selfadjoint if AT is selfadjoint, that is, $AT = T^*A$. In addition, an operator T is called A-positive if $AT \geq 0$ and we write $T \geq_A 0$.

For the sequal, the Hilbert space $\left(\mathcal{R}\left(A^{1/2}\right), \langle \cdot, \cdot \rangle_{\mathbb{R}\left(A^{1/2}\right)}\right)$ will be simply denoted by $\mathbb{R}\left(A^{1/2}\right)$.

Using these notions, in the study in [30], the author investigated upper and lower bounds of the A-numerical radius of operators.

The paper is organized as follows.

In Section 2, inspired by the A-numerical radius inequalities of bounded linear operators in the study in [30] and by using (A,r)-adjoint of operators, we state a useful characterization of the A-Berezin number and A-Berezin norm for $T \in \mathcal{B}_{A,r}(\mathcal{H})$ as follows :

$$\operatorname{ber}_A(T) = \sup_{x^2+y^2=1} \|x\mathcal{R}_A(T) + y\zeta_A(T)\|_{A-\mathrm{Ber}}.$$

In particular, for $T \in \mathcal{B}_{A,r}(\mathcal{H})$ we prove that

$$\operatorname{ber}_A(T) = \|T\|_{A-\mathrm{Ber}},$$

$$\operatorname{ber}_A(T) \leq \frac{\sqrt{2}}{2}\sqrt{\|TT^{*A} + T^{*A}T\|_{A-\mathrm{ber}}} \leq \|T\|_{A-\mathrm{Ber}},$$

$$\frac{1}{2}\|T\|_{A-\mathrm{Ber}} \leq \frac{1}{2}\sqrt{\|TT^{*A} + T^{*A}T\|_{A-\mathrm{Ber}} + 2\widetilde{c}_A(T^2)} \leq \operatorname{ber}_A(T)$$

and

$$\frac{1}{2}\|T\|_{A-\mathrm{Ber}} \leq \max\left\{|\sin|_A T, \frac{\sqrt{2}}{2}\right\}\operatorname{ber}_A(T) \leq \operatorname{ber}_A(T).$$

Our results generalize recent Berezin number inequalities of bounded linear operators thanks to Başaran and Gürdal [6] and Huban et al. [19, 20].

2. Main results

Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a reproducing kernel Hilbert space with reproducing kernel K_λ. It is natural to define the A-Berezin symbol of operator $T \in \mathcal{B}(\mathcal{H}(\Omega))$ by the formula

$$\widetilde{T}^A(\lambda) := \left\langle T\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A = \left\langle AT\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle, \ \lambda \in \Omega.$$

We denote the set of all operators in $\mathcal{B}(\mathcal{H}(\Omega))$ admitting (A,r)-adjoints by $\mathcal{B}_{A,r}(\mathcal{H}) := \mathcal{B}_{A,r}(\mathcal{H}(\Omega))$.

Throughout this section, for an arbitrary operator $T \in \mathcal{B}_{A,r}(\mathcal{H})$, we write

$$\mathcal{R}_A(T) := \frac{T + T^{*A}}{2} \text{ and } \zeta_A(T) := \frac{T - T^{*A}}{2i}.$$

For $T \in \mathcal{B}_{A,r}(\mathcal{H})$, its Crawford number $c_A(T)$ is defined by

$$c_A(T) := \inf\left\{\left|\langle Tx, x\rangle_A\right| : x \in \mathcal{H}, \ \|x\|_A = 1\right\}$$

(see [30]). We also introduce the number $\widetilde{c}_A(T) := \inf_{\lambda\in\Omega}\left|\widetilde{T}^A(\lambda)\right|$. It is clear that

$$c_A(T) \le \widetilde{c}_A(T) \le \mathrm{ber}_A(T).$$

For $T \in \mathcal{B}_{A,r}(\mathcal{H})$, let us recall the abbreviated notion

$$|\cos|_A T := \inf\left\{\frac{|\langle Tx, x\rangle|}{\|Tx\|_A \|x\|_A} : x \in \mathcal{H}, \ \|Tx\|_A \|x\|_A \ne 0\right\}$$

and

$$|\sin|_A T := \sqrt{1 - |\cos|_A^2 T}.$$

Our first result proves inequalities between $\|T\|_{A-\mathrm{ber}}$ and $\mathrm{ber}_A(T)$ with A in $\mathcal{B}_{A,r}(\mathcal{H})$.

Lemma 2.1 *Let $T \in \mathcal{B}_{A,r}(\mathcal{H})$ be an (A,r)-selfadjoint operator. Then*

$$\mathrm{ber}_A(T) = \|T\|_{A-\mathrm{Ber}}.$$

Proof Let $\lambda \in \Omega$ be arbitrary. By the Cauchy-Schwarz inequality, we get

$$\left|\widetilde{T}^A\right| = \left|\left\langle T\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A\right| \le \left\|T\widehat{K}_\lambda\right\|_{A-\mathrm{Ber}} \left\|\widehat{K}_\lambda\right\|_{A-\mathrm{Ber}}$$

and

$$\sup_{\lambda\in\Omega}\left|\left\langle T\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A\right| \le \sup_{\lambda\in\Omega}\left\|T\widehat{K}_\lambda\right\|_{A-\mathrm{Ber}}$$

and hence,

$$\mathrm{ber}_A(T) \le \|T\|_{A-\mathrm{Ber}}.$$

In addition, since T is an (A,r)-selfadjoint operator, for every $\lambda, \mu \in \Omega$, we have

$$\left\langle T\left(\widehat{K}_\lambda + \widehat{K}_\mu\right), \widehat{K}_\lambda + \widehat{K}_\mu \right\rangle_A = \left\langle T\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A + 2\,\mathrm{Re}\left\langle T\widehat{K}_\lambda, \widehat{K}_\mu \right\rangle_A + \left\langle T\widehat{K}_\mu, \widehat{K}_\mu \right\rangle_A$$

and

$$\left\langle T\left(\widehat{K}_\lambda-\widehat{K}_\mu\right),\widehat{K}_\lambda-\widehat{K}_\mu\right\rangle_A=\left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A-2\operatorname{Re}\left\langle T\widehat{K}_\lambda,\widehat{K}_\mu\right\rangle_A+\left\langle T\widehat{K}_\mu,\widehat{K}_\mu\right\rangle_A.$$

Consequently, we conclude

$$4\operatorname{Re}\left\langle T\widehat{K}_\lambda,\widehat{K}_\mu\right\rangle_A=\left\langle T\left(\widehat{K}_\lambda+\widehat{K}_\mu\right),\widehat{K}_\lambda+\widehat{K}_\mu\right\rangle_A-\left\langle T\left(\widehat{K}_\lambda-\widehat{K}_\mu\right),\widehat{K}_\lambda-\widehat{K}_\mu\right\rangle_A.$$

So, we attain

$$4\left|\operatorname{Re}\left\langle T\widehat{K}_\lambda,\widehat{K}_\mu\right\rangle_A\right|\le \operatorname{ber}_A(T)\left(\left\|\widehat{K}_\lambda+\widehat{K}_\mu\right\|_{A-\mathrm{Ber}}^2+\left\|\widehat{K}_\lambda-\widehat{K}_\mu\right\|_{A-\mathrm{Ber}}^2\right).$$

Then this paralellogram law as observed by that:

$$\left|\operatorname{Re}\left\langle T\widehat{K}_\lambda,\widehat{K}_\mu\right\rangle_A\right|\le\frac{\operatorname{ber}_A(T)}{4}\left(2\left\|\widehat{K}_\lambda\right\|_{A-\mathrm{Ber}}^2+2\left\|\widehat{K}_\mu\right\|_{A-\mathrm{Ber}}^2\right)=\operatorname{ber}_A(T). \tag{2.1}$$

Now, contemplate the polar decomposition $\left\langle T\widehat{K}_\lambda,\widehat{K}_\mu\right\rangle_A=e^{i\theta}\left|\left\langle T\widehat{K}_\lambda,\widehat{K}_\mu\right\rangle_A\right|$ with $\theta\in\mathbb{R}$. In (2.1) by replacing $\widehat{K}_\mu$ by $e^{i\theta}\widehat{K}_\mu$, we get

$$\left|\left\langle T\widehat{K}_\lambda,\widehat{K}_\mu\right\rangle_A\right|=\operatorname{Re}\left\langle T\widehat{K}_\lambda,e^{i\theta}\widehat{K}_\mu\right\rangle_A\le\operatorname{ber}_A(T).$$

By taking the supremum over $\lambda,\mu\in\Omega$ above inequality, we have

$$\|T\|_{A-\mathrm{Ber}}\le\operatorname{ber}_A(T).$$

This finalizes the proof. □

Corollary 2.2 *Let $T\in\mathcal{B}_{A,r}(\mathcal{H})$. Then, for $\gamma\in\mathbb{C}$,*

$$0\le\|T\|_{A-\mathrm{Ber}}^2-\operatorname{ber}_A^2(T)\le\inf_{\gamma\in\mathbb{C}}\left\{\|T-\gamma I\|_{A-\mathrm{Ber}}^2-\widetilde{c}_A^2(T-\gamma I)\right\}.$$

Proof Let $\lambda\in\Omega$ be arbitrary. An elementary calculus shows that

$$\left\|T\widehat{K}_\lambda\right\|_{A-\mathrm{Ber}}^2-\left|\left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right|^2=\left\|T\widehat{K}_\lambda-\gamma\widehat{K}_\lambda\right\|_{A-\mathrm{Ber}}^2-\left|\left\langle T\widehat{K}_\lambda-\gamma\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right|^2$$

whence

$$\begin{aligned}\left\|T\widehat{K}_\lambda\right\|_{A-\mathrm{Ber}}^2-\left|\left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right|^2&\le\left\|T\widehat{K}_\lambda\right\|_{A-\mathrm{Ber}}^2-\inf_{\lambda\in\Omega}\left|\left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right|^2\\&=\|T-\gamma I\|_{A-\mathrm{Ber}}^2-\widetilde{c}_A^2(T-\gamma I).\end{aligned}$$

Accordingly

$$\left\|T\widehat{K}_\lambda\right\|_{A-\mathrm{Ber}}^2-\left|\left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right|^2\le\inf_{\gamma\in\mathbb{C}}\left\{\|T-\gamma I\|_{A-\mathrm{Ber}}^2-\widetilde{c}_A^2(T-\gamma I)\right\}$$

and by taking supremum over $\lambda\in\Omega$,

$$\sup_{\lambda\in\Omega}\left(\left\|T\widehat{K}_\lambda\right\|_{A-\mathrm{Ber}}^2-\left|\left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right|^2\right)\le\inf_{\gamma\in\mathbb{C}}\left\{\|T-\gamma I\|_{A-\mathrm{Ber}}^2-\widetilde{c}_A^2(T-\gamma I)\right\}$$

which is equivalent to

$$\|T\|_{A-\mathrm{Ber}}^2 - \mathrm{ber}_A^2(T) \leq \inf_{\gamma\in\mathbb{C}} \left\{ \|T-\gamma I\|_{A-\mathrm{Ber}}^2 - \widetilde{c}_A^2(T-\gamma I) \right\}$$

as required. □

Lemma 2.3 *Let $T \in \mathcal{B}_{A,r}(\mathcal{H})$. For every $\theta \in \mathbb{R}$,*

$$\mathrm{ber}_A\left(\frac{e^{i\theta}T + \left(e^{i\theta}T\right)^{*A}}{2}\right) = \left\|\frac{e^{i\theta}T + \left(e^{i\theta}T\right)^{*A}}{2}\right\|_{A-\mathrm{Ber}}.$$

Proof Let $\theta \in \mathbb{R}$. We have $\left(\frac{\left(e^{i\theta}T\right)^{*A} + \left(\left(e^{i\theta}T\right)^{*A}\right)^{*A}}{2}\right)^{*A} = \frac{\left(\left(e^{i\theta}T\right)^{*A}\right)^{*A} + \left(e^{i\theta}T\right)^{*A}}{2}$. By appling the property $AT^{*A} = T^*A$, then we can reach the following easily.

$$\begin{aligned}
&\left\langle \frac{\left(e^{i\theta}T\right)^{*A} + \left(\left(e^{i\theta}T\right)^{*A}\right)^{*A}}{2}\widehat{K}_\lambda, \widehat{K}_\mu \right\rangle_A \\
&= \left\langle A\left(\frac{\left(e^{i\theta}T\right)^{*A} + \left(\left(e^{i\theta}T\right)^{*A}\right)^{*A}}{2}\right)\widehat{K}_\lambda, \widehat{K}_\mu \right\rangle \\
&= \left\langle \left(A\frac{\left(e^{i\theta}T\right)^{*A}}{2} + A\frac{\left(\left(e^{i\theta}T\right)^{*A}\right)^{*A}}{2}\right)\widehat{K}_\lambda, \widehat{K}_\mu \right\rangle \\
&= \left\langle \frac{\left(\left(\left(e^{i\theta}T\right)^{*A}\right)^{*A}\right)^{*A}}{2}A\widehat{K}_\lambda, \widehat{K}_\mu \right\rangle + \left\langle \frac{\left(\left(e^{i\theta}T\right)^{*A}\right)^{*}}{2}A\widehat{K}_\lambda, \widehat{K}_\mu \right\rangle \\
&= \left\langle \left(\frac{\left(e^{i\theta}T\right)^{*A} + \left(\left(e^{i\theta}T\right)^{*A}\right)^{*A}}{2}\right)^{*A} A\widehat{K}_\lambda, \widehat{K}_\mu \right\rangle \\
&= \left\langle A\widehat{K}_\lambda, \left(\frac{\left(e^{i\theta}T\right)^{*A} + \left(\left(e^{i\theta}T\right)^{*A}\right)^{*A}}{2}\right)\widehat{K}_\mu \right\rangle
\end{aligned}$$

for all $\lambda, \mu \in \Omega$. This indicates that $\frac{\left(e^{i\theta}T\right)^{*A} + \left(\left(e^{i\theta}T\right)^{*A}\right)^{*A}}{2}$ is an (A,r)-selfadjoint operator. So by Lemma 2.1 we get

$$\mathrm{ber}_A\left(\frac{\left(e^{i\theta}T\right)^{*A} + \left(\left(e^{i\theta}T\right)^{*A}\right)^{*A}}{2}\right) = \left\|\frac{\left(e^{i\theta}T\right)^{*A} + \left(\left(e^{i\theta}T\right)^{*A}\right)^{*A}}{2}\right\|_{A-\mathrm{Ber}}. \tag{2.2}$$

Since $\operatorname{ber}_A\left(\mathcal{E}^{*A}\right)=\operatorname{ber}_A(\mathcal{E})$ and $\left\|\mathcal{E}^{*A}\right\|_{A-\text{Ber}}=\|\mathcal{E}\|_{A-\text{Ber}}$ for every $\mathcal{E}\in\mathcal{B}_{A,r}(\mathcal{H})$, from (2.2) it follows that

$$\operatorname{ber}_A\left(\frac{e^{i\theta}T+\left(e^{i\theta}T\right)^{*A}}{2}\right)=\left\|\frac{e^{i\theta}T+\left(e^{i\theta}T\right)^{*A}}{2}\right\|_{A-\text{Ber}}.$$

□

Now, we explain the third lemma which will be applied to prove Theorem 2.5.

Lemma 2.4 *Let $T\in\mathcal{B}_{A,r}(\mathcal{H})$ and $\lambda\in\Omega$. Then*

$$\sup_{\theta\in\mathbb{R}}\left|\left(\frac{e^{i\theta}T+\left(e^{i\theta}T\right)^{*A}}{2}\right)^{\sim_A}(\lambda)\right|=\left|\widetilde{T}^A(\lambda)\right|. \tag{2.3}$$

Proof Let $\theta\in\mathbb{R}$. We have

$$\begin{aligned}\left|\left\langle\frac{e^{i\theta}T+\left(e^{i\theta}T\right)^{*A}}{2}\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right|&=\frac{1}{2}\left|e^{i\theta}\left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A+e^{-i\theta}\left\langle T^{*A}\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right|\\&=\frac{1}{2}\left|e^{i\theta}\left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A+e^{-i\theta}\left\langle\widehat{K}_\lambda,T\widehat{K}_\lambda\right\rangle_A\right|.\end{aligned}$$

Thus

$$\left|\left\langle\frac{e^{i\theta}T+\left(e^{i\theta}T\right)^{*A}}{2}\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right|=\left|\operatorname{Re}\left(e^{i\theta}\left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right)\right|. \tag{2.4}$$

Now, it is observed that

$$\left|\left\langle\frac{e^{i\theta}T+\left(e^{i\theta}T\right)^{*A}}{2}\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right|\leq\left|\left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle\right|_A,$$

whence

$$\sup_{\theta\in\mathbb{R}}\left|\left(\frac{e^{i\theta}T+\left(e^{i\theta}T\right)^{*A}}{2}\right)^{\sim_A}(\lambda)\right|\leq\left|\widetilde{T}^A(\lambda)\right|. \tag{2.5}$$

Now, if $\left|\left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right|=0$, the result becomes clear. Suppose that $\left|\left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right|\neq 0$. Then we state $e^{i\theta_0}=\frac{\left\langle\widehat{K}_\lambda,T\widehat{K}_\lambda\right\rangle_A}{\left|\left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right|}$. Therefore, by (2.4), we find

$$\left|\left\langle\frac{e^{i\theta_0}T+\left(e^{i\theta_0}T\right)^{*A}}{2}\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right|=\left|\operatorname{Re}\left(e^{i\theta_0}\left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right)\right|=\left|\left\langle T\widehat{K}_\lambda,\widehat{K}_\lambda\right\rangle_A\right|. \tag{2.6}$$

From (2.5) and (2.6) it follows that

$$\sup_{\theta\in\mathbb{R}}\left|\left(\frac{e^{i\theta}T+\left(e^{i\theta}T\right)^{*A}}{2}\right)^{\sim_A}(\lambda)\right|=\left|\widetilde{T}^A(\lambda)\right|.$$

□

It is time to explain a useful characterization of the A-Berezin number for semi-Hilbertian space operators.

Theorem 2.5 *Let* $T \in \mathcal{B}_{A,r}(\mathcal{H})$ *and* $\lambda \in \Omega$. *Then*

$$\operatorname{ber}_A(T) = \sup_{\theta \in \mathbb{R}} \left\| \frac{e^{i\theta}T + \left(e^{i\theta}T\right)^{*A}}{2} \right\|_{A-\mathrm{Ber}}.$$

Proof Let $\widehat{K}_\lambda$ be a reproducing kernel of space $\mathcal{H}(\Omega)$ and $\theta \in \mathbb{R}$. By Lemma 2.3 it follows that

$$\operatorname{ber}_A\left(\frac{e^{i\theta}T + \left(e^{i\theta}T\right)^{*A}}{2}\right) = \left\| \frac{e^{i\theta}T + \left(e^{i\theta}T\right)^{*A}}{2} \right\|_{A-\mathrm{Ber}}.$$

Therefore, by Lemma 2.4 we come to this conclusion

$$\begin{aligned}\sup_{\theta \in \mathbb{R}} \left\| \frac{e^{i\theta}T + \left(e^{i\theta}T\right)^{*A}}{2} \right\|_{A-\mathrm{Ber}} &= \sup_{\theta \in \mathbb{R}} ber_A\left(\frac{e^{i\theta}T + \left(e^{i\theta}T\right)^{*A}}{2}\right) \\ &= \sup_{\theta \in \mathbb{R}} \sup_{\lambda \in \Omega} \left| \left\langle \frac{e^{i\theta}T + \left(e^{i\theta}T\right)^{*A}}{2} \widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A \right| \\ &\quad \text{(by the equation (2.3)} \\ &= \sup_{\lambda \in \Omega} \left| \widetilde{T}^A(\lambda) \right|_A = \operatorname{ber}_A(T).\end{aligned}$$

□

We introduce one of the main results of this section.

Theorem 2.6 *Let* $T \in \mathcal{B}_{A,r}(\mathcal{H})$. *Then for* $x,\ y \in \mathbb{R}$,

$$\operatorname{ber}_A(T) = \sup_{x^2+y^2=1} \left\| x\mathcal{R}_A(T) + y\zeta_A(T) \right\|_{A-\mathrm{Ber}}.$$

Proof Let $\theta \in \mathbb{R}$. Put $x = \cos\theta$ and $y = -\sin\theta$. We have

$$\begin{aligned}\frac{e^{i\theta}T + \left(e^{i\theta}T\right)^{*A}}{2} &= \frac{(\cos\theta + i\sin\theta)T + (\cos\theta - i\sin\theta)T^{*A}}{2} \\ &= x\frac{T + T^{*A}}{2} + y\frac{T - T^{*A}}{2i}.\end{aligned}$$

Therefore

$$\sup_{\theta \in \mathbb{R}} \left\| \frac{e^{i\theta}T + \left(e^{i\theta}T\right)^{*A}}{2} \right\|_{A-\mathrm{Ber}} = \sup_{x^2+y^2=1} \left\| x\frac{T + T^{*A}}{2} + y\frac{T - T^{*A}}{2i} \right\|_{A-\mathrm{Ber}}$$

and hence, by Theorem 2.5, we obtain

$$\operatorname{ber}_A(T) = \sup_{x^2+y^2=1} \left\| x\frac{T + T^{*A}}{2} + y\frac{T - T^{*A}}{2i} \right\|_{A-\mathrm{Ber}}.$$

□

By establishing $(x, y) = (1, 0)$ and $(x, y) = (0, 1)$ in Theorem 2.6, the following result is achieved.

Corollary 2.7 *Let* $T \in \mathcal{B}_{A,r}(\mathcal{H})$. *Then*

$$\max\left\{\left\|\frac{T+T^{*A}}{2}\right\|_{A-\mathrm{Ber}},\left\|\frac{T-T^{*A}}{2i}\right\|_{A-\mathrm{Ber}}\right\}\leq \mathrm{ber}_A(T).$$

Another consequence of Theorem 2.6 is shown as this:

Corollary 2.8 *Let* $T \in \mathcal{B}_{A,r}(\mathcal{H})$ *and* $\lambda,\mu \in \Omega$. *Then*

$$\frac{1}{2}\|T\|_{A-\mathrm{Ber}} \leq \mathrm{ber}_A(T) \leq \|T\|_{A-\mathrm{Ber}}. \tag{2.7}$$

Proof Let $\lambda \in \Omega$ be arbitrary. By the Cauchy-Schwarz inequality, we get

$$\left|\widetilde{T}^A\right| = \left|\left\langle T\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A\right| \leq \left\|T\widehat{K}_\lambda\right\|_{A-\mathrm{Ber}}\left\|\widehat{K}_\lambda\right\|_{A-\mathrm{Ber}}.$$

By taking the supremum over $\lambda \in \Omega$ above inequality, we have

$$\mathrm{ber}_A(T) \leq \|T\|_{A-\mathrm{Ber}}.$$

Hence, by using Corollary 2.7, we get

$$T = \frac{T+T^{*A}}{2} - \frac{T^{*A}}{2} = \frac{T+T^{*A}}{2} + i\frac{T-T^{*A}}{2i}$$

and

$$\begin{aligned}\|T\|_{A-\mathrm{Ber}} &\leq \left\|\frac{T+T^{*A}}{2}\right\|_{A-\mathrm{Ber}} + \left\|\frac{T-T^{*A}}{2}\right\|_{A-\mathrm{Ber}}\\ &\leq 2\mathrm{ber}_A(T).\end{aligned}$$

So, we have $\|T\|_{A-\mathrm{Ber}} \leq 2\mathrm{ber}_A(T)$. □

We improve the second inequality in (2.7) in the following theorem.

Theorem 2.9 *Let* $T \in B_{A,r}(\mathcal{H})$ *and* $\lambda \in \Omega$. *Then*

$$\mathrm{ber}_A(T) \leq \frac{\sqrt{2}}{2}\sqrt{\left\|TT^{*A}+T^{*A}T\right\|_{A-\mathrm{ber}}} \leq \|T\|_{A-\mathrm{Ber}}.$$

Proof Put $P := \frac{T^{*A}+(T^{*A})^{*A}}{2}$ and $Q := \frac{T^{*A}-(T^{*A})^{*A}}{2i}$. Then $T^{*A} = P + iQ$. Also, an elementary calculus shows that

$$P^2+Q^2 = \frac{\left(T^{*A}\right)^{*A}T^{*A}+T^{*A}\left(T^{*A}\right)^{*A}}{2} = \left(\frac{TT^{*A}+T^{*A}T}{2}\right)^{*A}.$$

Since $\left\|\mathcal{E}^{*A}\right\|_{A-Ber} = \|\mathcal{E}\|_{A-Ber}$ for every $\mathcal{E} \in \mathcal{B}_{A,r}(\mathcal{H})$, hence

$$\left\|P^2+Q^2\right\|_{A-\mathrm{Ber}} = \frac{1}{2}\left\|TT^{*A}+T^{*A}T\right\|_{A-\mathrm{Ber}}. \tag{2.8}$$

Now, let $\lambda \in \Omega$ be arbitrary. We have

$$\begin{aligned}
\left|\left\langle \widehat{K}_\lambda, T\widehat{K}_\lambda \right\rangle_A\right|^2 &= \left|\left\langle T^{*A}\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A\right|^2 \\
&= \left\langle (P+iQ)\,\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A \left\langle \widehat{K}_\lambda, (P+iQ)\,\widehat{K}_\lambda \right\rangle_A \\
&= \left(\left\langle P\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A + i\left\langle Q\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A\right)\left(\left\langle \widehat{K}_\lambda, P\widehat{K}_\lambda \right\rangle_A - i\left\langle \widehat{K}_\lambda, Q\widehat{K}_\lambda \right\rangle_A\right) \\
&= \left|\left\langle P\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A\right|^2 + \left|\left\langle Q\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A\right|^2 \\
&\leq \left\langle P\widehat{K}_\lambda, P\widehat{K}_\lambda \right\rangle_A + \left\langle Q\widehat{K}_\lambda, Q\widehat{K}_\lambda \right\rangle_A \\
&\quad \text{(by Cauchy-Schwarz inequality} \\
&= \left\langle P^2\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A + \left\langle Q^2\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A \\
&\text{(since } P^{*A} = P \text{ and } Q^{*A} = Q\text{)} \\
&= \left\langle \left(P^2+Q^2\right)\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A \\
&= \frac{1}{2}\left\langle \left(TT^{*A}+T^{*A}T\right)^{*A}\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A \\
&\quad \text{(by the inequality (2.8)}.
\end{aligned}$$

and

$$\sup_{\lambda\in\Omega}\left|\left\langle \widehat{K}_\lambda, T\widehat{K}_\lambda \right\rangle_A\right|^2 \leq \frac{1}{2}\sup_{\lambda\in\Omega}\left\langle \left(TT^{*A}+T^{*A}T\right)\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle_A,$$

or equivalently,

$$\mathrm{ber}_A(T) \leq \frac{\sqrt{2}}{2}\sqrt{\left\|TT^{*A}+T^{*A}T\right\|_{A-ber}}. \tag{2.9}$$

In addition, since $\left\|TT^{*A}\right\|_{A-Ber} = \left\|T^{*A}T\right\|_{A-Ber} = \|T\|_{A-Ber}^2$, by the triangle inequality we reach

$$\frac{\sqrt{2}}{2}\sqrt{\left\|TT^{*A}+T^{*A}T\right\|_{A-\mathrm{Ber}}} \leq \frac{\sqrt{2}}{2}\sqrt{\left\|TT^{*A}\right\|_{A-\mathrm{Ber}}+\left\|T^{*A}T\right\|_{A-\mathrm{Ber}}} = \|T\|_{A-\mathrm{Ber}}.$$

Thus

$$\frac{\sqrt{2}}{2}\sqrt{\left\|TT^{*A}+T^{*A}T\right\|_{A-\mathrm{Ber}}} \leq \|T\|_{A-\mathrm{Ber}}. \tag{2.10}$$

By the inequalities (2.9) and (2.10), we get

$$\mathrm{ber}_A(T) \leq \frac{\sqrt{2}}{2}\sqrt{\left\|TT^{*A}+T^{*A}T\right\|_{A-\mathrm{Ber}}} \leq \|T\|_{A-\mathrm{Ber}}$$

as required. □

We express another improvement of the second inequality in (2.8).

Theorem 2.10 *Let $T \in \mathcal{B}_{A,r}(\mathcal{H})$. Then*

$$\mathrm{ber}_A(T) \leq \frac{1}{2}\sqrt{\|TT^{*A} + T^{*A}T\|_{A-\mathrm{Ber}} + 2\mathrm{ber}_A(T^2)} \leq \|T\|_{A-\mathrm{Ber}}.$$

Proof Since $\left\|\mathcal{E}^{*A}\right\|_{A-\mathrm{Ber}} = \|\mathcal{E}\|_{A-\mathrm{Ber}}$ and $\left\|\mathcal{E}\mathcal{E}^{*A}\right\|_{A-\mathrm{Ber}} = \|\mathcal{E}\|^2_{A-\mathrm{Ber}}$ for every $\mathcal{E} \in \mathcal{B}_{A,r}(\mathcal{H})$, by Theorem 2.5, we have

$$\begin{aligned}
&\mathrm{ber}_A(T) \\
&= \sup_{\theta\in\mathbb{R}} \left\| \frac{e^{i\theta}T + \left(e^{i\theta}T\right)^{*A}}{2} \right\|_{A-\mathrm{Ber}} \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| \left(e^{i\theta}T\right)^{*A} + \left(\left(e^{i\theta}T\right)^{*A}\right)^{*A} \right\|_{A-\mathrm{Ber}} \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \sqrt{\left\| \left(e^{i\theta}T\right)^{*A} + \left(\left(e^{i\theta}T\right)^{*A}\right)^{*A} \left(\left(\left(e^{i\theta}T\right)^{*A}\right)^{*A} + \left(e^{i\theta}T\right)^{*A}\right) \right\|_{A-\mathrm{Ber}}} \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \sqrt{\left\| T^{*A}\left(T^{*A}\right)^{*A} + \left(T^{*A}\right)^{*A}T^{*A} + \left(\left(e^{i\theta}T\right)^{*A}\right)^2 + \left(\left(\left(e^{i\theta}T\right)^{*A}\right)^{*A}\right)^2 \right\|_{A-\mathrm{Ber}}} \\
&\leq \frac{1}{2}\sup_{\theta\in\mathbb{R}} \sqrt{\left\| T^{*A}\left(T^{*A}\right)^{*A} + \left(T^{*A}\right)^{*A}T^{*A} \right\|_{A-\mathrm{Ber}} + \left\| \left(\left(e^{i\theta}T\right)^{*A}\right)^2 + \left(\left(\left(e^{i\theta}T\right)^{*A}\right)^{*A}\right)^2 \right\|_{A-\mathrm{Ber}}} \\
&\leq \frac{1}{2}\sup_{\theta\in\mathbb{R}} \sqrt{\|TT^{*A} + T^{*A}T\|_{A-\mathrm{Ber}} + 2\left\| \frac{e^{2i\theta}T^2 + \left(e^{2i\theta}T^2\right)^{*A}}{2} \right\|_{A-\mathrm{Ber}}} \\
&\leq \frac{1}{2}\sqrt{\|TT^{*A} + T^{*A}T\|_{A-\mathrm{Ber}} + 2\sup_{\theta\in\mathbb{R}}\left\| \frac{e^{2i\theta}T^2 + \left(e^{2i\theta}T^2\right)^{*A}}{2} \right\|_{A-\mathrm{Ber}}} \\
&= \frac{1}{2}\sqrt{\|TT^{*A} + T^{*A}T\|_{A-\mathrm{Ber}} + 2\mathrm{ber}_A(T^2)} \text{ (by Theorem 2.5)} \\
&\leq \frac{1}{2}\sqrt{\|TT^{*A}\|_{A-Ber} + \|T^{*A}T\|_{A-Ber} + 2ber_A(T^2)} \\
&= \frac{\sqrt{2}}{2}\sqrt{\|T\|^2_{A-Ber} + ber_A(T^2)} \\
&\leq \frac{\sqrt{2}}{2}\sqrt{\|T\|^2_{A-Ber} + \|T^2\|_{A-Ber}} \quad \text{(by Corollary 2.8)} \\
&\leq \frac{\sqrt{2}}{2}\sqrt{\|T\|^2_{A-Ber} + \|T\|^2_{A-Ber}} = \|T\|_{A-Ber}
\end{aligned}$$

which proves the desired inequalities. □

We achieved an improvement of the first inequality in (2.7) in the theorem below.

Theorem 2.11 *Let $T \in \mathcal{B}_{A,r}(\mathcal{H})$. Then*

$$\frac{1}{2}\|T\|_{A-\mathrm{Ber}} \leq \frac{1}{2}\sqrt{\|TT^{*A}+T^{*A}T\|_{A-\mathrm{Ber}}+2\widetilde{c}_A\left(T^2\right)} \leq \mathrm{ber}_A(T).$$

Proof Let $\lambda \in \Omega$ be an arbitrary. Suppose that $\left|\left\langle T^{*A}T^{*A}\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A\right| = e^{-2i\theta}\left\langle T^{*A}T^{*A}\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A$ for some real number θ. Since $\left\|\mathcal{E}^{*A}\right\|_{A-\mathrm{Ber}} = \|\mathcal{E}\|_{A-\mathrm{Ber}}$ and $\left\|\mathcal{R}\mathcal{R}^{*A}\right\|_{A-\mathrm{Ber}} = \|\mathcal{R}\|^2_{A-\mathrm{Ber}}$ for every $\mathcal{E} \in \mathcal{B}_{A,r}(\mathcal{H})$. Then, we have

$$\begin{aligned}
& e^{2i\theta}\left\langle \left(T^{*A}\right)^{*A}\left(T^{*A}\right)^{*A}\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A \\
& = e^{-2i\theta}\overline{\left\langle T^{*A}T^{*A}\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A} = \left|\left\langle T^{*A}T^{*A}\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A\right| = \left|\left\langle \widehat{K}_\lambda, T^2\widehat{K}_\lambda\right\rangle_A\right|.
\end{aligned}$$

Thus

$$e^{-2i\theta}\left\langle T^{*A}T^{*A}\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A = \left|\left\langle \widehat{K}_\lambda, T^2\widehat{K}_\lambda\right\rangle_A\right| = e^{2i\theta}\left\langle \left(T^{*A}\right)^{*A}\left(T^{*A}\right)^{*A}\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A. \tag{2.11}$$

So, by Theorem 2.5, we obtain

$$\begin{aligned}
& 4\mathrm{ber}^2_A(T) \\
& \geq \left\|e^{i\theta}T + \left(e^{i\theta}T\right)^{*A}\right\|^2_{A-\mathrm{Ber}} \\
& = \left\|\left(e^{i\theta}T\right)^{*A} + \left(\left(e^{i\theta}T\right)^{*A}\right)^{*A}\right\|^2_{A-\mathrm{Ber}} \\
& = \left\|\left(e^{i\theta}T\right)^{*A} + \left(\left(e^{i\theta}T\right)^{*A}\right)^{*A}\left(\left(e^{i\theta}T\right)^{*A} + \left(\left(e^{i\theta}T\right)^{*A}\right)^{*A}\right)^{*A}\right\|_{A-\mathrm{Ber}} \\
& = \left\|T^{*A}\left(T^{*A}\right)^{*A} + \left(T^{*A}\right)^{*A}T^{*A} + e^{-2i\theta}T^{*A}T^{*A} + e^{2i\theta}\left(T^{*A}\right)^{*A}\left(T^{*A}\right)^{*A}\right\|_{A-\mathrm{Ber}} \\
& \geq \left|\left\langle T^{*A}\left(T^{*A}\right)^{*A} + \left(T^{*A}\right)^{*A}T^{*A} + e^{-2i\theta}T^{*A}T^{*A} + e^{2i\theta}\left(T^{*A}\right)^{*A}\left(T^{*A}\right)^{*A}\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A\right| \\
& = \left|\left\langle\left(T^{*A}\left(T^{*A}\right)^{*A} + \left(T^{*A}\right)^{*A}T^{*A}\widehat{K}_\lambda, \widehat{K}_\lambda\right)\right\rangle_A\right. \\
& \left. + e^{-2i\theta}\left\langle T^{*A}T^{*A}\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A + e^{2i\theta}\left\langle\left(T^{*A}\right)^{*A}\left(T^{*A}\right)^{*A}\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A\right| \\
& = \left\langle\left(T^{*A}\left(T^{*A}\right)^{*A} + \left(T^{*A}\right)^{*A}T^{*A}\right)\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A + 2\left|\left\langle \widehat{K}_\lambda, T^2\widehat{K}_\lambda\right\rangle_A\right| \\
& \text{(by inequality (2.11))} \\
& \geq \left\langle\left(T^{*A}\left(T^{*A}\right)^{*A} + \left(T^{*A}\right)^{*A}T^{*A}\right)\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A + 2\inf_{\lambda\in\Omega}\left|\left\langle \widehat{K}_\lambda, T^2\widehat{K}_\lambda\right\rangle_A\right| \\
& = \left\langle\left(T^{*A}\left(T^{*A}\right)^{*A} + \left(T^{*A}\right)^{*A}T^{*A}\right)\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A + 2\widetilde{c}_A\left(T^2\right).
\end{aligned}$$

From this it follows that

$$\frac{1}{2}\sqrt{\left\langle\left(T^{*A}\left(T^{*A}\right)^{*A} + \left(T^{*A}\right)^{*A}T^{*A}\right)\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A + 2\widetilde{c}_A\left(T^2\right)} \leq \mathrm{ber}_A(T).$$

Taking the supremum over $\lambda \in \Omega$ in the above inequality we get

$$\frac{1}{2}\sqrt{\|TT^{*A}+T^{*A}T\|_{A-\text{ber}}+2\widetilde{c}_A\left(T^2\right)} \leq \text{ber}_A\left(T\right). \tag{2.12}$$

Furthermore, since $T^{*A}T$ is an A-positive operator, from $\left\|TT^{*A}+T^{*A}T\right\|_{A-\text{Ber}} \geq \left\|TT^{*A}\right\|_{A-\text{Ber}} = \|T\|_{A-\text{Ber}}^2$ it follows that

$$\frac{1}{2}\sqrt{\|TT^{*A}+T^{*A}T\|_{A-\text{ber}}+2\widetilde{c}_A\left(T^2\right)} \geq \frac{1}{2}\sqrt{\|TT^{*A}+T^{*A}T\|_{A-\text{Ber}}} \geq \frac{1}{2}\|T\|_{A-\text{Ber}}. \tag{2.13}$$

Now, by (2.12) and (2.13) we conclude that

$$\frac{1}{2}\|T\|_{A-\text{Ber}} \leq \frac{1}{2}\sqrt{\|TT^{*A}+T^{*A}T\|_{A-\text{Ber}}+2\widetilde{c}_A\left(T^2\right)} \leq \text{ber}_A\left(T\right).$$

□

Another improvement of the first inequality in (2.7) is stated.

Theorem 2.12 *Let $T \in \mathcal{B}_{A,r}\left(\mathcal{H}\right)$ and $\lambda \in \Omega$. Then*

$$\frac{1}{2}\|T\|_{A-\text{Ber}} \leq \sqrt{\frac{\text{ber}_A^2\left(T\right)}{2}+\frac{\text{ber}_A\left(T\right)}{2}\sqrt{\text{ber}_A^2\left(T\right)-\widetilde{c}_A^2\left(T\right)}} \leq \text{ber}_A\left(T\right).$$

Proof Clearly, $\sqrt{\frac{\text{ber}_A^2(T)}{2}+\frac{\text{ber}_A(T)}{2}\sqrt{\text{ber}_A^2\left(T\right)-\widetilde{c}_A^2\left(T\right)}} \leq \text{ber}_A\left(T\right)$. Now, let $\lambda \in \Omega$ be arbitrary. Suppose that $\left|\left\langle T\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A\right| = e^{i\theta}\left\langle T\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A$ for some real number θ. Put $P := \frac{e^{i\theta}T+\left(e^{i\theta}T\right)^{*A}}{2}$ and $Q := \frac{e^{i\theta}T-\left(e^{i\theta}T\right)^{*A}}{2i}$. Then $P+iQ = e^{i\theta}T$ and

$$\left\langle P\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A + i\left\langle Q\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A = e^{i\theta}\left\langle T\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A = \left|\left\langle T\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A\right| \geq 0.$$

It follows from $\left\langle Q\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A = \text{Im}\left\langle e^{i\theta}T\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A \in \mathbb{R}$ that

$$\left\langle e^{i\theta}T\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A = \left\langle P\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A$$

$$\text{since } \left\langle Q\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A = 0$$

So, we have

$$\frac{1}{4}\left\|T\widehat{K}_{\lambda}\right\|_{A}^{2}=\frac{1}{4}\left(\left\|e^{i\theta}T\widehat{K}_{\lambda}-\left\langle e^{i\theta}T\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\widehat{K}_{\lambda}\right\|_{A-\mathrm{Ber}}^{2}+\left|\left\langle T\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\right|^{2}\right)$$

$$=\frac{1}{4}\left(\left\|P\widehat{K}_{\lambda}-\left\langle P\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\widehat{K}_{\lambda}+i\left\langle Q\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\widehat{K}_{\lambda}\right\|_{A-\mathrm{Ber}}^{2}+\left|\left\langle T\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\right|^{2}\right)$$

since $\left\langle Q\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}=0$

$$\leq\frac{1}{4}\left(\left(\left\|P\widehat{K}_{\lambda}-\left\langle P\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\widehat{K}_{\lambda}\right\|_{A-\mathrm{Ber}}+\left\|Q\widehat{K}_{\lambda}\right\|_{A}\right)^{2}+\left|\left\langle T\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\right|^{2}\right)$$

$$=\frac{1}{4}\left(\left(\sqrt{\left\|P\widehat{K}_{\lambda}\right\|_{A}^{2}-\left|\left\langle P\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\right|^{2}}+\left\|Q\widehat{K}_{\lambda}\right\|_{A}\right)^{2}+\left|\left\langle T\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\right|^{2}\right)$$

$$=\frac{1}{4}\left(\left(\sqrt{\left\|P\widehat{K}_{\lambda}\right\|_{A}^{2}-\left|\left\langle e^{i\theta}T\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\right|^{2}}+\left\|Q\widehat{K}_{\lambda}\right\|_{A}\right)^{2}+\left|\left\langle T\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\right|^{2}\right)$$

since $\left\langle P\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}=\left\langle e^{i\theta}T\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}$

$$\leq\frac{1}{4}\left(\left(\sqrt{\|P\|_{A-\mathrm{Ber}}^{2}-\left|\left\langle T\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\right|^{2}}+\|Q\|_{A-\mathrm{Ber}}\right)^{2}+\left|\left\langle T\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\right|^{2}\right)$$

$$\leq\frac{1}{4}\left(\left(\sqrt{\mathrm{ber}_{A}^{2}(T)-\left|\left\langle T\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\right|^{2}}+\mathrm{ber}_{A}(T)\right)^{2}+\left|\left\langle T\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\right|^{2}\right)$$

$(\textit{since}\ \|P\|_{A-\mathrm{Ber}},\|Q\|_{A-\mathrm{Ber}}\leq\mathrm{ber}_{A}\left(e^{i\theta}T\right)=\mathrm{ber}_{A}(T))$

$$=\frac{\mathrm{ber}_{A}^{2}(T)}{2}+\frac{\mathrm{ber}_{A}(T)}{2}\sqrt{\mathrm{ber}_{A}^{2}(T)-\left|\left\langle T\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\right|^{2}}.$$

Hence

$$\frac{1}{2}\left\|T\widehat{K}_{\lambda}\right\|_{A}\leq\sqrt{\frac{\mathrm{ber}_{A}^{2}(T)}{2}+\frac{\mathrm{ber}_{A}(T)}{2}\sqrt{\mathrm{ber}_{A}^{2}(T)-\inf_{\lambda\in\Omega}\left|\left\langle T\widehat{K}_{\lambda},\widehat{K}_{\lambda}\right\rangle_{A}\right|^{2}}},\tag{2.14}$$

which implies

$$\frac{1}{2}\left\|T\widehat{K}_{\lambda}\right\|_{A}\leq\sqrt{\frac{\mathrm{ber}_{A}^{2}(T)}{2}+\frac{\mathrm{ber}_{A}(T)}{2}\sqrt{\mathrm{ber}_{A}^{2}(T)-\widetilde{c}_{A}^{2}(T)}}.$$

Taking the supremum over $\lambda\in\Omega$ in the above inequality we get

$$\frac{1}{2}\|T\|_{A-\mathrm{Ber}}\leq\sqrt{\frac{\mathrm{ber}_{A}^{2}(T)}{2}+\frac{\mathrm{ber}_{A}(T)}{2}\sqrt{\mathrm{ber}_{A}^{2}(T)-\widetilde{c}_{A}^{2}(T)}}.$$

□

A considerable improvement of the first inequality in (2.7) is said in the last theorem.

Theorem 2.13 *Let $T \in \mathcal{B}_{A,r}(\mathcal{H})$ and $\lambda \in \Omega$. Then*

$$\frac{1}{2}\|T\|_{A-\text{Ber}} \leq \max\left\{|\sin|_A T, \frac{\sqrt{2}}{2}\right\} \text{ber}_A(T) \leq \text{ber}_A(T).$$

Proof Obviously,

$$\max\left\{|\sin|_A T, \frac{\sqrt{2}}{2}\right\} \text{ber}_A(T) \leq \text{ber}_A(T).$$

Furthermore, by (2.14) we have

$$\frac{1}{2}\left\|T\widehat{K}_\lambda\right\|_A \leq \sqrt{\frac{\text{ber}_A^2(T)}{2} + \frac{\text{ber}_A(T)}{2}\sqrt{\text{ber}_A^2(T) - \left|\left\langle T\widehat{K}_\lambda, \widehat{K}_\lambda\right\rangle_A\right|^2}}$$

and hence

$$\frac{1}{2}\left\|T\widehat{K}_\lambda\right\|_A \leq \sqrt{\frac{\text{ber}_A^2(T)}{2} + \frac{\text{ber}_A(T)}{2}\sqrt{\text{ber}_A^2(T) - \left\|T\widehat{K}_\lambda\right\|_A^2 |\cos|_A^2 T}}.$$

From this it follows that

$$\left\|T\widehat{K}_\lambda\right\|_A^2 - 2\text{ber}_A^2(T) \leq 2\text{ber}_A(T)\sqrt{\text{ber}_A^2(T) - \left\|T\widehat{K}_\lambda\right\|_A^2 |\cos|_A^2 T} \tag{2.15}$$

We contemplate two cases.

Case1. $\left\|T\widehat{K}_\lambda\right\|_A^2 - 2\text{ber}_A^2(T) \leq 0$. Then we reach that $\left\|T\widehat{K}_\lambda\right\|_A \leq \sqrt{2}\text{ber}_A(T)$ and so

$$\frac{1}{2}\|T\|_{A-\text{Ber}} \leq \frac{\sqrt{2}}{2}\text{ber}_A(T). \tag{2.16}$$

Case 2. $\left\|T\widehat{K}_\lambda\right\|_A^2 - 2\text{ber}_A^2(T) > 0$. By (2.15) it follows that

$$\begin{aligned}&\left\|T\widehat{K}_\lambda\right\|_{A-\text{Ber}}^4 - 4\left\|T\widehat{K}_\lambda\right\|_{A-\text{Ber}}^2 \text{ber}_A^2(T) + 4\text{ber}_A^4(T)\\ &\leq 4\text{ber}_A^4(T) - 4\text{ber}_A^2(T)\left\|T\widehat{K}_\lambda\right\|_{A-\text{Ber}}^2 |\cos|_A^2 T.\end{aligned}$$

Thus

$$\left\|T\widehat{K}_\lambda\right\|_A^2 \leq 4\left(1 - |\cos|_A^2 T\right)\text{ber}_A^2(T).$$

This yields

$$\frac{1}{2}\left\|T\widehat{K}_\lambda\right\|_A \leq |\sin|_A T\text{ber}_A(T),$$

and hence

$$\frac{1}{2}\|T\|_{A-\text{Ber}} \leq |\sin|_A T\text{ber}_A(T). \tag{2.17}$$

Now, by (2.16) and (2.17) we obtain

$$\frac{1}{2}\|T\|_{A-\mathrm{Ber}} \leq \max\left\{|\sin|_A T, \frac{\sqrt{2}}{2}\right\} \mathrm{ber}_A(T).$$

□

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