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To the solution of integro-differential equations with nonlocal conditions

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Abstract: We investigate linear integro-differential equations with ordinary derivatives. The kernels of the integrands depend only on the variable of integration, and the conditions involve the terms with the point and integral values of the unknown function. We drive necessary and sufficient conditions for the existence and uniqueness of the solution of the problem, which can be used both for analytical and numerical solutions. We present the results of solving an illustrative test problem.

Key words: Integro-differential equations, loaded differential equations, integral conditions, nonlocal conditions, existence and uniqueness conditions

1. Introduction

The paper investigates a class of systems of linear integro-differential equations with ordinary derivatives and nonlocal conditions. The specificity of the problems lies, first, in the fact that the kernels of the integral parts depend on only one variable of integration; second, linear nonlocal conditions contain terms with point and integral values of the sought for phase variables. Such nonlocal problems arise due to the practical impossibility of measuring or influencing dynamic objects and processes instantly or at separate lumped points. The measurements taken and actions carried out, as a rule, are distributed in space and/or in time and characterize, on average, the state of the object in the vicinity of the measured point or moment in time. Problems described by integro-differential equations with kernels, depending on only one variable of integration, are found, for example, in optimal control problems feedback for dynamic objects with feedback, when measurements in time or in space are not of a point, but of an interval nature. Therefore, the values of the controls actions at the current time, assigned according to the results of measurements, participate in the equation in the interval sense, i.e. on average over the intervals in which the measurements were carried out [4, 15].

Note that studies of differential and integro-differential equations with nonlocal conditions began at the beginning of the 20th century in the works of Ch.J. Valle-Poussin and J.D. Tamarkin [9, 20]. Later, existence and uniqueness conditions for solutions were obtained for various classes of these problems [12, 13, 21], and numerical approaches and schemes were proposed [6, 8, 11]. The greatest complexity of the study of these

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problems is associated with obtaining constructive, necessary and sufficient conditions for the existence and uniqueness of the solution easily verified directly from the data of the problem, and its qualitative properties [7], [16]-[19].

In this paper, to study the existence and uniqueness conditions of the considered class of problems, in essence, the fundamental matrix of solutions of a system of differential equations is used. Its construction in the case of nonautonomous systems is itself difficult. Therefore, in practice, these issues are often resolved by conducting numerical experiments. The approach proposed in the paper to the study of the problems under consideration can be used as a constructive approach to the numerical solution.

The paper describes the proposed approach by the example of solving an illustrative problem.

2. Formulation of the problem

Consider the following system of n linear integro-differential equations

$$\dot{x}(t) = A^0(t)x(t) + \sum_{i=1}^{l_1} B^i(t) \int_{\bar{t}_{i1}}^{\bar{t}_{i2}} A^i(\tau)x(\tau)d\tau + C(t),\ t \in [t_0, t_f], \tag{2.1}$$

with n conditions containing integral and point values of the phase function:

$$\sum_{j=1}^{l_2} \alpha^j x(\breve{t}_j) + \sum_{k=1}^{l_3} \int_{\widehat{t}_{k1}}^{\widehat{t}_{k2}} \beta^k(\tau)x(\tau)d\tau = \gamma. \tag{2.2}$$

Here $x(t) \in R^n$ is an unknown continuously differentiable vector-function. Piecewise continuous n-dimensional vector-function $C(t)$, $(n \times n)$ matrix functions $A^0(t)$, $B^i(t)$ at $t \in [t_0, t_f]$, $A^i(t)$ at $t \in [\bar{t}_{i1}, \bar{t}_{i2}]$, $i = 1, 2, ..., l_1$, $\beta^k(t)$, at $t \in [\widehat{t}_{k1}, \widehat{t}_{k2}]$, $k = 1, 2, ..., l_3$, $(n \times n)$ constant matrices α^j, $j = 1, 2, ..., l_2$, and n -dimensional vector γ are given. $\bar{t}_{i1}, \bar{t}_{i2}, \widehat{t}_{k1}, \widehat{t}_{k2}$ are given points of time from $[t_0, t_f]$ and for simplicity, without loss of generality, we will consider them ordered:

$$\bar{t}_{i1} < \bar{t}_{i2},\ \bar{t}_{i+1,1} > \bar{t}_{i2},\ i = 1, 2, ..., l_1 - 1,$$

$$\breve{t}_j < \breve{t}_{j+1},\ \ j = 1, 2, ..., l_2 - 1,\ \breve{t}_1 = t_0,\ \ \breve{t}_{l_2} = t_f,$$

$$\widehat{t}_{k1} < \widehat{t}_{k2},\ \widehat{t}_{k1+1,1} > \widehat{t}_{k2}\ \ k = 1, 2, ..., l_3 - 1.$$

Integro-differential equations (2.1) are also called integrally loaded equations [15]. If we introduce new phase variables:

$$y^i(t) = \begin{cases} 0_n, & t \le \bar{t}_{i1}, \\ \int_{\bar{t}_{i1}}^{t} A^i(\tau)x(\tau)d\tau, & \bar{t}_{i1} < t < \bar{t}_{i2}, \end{cases}$$

satisfying the system of differential equations

$$\dot{y}^i(t) = A^i(t)x(t),\ \ \ t \in [\bar{t}_{i1},\ \bar{t}_{i2}], \tag{2.3}$$

and conditions:

$$y^i(\bar{t}_{i1}) = 0_n,\ i = 1, 2, ..., l_1, \tag{2.4}$$

then system (2.1) will take the form:

$$\dot{x}(t) = A^0(t)x(t) + \sum_{i=1}^{l_1} B^i(t)y^i(\bar{t}_{i2}) + C(t). \tag{2.5}$$

Problem (2.5), (2.3), (2.2), (2.4) of size $(l_1+1)n$ relates to point-loaded problems with nonseparated conditions (2.2), including terms with point and integral values of unknown functions. Such problems have been studied by many authors, in particular, in [1, 6]. If we also introduce $l_3 n$ variables

$$z^k(t) = \begin{cases} 0_n, & t \le \widehat{t}_{k1}, \\ \int\limits_{\widehat{t}_{k1}}^{t} \beta^k(\tau)x(\tau)d\tau, & \widehat{t}_{k1} < t < \widehat{t}_{k2}, \end{cases}$$

then with respect to them we obtain a system of differential equations:

$$\dot{z}^k(t) = \beta^k(t)x(t),\quad t \in [\widehat{t}_{k1},\ \widehat{t}_{k2}],\ k = 1, 2, ..., l_3, \tag{2.6}$$

with conditions

$$z^k(\widehat{t}_{k1}) = 0_n,\ k = 1, 2, ..., l_3. \tag{2.7}$$

Then conditions (2.2) take the form

$$\sum_{j=1}^{l_2} \alpha^j x(\breve{t}_j) + \sum_{k=1}^{l_3} z^k(\widehat{t}_{k2}) = \gamma, \tag{2.8}$$

in which there are no integral terms with respect to unknown functions.

The resulting system of differential equations (2.5), (2.3), (2.6) with conditions (2.8), (2.4), (2.7) is of dimension $(l_1+l_3+1)n$ and is pointwise loaded, but the problem is already free from integral terms.

By combining and arranging the points $\breve{t}_j$, $\widehat{t}_k$, $j = 1, 2, ..., l_2$, $k = 1, 2, ..., l_3$, we denote these points by $\tilde{t}_i$, $i = 1, 2, ..., (l_2+l_3)$. After ordering them let us combine vector functions $(x(t), (y(t), z(t))$ into one and denote it by $u(t) \in R^{(l_1+l_3+1)n}$. Then conditions (2.8), (2.4), (2.7) in the general case can be written in the form:

$$\sum_{i=1}^{l_2+l_3} \tilde{\alpha}^i u(\tilde{t}_i) = \tilde{\gamma}, \tag{2.9}$$

where $\tilde{\alpha}^i$, $i = 1, 2, ..., (l_2+l_3)$, and $\tilde{\gamma}$ are square matrices and a vector of size $(l_2+l_3+1)n$, respectively.

Using the methodology of [14], conditions (2.9) can be reduced to two-point separated conditions. To do this, new variables $\vartheta(t)$ are introduced, related to functions $(x(t), (y(t), z(t))$, with respect to which from (2.3), (2.5), (2.6) for each of the intervals $(\tilde{t}_i,\ \tilde{t}_{i+1})$, $i = 0, 1, ..., (l_2+l_3-1)$, differential equations are introduced with conditions at the ends of each of the corresponding intervals. Scaling the time variables at each of the intervals

$(\tilde{t}_i,\, \tilde{t}_{i+1})$, we translate them into the interval (0;1). As a result, we get a two-point loaded problem, which can be conventionally written, generally forms as follows:

$$\dot{\vartheta}(t) = \mathbf{A}^0(t)\vartheta(t) + \sum_{i=1}^{l_1} \mathbf{B}^i(t)\vartheta(\xi_i) + \mathbf{C}(t), \;\; t \in (0,1), \tag{2.10}$$

$$\vartheta(0) = \vartheta_0, \;\; \vartheta(1) = \vartheta_1,$$

where ξ_i from $(0,1)$ corresponds to points $\bar{t}_{i2}, \;\; i = 1,2,...,l_1$.

Here $n(l_2 + l_3)(l_1 + l_3 + 1)$-dimensional, square matrices $\mathbf{A}^0(t), \;\; \mathbf{B}^i(t)$, and vector $\mathbf{C}(t)$ are determined directly by the original matrices $A^0(t)$, $B^0(t)$, and vector $C(t)$, respectively, taking into account the above changes of phase and time variables. Loaded differential equations of the form (2.10) with various other types of initial-boundary conditions, as well as the corresponding inverse problems, optimal control problems in various settings have been studied by many authors [4, 5, 10]. For them, the necessary conditions for the existence and uniqueness of the solution were obtained [7], [16]-[18] and various numerical solution schemes were proposed [3, 6, 8]. Considering a significant increase in the order of the original problem (2.1) and (2.2), the use of such approaches to the study and solution of practical problems of the form (2.1) and (2.2) and related inverse problems, optimal control problems, causes serious difficulties [4, 5, 10].

Further in the article, we propose an approach to the study and solution of problem (2.1) and (2.2), which does not require an increase in the order of the original system of differential equations and a transition to other phase variables.

3. Research of problem (2.1) and (2.2)

We introduce the following two systems of differential equations:

$$\dot{x}(t) = A^0(t)x(t) + C(t), \;\; t \in [t_0, t_f], \tag{3.1}$$

$$\dot{x}(t) = A^0(t)x(t) + \sum_{i=1}^{l_1} B^i(t)\lambda^i + C(t), \;\; t \in [t_0, t_f], \tag{3.2}$$

and consider two corresponding auxiliary problems: (3.1), (2.2) and (3.2), (2.2).

Here $\lambda^i \in R^n, \;\; i = 1,2,...,l_1$ are arbitrary scalar vectors, and the remaining functions and parameters are the same as in problem (2.1) and (2.2).

Let the $(n \times n)$ matrix function $\Phi(t,\tau)$ be the fundamental matrix of solutions to the homogeneous system corresponding to systems (3.1) and (3.2), i.e.

$$\dot{\Phi}(t,\tau) = A^0(t)\Phi(t,\tau), \;\;\; \Phi(t_0,t_0) = E_n, \;\; t,\tau \in [t_0,t_f],$$

where E_n is the n-dimensional identity matrix.

Theorem 3.1 *Let the functions* $A^0(t)$ *and* $C(t)$ *be continuous at* $t \in [t_0, t_f], \beta^k(t)$ *be continuous at* $t \in [\widehat{t}_{k1}, \widehat{t}_{k2}], \quad k = 1, 2, ..., l_3$. *For the existence and uniqueness of the solution to problem (3.1) and (2.2), it is necessary and sufficient that*

$$rank\left(\sum_{j=1}^{l_2}\alpha^j\Phi(\breve{t}_j, t_0) + \sum_{k=1}^{l_3}\int_{\widehat{t}_{k1}}^{\widehat{t}_{k2}}\beta^k(\tau)\Phi(\tau, t_0)d\tau\right) = n. \tag{3.3}$$

Proof According to Cauchy's well-known formula, the solution of the system of differential equations (3.1) with some yet unknown arbitrary initial condition

$$x(t_0) = x_0, \tag{3.4}$$

is of the form:

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t}\Phi(t, \xi)C(\xi)d\xi. \tag{3.5}$$

Substituting (3.5) into (2.2), after grouping, we obtain

$$\left[\sum_{j=1}^{l_2}\alpha^j\Phi(\breve{t}_j, t_0) + \sum_{k=1}^{l_3}\int_{\widehat{t}_{k1}}^{\widehat{t}_{k2}}\beta^k(\xi)\Phi(\xi, t_0)d\xi\right]x_0 =$$

$$= \gamma - \sum_{j=1}^{l_2}\alpha^j\int_{t_0}^{\breve{t}_j}\Phi(\breve{t}_j, \xi)C(\xi)d\xi - \sum_{k=1}^{l_3}\int_{\widehat{t}_{k1}}^{\widehat{t}_{k2}}\beta^k(\tau)\int_{t_0}^{\tau}\Phi(\tau, \xi)C(\xi)d\xi d\tau. \tag{3.6}$$

Formula (3.6) represents a n-th order linear algebraic system with respect to the unknown initial condition $x(t_0)$. For the existence and uniqueness of the solution of this system, as is known, it is necessary and sufficient that condition (3.3) of Theorem 3.1 be satisfied. □

It is clear that for an arbitrary given initial condition x_0 under the accepted conditions on the functions participating in equation (2.1), the Cauchy problem (3.1), (3.4) has a unique solution. Due to the uniqueness of the solution x_0 of the algebraic system (3.6) under condition (3.3), the problem (3.1) and (2.2) has a unique solution represented in the form (3.5).

The following theorem holds for problem (3.2) and (2.2).

Theorem 3.2 *Let the matrices* $A^0(t), B^i(t)$, $t \in [t_0, t_f]$, $i = 1, 2, ..., l_1$, $\beta^k(t)$, $t \in [\widehat{t}_{k1}, \widehat{t}_{k2}]$, $k = 1, 2, ..., l_3$, *vector* $C(t)$, $t \in [t_0, t_f]$, *be continuous and condition (3.3) be satisfied. Then, for arbitrarily given vectors* $\lambda^i \in R^n$, $i = 1, 2, ..., l_1$, *the solution to problem (3.2) and (2.2) exists, is unique and can be represented in the form*

$$x(t) = x^0(t) + \sum_{i=1}^{l_1}X^i(t)\lambda^i, \tag{3.7}$$

where n *-dimensional vector-function* $x^0(t)$ *and* $(n\times n)$ *matrix functions* $X^i(t),\ i=1,2,...,l_1,$ *are solutions to the following problems*

$$\dot{x}^0(t)=A^0(t)x^0(t)+C(t),\ \ t\in[t_0,t_f], \tag{3.8}$$

$$\sum_{j=1}^{l_2}\alpha^j x^0(\breve{t}_j)+\sum_{k=1}^{l_3}\int_{\widehat{t}_{k1}}^{\widehat{t}_{k2}}\beta^k(\tau)x^0(\tau)d\tau=\gamma, \tag{3.9}$$

$$\dot{X}^i(t)=A^0(t)X^i(t)+B^i(t),\ \ t\in[t_0,t_f], \tag{3.10}$$

$$\sum_{j=1}^{l_2}\alpha^j X^i(\breve{t}_j)+\sum_{k=1}^{l_3}\int_{\widehat{t}_{k1}}^{\widehat{t}_{k2}}\beta^k(\tau)X^i(\tau)d\tau=0_{n\times n},\ \ i=1,2,...,l_1. \tag{3.11}$$

Representation (3.7) for the solution of problem (3.2) and (2.2) is unique.

Proof Existence and uniqueness of a solution to problem (3.2) and (2.2) for arbitrarily given vectors $\lambda^i\in R^n,\ i=1,2,...,l_1$, under condition (3.3) is proved similarly to the proof of Theorem 3.1.

Differentiating representation (3.7) and substituting the result into (3.2), after simple transformations and groupings we will have

$$\left[\dot{x}^0(t)-A^0(t)x^0(t)-C(t)\right]+\sum_{i=1}^{l_1}\left[\dot{X}^i(t)-A^0(t)X^i(t)-B^i(t)\right]\lambda^i=0_n.$$

Due to the arbitrariness of the vectors $\lambda^i\in R^n,\ i=1,2,...,l_1$ for the last equality to be satisfied, it is necessary and sufficient that the expressions in square brackets be equal to zero, and therefore $x^0(t)$ and $X^i(t),\ i=1,2,...,l_1$, satisfy differential equations (3.8) and (3.10).

Substituting the representation of the solution to problem (3.2) and (2.2) in the form (3.7) into condition (2.2), after grouping we will have

$$\left[\sum_{j=1}^{l_2}\alpha^j x^0(\breve{t}_j)+\sum_{k=1}^{l_3}\int_{\widehat{t}_{k1}}^{\widehat{t}_{k2}}\beta^k(\tau)x^0(\tau)d\tau-\gamma\right]+$$

$$+\sum_{i=1}^{l_1}\left[\sum_{j=1}^{l_2}\alpha^j X^i(\breve{t}_j)+\sum_{k=1}^{l_3}\int_{\widehat{t}_{k1}}^{\widehat{t}_{k2}}\beta^k(\tau)X^i(\tau)d\tau\right]\lambda^i=0_n.$$

Taking into account the arbitrariness of the vectors $\lambda^i\in R^n,\ i=1,2,...,l_1$, it is necessary and sufficient that the expressions in square brackets be equal to zero, and therefore conditions (3.9) and (3.11) are satisfied. □

Let us prove the uniqueness of the representation of solution (3.7) for problem (3.2) and (2.2). Let there be one more representation of the solution to problem (3.2) and (2.2):

$$x(t) = \bar{x}^0(t) + \sum_{i=1}^{l_1} \bar{X}^i(t)\lambda^i.$$

Here $\bar{x}^0(t)$ and $\bar{X}^i(t)$ are $n-$dimensional vector-function and $n-$dimensional square matrix functions $i = 1, 2, ..., l_1$, respectively. Then the difference function

$$\Delta x(t) = x(t) - \bar{x}(t) = \Delta x^0(t) + \sum_{i=1}^{l_1} \Delta X^i(t)\lambda^i, \tag{3.12}$$

$$\Delta x^0(t) = x^0(t) - \bar{x}(t),\ \Delta X^i(t) = X^i(t) - \bar{X}^i(t)\,,\ i = 1, 2, ..., l_1,\ t \in [t_0, t_f]$$

must satisfy the homogeneous system of differential equations

$$\Delta\dot{x}(t) = A(t)x(t), \tag{3.13}$$

homogeneous conditions

$$\sum_{j=1}^{l_2} \alpha^j \Delta x(\breve{t}_j) + \sum_{k=1}^{l_3} \int\limits_{\widehat{t}_{k1}}^{\widehat{t}_{k2}} \beta^k(\tau)\Delta x(\tau)d\tau = 0_n, \tag{3.14}$$

for arbitrary vectors $\lambda^i \in R^n,\ i = 1, 2, ..., l_1$.

According to Theorem 3.1, problem (3.13) and (3.14) has a unique solution, namely, in this case, the trivial one: $\Delta x(t) = 0_n,\ t \in [t_0, t_f]$.

Then, from (3.12), taking into account the arbitrariness of the vectors $\lambda^i \in R^n,\ i = 1, 2, ..., l_1$, it follows that:

$$\Delta x^0(t) = 0_n,\ \ \Delta X^i(t) = 0_{n\times n},\ \ \ t \in [t_0, t_f].$$

This implies the uniqueness of the representation of the solution to problem (3.10) and (3.11) in the form (3.7).

Let us present the main theorem concerning the considered problem (2.1) and (2.2). Let us introduce the notation for the Kronecker symbol δ_{ij}: $\delta_{ij} = 1$ and $\delta_{ij} = 0$, if $i \neq j$.

Theorem 3.3 *Let all conditions of Theorem 3.1 be satisfied. For the existence and uniqueness of a solution to problem (2.1) and (2.2), it is necessary and sufficient for the following condition to hold true*

$$rank\left(G : G = \left(\left(\delta_{ij}E_n - \int\limits_{\bar{t}_{i1}}^{\bar{t}_{i2}} A^i(\tau)X^j(\tau)d\tau\right)_{i,j=1}^{l_1}\right)\right) = nl, \tag{3.15}$$

where n-dimensional square matrices $X^i(t),\ i = 1, 2, ..., l_1$, are solutions to problem (3.10) and (3.11).

Proof We introduce the following yet unknown n-dimensional numeric vectors

$$\lambda^i = \int_{\bar{t}_{i1}}^{\bar{t}_{i2}} A^i(\tau)x(\tau)d\tau, \ i = 1, 2, ..., l_1, \tag{3.16}$$

where $x(t)$ is the sought for solution to problem (2.1) and (2.2). Then, according to Theorem 3.2, a solution to problem (2.1) and (2.2), if it exists, has a unique representation in the form (3.7). Participated in (3.7) are $n-$ dimensional vector $x^0(t)$ and $(n \times n)$ matrix functions $X^i(t)$, $i = 1, 2, ..., l_1$, which represent solutions of auxiliary problems (3.8) – (3.11), and they do not depend on λ^i. Under the accepted conditions, according to Theorem 3.2, solutions to these problems exist and are unique.

Using the representation in the form (3.7) to solve problem (2.1) and (2.2), taking into account the notation (3.16), we obtain the following algebraic system with respect to unknown vectors $\lambda^i \in R^n$, $i = 1, 2, ..., l_1$:

$$\lambda^i = \int_{\bar{t}_{i1}}^{\bar{t}_{i2}} A^i(\tau)x^0(\tau)d\tau + \sum_{j=1}^{l_1} \int_{\bar{t}_{j1}}^{\bar{t}_{j2}} A^i(\tau)X^j(\tau)d\tau \cdot \lambda^j.$$

Let us write this system in the form:

$$G\Lambda = Q, \tag{3.17}$$

Here, the following notations are used for nl-dimensional vectors:

$$Q = \left(\int_{\bar{t}_{11}}^{\bar{t}_{12}} A^1(\tau)x^0(\tau)d\tau, \int_{\bar{t}_{21}}^{\bar{t}_{22}} A^1(\tau)x^0(\tau)d\tau, ..., \int_{\bar{t}_{l1}}^{\bar{t}_{l2}} A^l(\tau)x^0(\tau)d\tau \right)^T,$$

$$\Lambda = (\lambda^1, \lambda^2, ..., \lambda^l)^T,$$

"T" is the transposition sign.

Algebraic system (3.17), whose Jacobian matrix is the matrix G from condition (3.15) will have a unique solution only if condition (3.15) of Theorem 3.3 is satisfied. If nl -dimensional vector Λ is defined, then, taking into account the notation (3.16), problem (2.1) and (2.2) will coincide with problem (3.1) and (3.2), for which the existence and uniqueness of a solution were proved in Theorem 3.1 under the above conditions. □

4. Scheme for solving problem (2.1) and (2.2)

When choosing a method for solving a specific problem of the form (2.1) and (2.2), two possible cases should be considered with respect to the system of differential equations (2.1).

In the case of an autonomous system of differential equations (2.1), i.e. when $A^0(t) = const$, $t \in [t_0, t_f]$, an effective approach is using the construction of the fundamental matrix $\Phi(t, \tau)$ using the found eigenvalues of the matrix A^0. Further, matrix $\Phi(t, \tau)$ is used to construct solutions to auxiliary problems (3.8) – (3.11), it does not matter whether numerical or analytical methods will be used to construct the fundamental matrix. Knowledge of the fundamental matrix greatly facilitates the work with conditions (3.9) and (3.11) containing point and integral values of the unknown functions. The solution of problems (3.8) – (3.11) in this case is first reduced to the solution of the linear algebraic system of equations (3.6) by determining the initial conditions, the vector $x^0(t_0)$ and matrices $X(t_0)$; then, using these initial conditions, we solve the Cauchy problems with respect to (3.8), (3.10), and determine $x^0(t_0)$ and $X^i(t_0)$, $i = 1, 2, ..., l_1$. If $x^0(t)$ and $X^i(t)$, $i = 1, 2, ..., l_1$, are obtained in an analytical form, then they are used to determine, from (3.16), the values of the parameters λ^i –the integrals in the right-hand sides of the system of differential equations (2.1), and the desired solution to problem (2.1), (2.2) from formula (3.7). If the systems (3.8) – (3.11) were solved using numerical methods and the values of solutions $x^0(t)$, $X^i(t)$, $i = 1, 2, ..., l_1$, and were not preserved at all points of the grid, then from formula (3.7), initial condition $x(t_0) = x^0(t_0) + \sum_{i=1}^{l_1} X^i(t_0)\lambda^i$ is determined and then the Cauchy problem is numerically solved with respect to system (2.1).

In case of a nonautonomous system of differential equations (2.1), i.e. when $A(t) \neq const$, the construction of the fundamental matrix $\Phi(t, \tau)$, even by numerical methods require, first, a large amount of computations, and, second a large amount of memory to store its values at all points of the grid the segment $[t_0, t_f]$ laid on. Therefore, to solve auxiliary problems (3.8) – (3.11) with nonseparated point and integral conditions, one should use special methods developed for such problems [1, 3]. Further procedures for solving an algebraic system with respect to $\lambda^i \in R^n$, $i = 1, 2, ..., l_1$ and finding a solution to the original problem (2.1), (2.2) are carried out according to the same scheme as described above.

Consider the solution to the following illustrative problem:

$$\dot{x}(t) = 2x(t) - 6\int_1^2 x(\tau)d\tau - 2t^2 + 24, \quad t \in [0, 4], \tag{4.1}$$

$$x(0) + 6\int_2^3 x(\tau)d\tau - x(4) = 33. \tag{4.2}$$

The differential equation is autonomous, as $A^0(t) = const = 2$, $t \in [0, 4]$, $B^1(t) = -6$, $C(t) = -2t^2 + 24$, $\alpha^1 = 1$, $\alpha^2 = -1$, $\beta = 6$.

It is easy to check that the solution to this problem is the function $x(t) = t^2 + t$.

Let us introduce the notation

$$\lambda^1 = \int_1^2 x(\tau)d\tau \tag{4.3}$$

and construct auxiliary problem (3.8) and(3.9):

$$\dot{x}^0(t) = 2x^0(t) - 2t^2 + 24, \tag{4.4}$$

$$x^0(0) + 6\int_2^3 x^0(\tau)d\tau - x^0(4) = 33. \tag{4.5}$$

and problem (3.10), (3.11):

$$\dot{X}^1(t) = 2X^1(t) - 6\,, \tag{4.6}$$

$$X^1(0) + 6\int\limits_2^3 X^1(\tau)d\tau - X^1(4) = 0. \tag{4.7}$$

Having solved the characteristic equations for (4.4) and (4.6), we obtain

$$x^0(t) = t^2 + t - c_0 e^{2t} - 11.5, \tag{4.8}$$

$$X^1(t) = c_1 e^{2t} + 3, \tag{4.9}$$

where c_0 and c_1 are constants, which must be determined, from conditions (4.5) and (4.7), respectively. Substituting (4.8) into (4.5) and (4.9) into (4.7), we obtain two independent linear equations for c_0 and c_1, from which we find

$$c_0 = \frac{69}{(e^8 - 3e^6 + 3e^4 - 1)}, \quad c_1 = \frac{18}{(e^8 - 3e^6 + 3e^4 - 1)}.$$

Then the solutions of auxiliary problems (3.8) – (3.11) are the functions:

$$x^0(t) = t^2 + t - \frac{69e^{2t}}{(e^8 - 3e^6 + 3e^4 - 1)} - 11.5, \quad X^1(t) = \frac{18e^{2t}}{(e^8 - 3e^6 + 3e^4 - 1)} + 3. \tag{4.10}$$

Using representation (3.7), we have

$$x(t) = x^0(t) + X^1(t) \cdot \lambda^1. \tag{4.11}$$

Taking into account the notation (3.16), we obtain the equation with respect to λ^1:

$$\lambda^1 = \int\limits_1^2 \left[x^0(t) + X^1(t) \cdot \lambda^1\right] dt.$$

Substituting into this equation the functions from (4.8) and (4.9), after integration we obtain: $\lambda^1 = 23/6$.

From representation (4.11), using (4.10), we obtain the desired solution to the original problem (4.1), (4.2):

$$x(t) = t^2 + t, \;\; t \in [0, 4]\,.$$

When solving the problems under consideration numerically, special attention should be paid to the solution auxiliary problems (4.4), (4.5) and (4.6), (4.7) involving conditions with nonseparated point and integral values of unknown solutions [3].

5. Conclusion

The paper investigates a class of problems described by integro-differential systems under conditions involving nonseparated point and integral values of the unknown functions. In practice, such problems often emerge when it is impossible to measure the state or the impact of the dynamics object at a point in space or a moment of time. The scheme for obtaining the existence and uniqueness conditions of the solution of the problem can be used to obtain both analytical and numerical solution of these problems.

Note that problems described by integro-differential equations with partial derivatives of, for example, parabolic or hyperbolic types, can be reduced to the class under consideration if the method of lines is applied to them.

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