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A new subclass of certain analytic univalent functions associated with hypergeometric functions

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Abstract: The main objective of the present paper is to give with using the linear operator theory and also hypergeometric representations of related functions a new special subclass $\mathcal{TS}_p(2^{-r}, 2^{-1}), r \in \mathbb{Z}^+$ of uniformly convex functions and in addition a suitable subclass of starlike functions with negative Taylor coefficients. Furthermore, the provided trailblazer outcomes in presented study are generalized to certain functions classes with fixed finitely many Taylor coefficients.

Key words: Convex function, hypergeometric function, starlike function, uniformly convex functions, univalent function.

1. Introduction

We begin with a very brief of some basic concepts on origin of the theory of univalent functions which will motivate us at this study. It is a known fact that there are many proofs of Riemann mapping theorem expressed in 1851 by the famous mathematician Riemann in his doctoral thesis. Indeed, unlike other famous theorems of mathematics, the Riemann theorem and proofs, throughout its history seeing influence on development of many mathematical concept and theory with the wide and variegated information. As the best illustrations of these cases can be given concept of family of functions and univalent function theory, respectively. We will now discuss these connections. Let us briefly recall the concept of the Riemann mapping theorem. The theorem states that any simply connected domain in the complex plane, that is not the entire plane($\mathbb{C}$), is bianalytic (or biholomorphic) to the unit disc $\mathcal{D}$. Furthermore given a simply connected $\mathcal{R} \subsetneq \mathbb{C}$ and any $z_0 \in \mathcal{R}$, there is a unique bianalytic map $f : \mathcal{R} \to \mathcal{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$. In fact, only the necessary condition of the relevant theorem was introduced by Riemann in 1851 [15]. Also, the sufficient condition of this theorem was introduced by Koebe in 1907 [9]. Under the influence of this theorem, analytical functions in a simple connected domain and their geometric properties were investigated separately. It is appreciated that it is very difficult to determine the common geometric properties of the analytical functions in the region $\mathcal{R}$ and collection a class of the functions that have these common properties. Under the contributions of Koebe, this problem has been evolved into an ordinary operation by conventional inverse transform techniques. Since $f : \mathcal{R} \subsetneq \mathbb{C} \to \mathcal{D}$ is bianalytic with analytic inverse, we have that $f^{-1} : \mathcal{D} \to \mathcal{R} \subsetneq \mathbb{C}$ is automatically analytic. In this way, it is possible to examine the geometrical properties of the field analytical functions, the area of which is reduced

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to the unit disc without disturbing the generality, which gives a new understanding to the theory of geometric function. Traditionally denote by $\mathcal{S}$ the subclass of univalent analytic functions f in the unit disc $\mathcal{D}$, and by $\mathcal{S}^*$ the subclass consisting of functions f in $\mathcal{S}$ which $f(\mathcal{D})$ is a starlike domain with respect to the origin, and by $\mathcal{C}$ the subclass consisting of functions f in $\mathcal{S}$ which $f(\mathcal{D})$ is a convex domain. In this case, a starlike domain can be expressed as a domain where each of its point can be seen when viewed from the origin point of the complex plane and also a convex domain is such a domain that is starlike with respect to each of its point. With equivalent and more mathematical phrases, a domain is starlike if every point of it can be connected to the origin with a line segment, and similarly, a domain is convex if it contained the line segment joining any two points in the domain. Furthermore, if extra conditions such as starlike, convex are imposed on f function which is analytic and univalent in the open unit disc, different classes of analytic functions and so on are obtained. As stated earlier, the images of functions in these classes describe various nice and interesting geometries and classical characterizations. The amazing facts discussed earlier have contributed to the convenience in the creation of formal function classes and analysis of the geometric properties of analytical functions as well as the development of the relationship between geometry and analysis. In this way, mathematicians have succeeded not only in describing those geometries in this field and characterized formal classes of analytic functions, but also in establishing close relations between certain prescribed properties of analytic functions and the geometries of their image regions. For example, if a function f maps the open unit disc $\mathcal{D}$ onto a starlike domain, then $Re\left(\frac{zf'}{f}\right)$ is positive. The converse is also true. After the emergence of these function families, mathematicians started to study different classes of conformal transformation and searched for features shared by all elements of the family.

As can be remembered, a complex-valued function $f(z)$ is said to be analytic in a simply connected domain $\mathcal{R}$ of the entire complex plane if $f(z)$ is complex differentiable at each point of $\mathcal{R}$ and if $f(z)$ is single-valued. Recall that, by an analytic function we mean a function that is analytic in certain domain. As a more modern term, the term holomorphic is sometimes used as a synonym for analytic. According to Weierstrass's definition (about 150 years ago), holomorphic functions in domain $\mathcal{R}$ are locally equal to sum functions of power series [27]. On the other hand, the holomorphicity or complex differentiability of a function in $\mathcal{R}$ implies that this holomorphic function has local representations by convergent power series. This amazing fact was already discovered by Cauchy in the years 1830-1840 and, as is well known, this development contributed to the classification of analytic (or holomorphic) functions using their rather fascinating common geometric characterizations.

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disc $\mathcal{D} = \{z \in \mathbb{C} :\mid z \mid< 1\}$ in the entire plane $\mathbb{C}$. If we impose to be normalized by conditions $f(0) = 0$ and $f'(0) - 1 = 0$ to the functions which are elements of class $\mathcal{H}$, we obtain a new class $\mathcal{A}$ which is subclass of $\mathcal{H}$. Furthermore, if these functions which are elements of class $\mathcal{A}$ are imposed to be univalent in the open unit disc $\mathcal{D}$, we obtain functions having the Taylor series form

$$w = f(z) = \sum_{n=1}^{\infty} a_n z^n = z + a_2 z^2 + ..., \tag{1.1}$$

where always $a_n \in \mathbb{C}$ with $a_1 = 1$, the class of these functions generally denoted by as $\mathcal{S}$. As will be remembered, a single-valued function $f(z)$ is said to be univalent in domain $\mathcal{R}$ if the images of the different elements in $\mathcal{R}$ are different under the function, that is, $z_1 \neq z_2$ in $\mathcal{R}$ implies that $f(z_1) \neq f(z_2)$. The fact that a function f, which is a member of class $\mathcal{S}$, has a power series representation in this way is very useful for studies in this field. Respectively, the classes reminded above are analytically characterized as follows:

$$\mathcal{H} = \{f : \mathcal{D} \to \mathbb{C} : f \text{ is an analytic function}\}$$

and

$$\mathcal{A} = \{f \in \mathcal{H} : f(0) = 0 \text{ and } f'(0) - 1 = 0\}$$

and

$$\mathcal{S} = \{f \in \mathcal{A} : f \text{ is an univalent function}\}.$$

Among all these classes, $\mathcal{S}$ class is the most intensely discussed class. There are also more special classes in the literature that are defined in this field and are studied extensively [5–7]. The first studies on analytical and univalent functions on the unit disc were initiated by Koebe in 1907. He discovered that the ranges of all functions in $\mathcal{S}$ contain a common disk $\mid w \mid \leq \frac{1}{4}$ with center in the origin and radius $\frac{1}{4}$(Koebe one-quarter theorem)[11]. Thus every univalent function f has a an inverse f^{-1} which $f^{-1}(f(z)) = z, z \in \mathcal{D}$. Furthermore, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + (5a_2^3 - 5a_2 a_3 + a^4)w^4 + \ldots.$$

In addition to the function f, which is an element of the $\mathcal{S}$ class is said to biunivalent in the open unit disk $\mathcal{D}$ if its inverse map f^{-1} is univalent too. Moreover, the class of all such functions is commonly denoted by Σ. Examples of biunivalent functions in $\mathcal{D}$ are

$$\frac{z}{1-z}, -log(1-z)$$

and so on. Algebraic sum of two functions belonging to class $\mathcal{S}$ may not be an element of this class. This pushed the researchers more intensively to investigate the properties of the Hadamard(or convolution) product of two functions in class $\mathcal{S}$. However, class $\mathcal{S}$ is protected under the junction of a certain number of elementary transformations which are conjugation, rotation and so on. The best example of this case is $k(x) = z(1-z)^{-2}$ Koebe function. On the other hand, Bieberbach conjecture, which is known as the coefficient estimate problem for the $\mathcal{S}$ class, is one of the very studied subjects on the $\mathcal{S}$ class. In 1985, Louis de Branges de Bourcia [3] proved the celebrated Bieberbach conjecture which was proposed in 1916 and which states that, for each $f \in \mathcal{S}$ given by the power series expansion (1.1), the following coefficient in equality holds true:

$$\mid a_n \mid \leq n, \text{ for all } n = 2, 3, \cdots.$$

However, this condition for the coefficients of a function of the form $w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ does not require it to be analytic or univalent in the unit disk. For example, the function $h(z) = 2z - k(z)$ satisfies famous the Bieberbach inequality $\mid a_n \mid \leq n (n \geq 2)$ but its $h'(z)$ derivative vanishes in $\mathcal{D}$ and so the function $h(z)$ is not univalent in $\mathcal{D}$. For many years, the problem of coefficient conjecture in univalent functions has challenged mathematician researchers.

The extremal example of this conjecture is also the Koebe function. However, while the $k(x) = z(1-z)^{-2}$ Koebe function is the most extremal example for univalent functions, the product $k^2(z)/z$ is not. This interesting situation led mathematician researchers to study the geometric properties of the products of univalent functions.

For example, while $k(x) = z(1-z)^{-2}$ Koebe function is the most important member of $\mathcal{S}$ class, $k^2(z)/z$ is not an example of this class. It is well known that, Hadamard products (or convolution) offer useful characterization for memberships in certain classes. For example, the convolution characterization allows us to take benefit from the distributive feature of the Hadamard product. Although the Hadamard product of two convex functions in class $\mathcal{A}$ is convex, Hadamard product of two convex functions in the same class $\mathcal{A}$ is not necessarily starlike . For $f(z) := z + \sum\limits_{n=2}^{\infty} a_n z^n$ and $g(z) := z + \sum\limits_{n=2}^{\infty} b_n z^n$ in class $\mathcal{A}$ is defined by

$$(f * g)(z) = f(z) * g(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n, \mid z \mid < 1.$$

For the above mentioned Hadamard product to be in the certain subclass of convex functions the same conditions have been studied and introduced by Ruscheweyh and Sheil-Small [19, 20].

In the following periods, depending on the developments in univalent function theory, more special subclasses of $\mathcal{S}$ class have been defined depending on the common geometric characterizations of $f(\mathcal{D})$ image sets of functions belonging to $\mathcal{S}$ class, which are also characterized analytically by the inequalities,

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \text{ or } Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$$

with $z \in \mathcal{D}$ and $\alpha \in [0,1)$. These subclasses are respectively the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathcal{D}$ and the class $\mathcal{C}(\alpha)$ of convex function of order α in $\mathcal{D}$, in both cases where $\alpha \in [0,1)$.These classes, which are formed according to the characterizations of $f(\mathcal{D})$ images of f functions, each of which is an element of class $\mathcal{C}$, were first introduced in 1981 by Robertson [16, 17] . Respectively, the classes reminded above are analytically characterized as follows:

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \alpha \in [0,1), z \in \mathcal{D} \right\}$$

and

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{A} : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \alpha \in [0,1), z \in \mathcal{D} \right\}.$$

In this sense, the concept of starlike function was first introduced in 1915 by Alexander [1] and the concept of convex function by Study [24] in 1913 . It is simple to see that $\mathcal{C}(\alpha) \subsetneq \mathcal{C}$, $\mathcal{S}^*(\alpha) \subsetneq \mathcal{S}$ and $\mathcal{C}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}$ for $0 < \alpha < 1$ and $\mathcal{C}(0) = \mathcal{C}$, $\mathcal{S}^*(0) = \mathcal{S}$ for $\alpha = 0$. Furthermore, it is easily observed that $f(z) \in \mathcal{C}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$ for $0 \leq \alpha < 1$(Alexander duality theorem).

Hypergeometric functions also take an important part in the theory of univalent function, as well as in other fields of mathematics. The studies in this area, which were slightly blocked in a certain period, gained new momentum with the use of related hypergeometric functions. It is well known that sometimes it is difficult to see the properties of functions given in a closed form. In such cases, expressing functions in terms of hypergeometric function eliminates this difficulty. The relationship between analytic univalent functions and hypergeometric functions has recently been investigated and studied by prominent mathematician researchers, which are accepted as initial studies in this sense. Merkes and Scott have proved very important and guiding features regarding the starlikeness of hypergeometric functions.

Hypergeometric functions were first introduced by Gauss in 1886. For this reason, these functions are commonly known as Gauss hypergeometric function. Most of the mathematical functions including all the univalent functions could be defined as hypergeometric functions. The Gaussian hypergeometric function $z \mapsto {}_1F_1(a;c;z)$ with complex parameters a, c $(c \neq 0, -1, -2, ...)$, $a \neq -1$ is defined for $\mid z \mid < 1$ by the power series as:

$$F(z) = {}_1F_1(a;c;z) = z + \sum_{n=0}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n, \tag{1.2}$$

where $(c)_n$ is the Pochhammer symbol or shifted factorial defined by

$$(c)_n := \frac{\Gamma(c+n)}{\Gamma(c)} = \begin{cases} c(c+1)(c+2)...(c+n-1), & \text{if } n = 1, 2, ..., \\ 1, & \text{if } n = 0. \end{cases}$$

Some of the properties of this symbol, which was first introduced and widely used by L. A. Pochhammer [13] in 1890 can be given as

$$(c)_n = c(c+1)_{n-1}, (c)_{m+n} = (c)_m (c+m)_n \text{ and } (1)_n = n!.$$

Hypergeometric functions are solutions to the the homogenous hypergeometric differential equation, which has a regular singular point at the starting point. A hypergeometric function can be derived from the hypergeometric differential equation. On the other hand, since $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1)$ (where Γ denotes Gamma function) we can write as

$$\begin{aligned} {}_1F_1(a,b;c;z) &= \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} z^n \\ &= 1 + \frac{a}{c1!} z + \frac{a(a+1)}{c(c+1)2!} z^2 + \cdots \\ &= \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)(1)_n} z^n. \end{aligned}$$

According to some mathematician researchers the function $F(z) = {}_1F_1(a;c;z)$ was first introduced by Kummer in 1837, hence it was also called the hypergeometric function. The Kummer confluent hypergeometric function is defined by the absolutely convergent infinite power series $\sum\limits_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} z^n$ where a, c and z are unrestricted. On the other hand, the term ”hypergeometric series” is usually used less to mean closed structure, but it is sometimes used to mean hypergeometric function. It is analytic at zero entire single-valued transcendental function of all $a \neq -1$ and $c \neq 0$ or a negative integer. The function $F(z) = {}_1F_1(a;c;z)$ has many extraordinary properties. Some of these can be listed as follows:

(i) $\frac{d}{dz} {}_1F_1(a;c;z) = \frac{a}{c} {}_1F_1(a+1;c+1;z)$

(ii) ${}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int\limits_0^1 e^{zt} t^{a-1} (1-t)^{c-a-1} dt, Re(c-a) > 0\,.$

Finally, let us briefly recall the main theme to be given so far. Considered the Riemann mapping theorem, in univalent function theory, the open unit disc $\mathcal{D}$ is well-understood to be considered as a standard domain. The numerous benefits this situation brings to the theory of geometric function, which aims to analyze the properties of analytical functions based on the geometric properties of their images, have been previously expressed. In this sense, one of the key tools used to study these geometric properties was the expression of analytical functions in terms of hypergeometric functions also. More information on mapping properties of the Gauss hypergeometric functions can be found in the studies in [2, 14, 23, 25].

Now we briefly collect some standard notations and basic concepts and certain subclasses of the $\mathcal{S}$ class used in this article. A function $f(z) \in \mathcal{S}$ is to be in uniformly convex(or starlike) in $\mathcal{D}$ if $f(z) \in \mathcal{C}$ $(\mathcal{S}^*)$ and has the property that for every circular arc λ contained in $\mathcal{D}$, with center ς also in $\mathcal{D}$, the arc $f(\lambda)$ is convex(or starlike) with respect to $f(\varsigma)$. The class of uniformly convex functions is denoted by $\mathcal{UCV}$ and the class of uniformly starlike functions by $\mathcal{UST}$. In some cases, it may be difficult to see geometrically whether a given function $f \in \mathcal{S}$ is in $\mathcal{UCV}$. There are some analytical conditions that can be used in such situations. One of these is as follows: a necessary and sufficient condition for a function $f \in \mathcal{S}$ to be in $\mathcal{UCV}$ is that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\}, z \in \mathcal{D} \tag{1.3}$$

Accordingly, the $\mathcal{UCV}$ class can be analytically described as

$$\mathcal{UCV} = \left\{f \in \mathcal{S} : \left|\frac{zf''(z)}{f'(z)}\right| \leq Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\}, z \in \mathcal{D}\right\}.$$

This class was first introduced by Goodman [8]. In addition, Rønning F. [18] and Ma and Minda [10] independently introduced a new class of parabolic starlike functions closely related to the $\mathcal{UCV}$, which is usually denoted by $\mathcal{S}_p$. A function $f \in \mathcal{S}$ is said to be in $\mathcal{S}_p$ if and only if

$$\left|\frac{zf'(z)}{f'(z)} - 1\right| \leq Re\left\{\frac{zf'(z)}{f(z)}\right\}, z \in \mathcal{D}. \tag{1.4}$$

Similarly, also the class $\mathcal{S}_p$ can be described analytically as

$$\mathcal{S}_p = \left\{f \in \mathcal{S} : \left|\frac{zf'(z)}{f(z)} - 1\right| \leq Re\left\{1 + \frac{zf'(z)}{f'(z)}\right\}, z \in \mathcal{D}\right\}.$$

Or equivalently, the class $\mathcal{S}_p$ is consisting of parabolic starlike functions which can be expressed as $f(z) = zg'(z)$ where $g \in \mathcal{UCV}, f \in \mathcal{S}$ and $z \in \mathcal{D}$. That is,

$$g(z) \in \mathcal{UCV} \Leftrightarrow f(z) = zg'(z) \in \mathcal{S}_p.$$

The class $\mathcal{S}_p$ geometrically describes the domain of values of the expression $\dfrac{zf'(z)}{f(z)}$, $z \in \mathcal{D}$, is the parabolic domain $(Imw)^2 < 2Rew - 1$.

In addition, Rønning generalized the class $\mathcal{S}_p$ using an α parameter, with $\alpha \in [-1,1)$,

$$\mathcal{S}_p(\alpha) = \left\{ f \in \mathcal{S} : \Big| \frac{zf'(z)}{f(z)} - 1 \Big| \leq Re\Big\{ 1 + \frac{zf'(z)}{f'(z)} - \alpha \Big\}, z \in \mathcal{D} \right\}.$$

As in many fields of mathematics, some linear operators, which have very powerful and very useful properties, lead many important developments in Geometric function theory. In particular, linear operators, which can be expressed in the convolution terms of certain analytic functions, stand out in this sense. At this stage, let us remember the certain linear operators, which are used frequently in this field and add excitement to the studies. In 1984, Carlson and Shaffer used the well-known Hadamard product to define the linear operator $\mathcal{L}(a,c) : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{L}(a,c)f(z) = {}_1F_1(a;c;z) * f(z),$$

where

$$\begin{aligned} {}_1F_1(a;c;z) &= \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \\ &= z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n, (c \neq 0, -1, -2, \cdots) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(c)\Gamma(a+n-1)}{\Gamma(a)\Gamma(c+n-1)} z^n. \end{aligned}$$

The Carlson and Shaffer linear operator maps $\mathcal{A}$ onto $\mathcal{A}$ with $\mathcal{L}(a,c)$ for $a \neq 0, -1, -2, \cdots$ [4]. Moreover, it is well-known that

$$z[\mathcal{L}(a,c)f(z)]' = a\mathcal{L}(a+1,c)f(z) - (a-1)\mathcal{L}(a,c)f(z), (c \neq 0, -1, -2, \cdots).$$

In 1975, Ruscheweyh used Hadamard(or convolution) product to define on class $\mathcal{A}$ of analytic functions a linear operator D^p, by

$$D^p f(z) = \frac{z}{(1-z)^{p+1}} * f(z), p \in \mathbb{R}, -1 < p.$$

For $p = n \in \{0, 1, 2, \cdots\}$, we obtain

$$D^p f(z) = \frac{z}{n!} \frac{d}{dz} \left(f(z) z^{n-1} \right).$$

The expression $D^n f(z)$ is called an nth-order Ruscheweyh derivative of $f(z)$ [19]. On the other hand, $D^0 f(z) = f(z)$ is identity operator, and $D^1 f(z) = zf'(z)$ is called as Alexander differential operator. If $a = p+1$ and $c = 1$, then

$$\mathcal{L}(p+1, 1)f(z) = D^p f(z)$$

which is the Ruscheweyh operator. Therefore, the Carlson and Shaffer derivative operator defined by $D^p f(z) = \dfrac{z}{(1-z)^{p+1}} * f(z)$ [22]. Moreover, for $\alpha \in [-1,1)$ and $\beta \geq 0$, we usually denoted by $\mathcal{S}(\alpha,\beta)$ the subclass of $\mathcal{S}$ consisting of the functions of the form (1.1) and satisfying the analytic criterion

$$Re\left\{ \frac{z\left(\mathcal{L}(a,c)f(z)\right)'}{\mathcal{L}(a,c)f(z)} - \alpha \right\} > \beta \left| \frac{z\left(\mathcal{L}(a,c)f(z)\right)'}{\mathcal{L}(a,c)f(z)} - 1 \right|, z \in \mathcal{D}. \tag{1.5}$$

We also let $\mathcal{TS}(\alpha,\beta)=\mathcal{S}(\alpha,\beta)\cap\lambda$ where λ, the class of $\mathcal{S}$ consisting of functions of the form

$$w=f(z)=z-\sum_{n=2}^{\infty}a_nz^n, a_n\geq 0, \tag{1.6}$$

was introduced and studied by Silverman [12, 21].

The ultimate aim of this study was to obtain the necessary and sufficient conditions for $f(z)$ to be in the class $\mathcal{TS}(2^{-r},2^{-1})$, where r is a positive integer.

2. Main results

Theorem 2.1 *Let $\beta=2^{-1}$ and $\alpha=2^{-r}$ for $r\in\{1,2,3,\cdots\}$. Then, a function $f(z)$ on the form (1.1) is in the class $\mathcal{S}$ if*

$$\sum_{n=2}^{\infty}\left[n\left(\frac{3}{2}\right)-\left(2^{-r}+\frac{1}{2}\right)\right]\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|\leq 1-2^{-r}. \tag{2.1}$$

Proof In view of equation (1.5), we need to show that

$$\frac{1}{2}\left|\frac{z\left(\mathcal{L}(a,c)f(z)\right)'}{\mathcal{L}(a,c)f(z)}-1\right|-Re\left\{\frac{z\left(\mathcal{L}(a,c)f(z)\right)'}{\mathcal{L}(a,c)f(z)}-1\right\}\leq 1-2^{-r}.$$

Now, after the necessary algebraic operations based on the expression of (1.6), we see that

$$\begin{aligned}&\frac{1}{2}\left|\frac{z\left(\mathcal{L}(a,c)f(z)\right)'}{\mathcal{L}(a,c)f(z)}-1\right|-Re\left\{\frac{z\left(\mathcal{L}(a,c)f(z)\right)'}{\mathcal{L}(a,c)f(z)}-1\right\}\\ &\leq\frac{3}{2}\left|\frac{z\left(\mathcal{L}(a,c)f(z)\right)'}{\mathcal{L}(a,c)f(z)}-1\right|\\ &\leq\frac{\frac{3}{2}\sum\limits_{n=2}^{\infty}(n-1)\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|}{1-\sum\limits_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|}.\end{aligned}$$

But this last expression is bounded by $1-2^{-r}$ if and only if (2.1) holds and hence proof is complete. □

Theorem 2.2 *Let $\beta=2^{-1}$ and $\alpha=2^{-r}$ for $r\in\{1,2,3,\cdots\}$. Then, a function $f(z)$ on the form (1.6) is in the class $\mathcal{TS}(2^{-r},2^{-1})$ if*

$$\sum_{n=2}^{\infty}\left[n\left(\frac{3}{2}\right)-\left(2^{-r}+\frac{1}{2}\right)\right]\frac{(a)_{n-1}}{(c)_{n-1}}a_n\leq 1-2^{-r}. \tag{2.2}$$

Proof According to Theorem (2.1), it will be enough to show only the necessity. If $f(z)\in\mathcal{TS}(2^{-r},2^{-1})$ and z is a real number then

$$\frac{1-\sum\limits_{n=2}^{\infty}n\frac{(a)_{n-1}}{(c)_{n-1}}a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}a_nz^{n-1}}-2^{-r}\geq\frac{1}{2}\left|\frac{1-\sum\limits_{n=2}^{\infty}n\frac{(a)_{n-1}}{(c)_{n-1}}a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}a_nz^{n-1}}\right|.$$

In the last expression obtained above, if $z \to 1$, the desired inequality (2.2) is obtained. This completes the proof of our theorem. □

Corollary 2.3 *A function $f(z)$ of the form (1.6) is in the class $\mathcal{TS}(2^{-r}, 2^{-1})$, then*

$$a_n \le \frac{(1-2^{-r})(c)_{n-1}}{\left[n\left(\frac{3}{2}\right) - \left(2^{-r} + \frac{1}{2}\right)\right](a)_{n-1}}, n \ge 2, r \in \{1,2,3,\dots\}.$$

Remark 2.4 *According to the Theorem (2.2), we can see that, if $f(z) \in \mathcal{TS}(2^{-r}, 2^{-1})$ is of the form (1.6) then*

$$a_2 = \frac{(1-2^{-r})(c)}{\left(\frac{5}{2} - 2^{-r}\right)(a)}$$

At this stage, additionally we defined a new subclass $\mathcal{TS}_p(2^{-r}, 2^{-1})$ of the class $\mathcal{TS}(2^{-r}, 2^{-1})$ with fixing the second coefficient and also obtained the following theorems. Here, the subclass $\mathcal{TS}_p(2^{-r}, 2^{-1})$ is the class of all functions to be in the $\mathcal{TS}(2^{-r}, 2^{-1})$ which has the form

$$f(z) = z - \frac{p(1-2^{-r})(c)}{\left(\frac{5}{2} - 2^{-r}\right)(a)} z^2 - \sum_{n=3}^{\infty} a_n z^n, a_n \ge 0, p \in [0,1]. \tag{2.3}$$

Theorem 2.5 *Let function $f(z)$ is the form (2.3) then $f(z) \in \mathcal{TS}_p(2^{-r}, 2^{-1})$ if and only if*

$$\sum_{n=2}^{\infty} \left[n\left(\frac{3}{2}\right) - \left(2^{-r} + \frac{1}{2}\right)\right] \frac{(a)_{n-1}}{(c)_{n-1}} a_n \le (1-p)(1-2^{-r}), \tag{2.4}$$

where $r \in \{1,2,3,\dots\}, p \in [0,1]$.

Proof We let

$$a_2 = \frac{p(1-2^{-r})(c)}{\left(\frac{5}{2} - 2^{-r}\right)(a)}$$

in expression (2.2) and by simplification, we see that the desired results hold. □

Corollary 2.6 *A function $f(z)$ of the form (2.3) is in the class $\mathcal{TS}(2^{-r}, 2^{-1})$, then*

$$a_n \le \frac{(1-p)(1-2^{-r})(c)_{n-1}}{\left[n\left(\frac{3}{2}\right) - \left(2^{-r} + \frac{1}{2}\right)\right](a)_{n-1}}, n \ge 3, r \in \{1,2,3,\dots\}.$$

Theorem 2.7 *The new subclass $\mathcal{TS}_p(2^{-r}, 2^{-1})$ is closed under convex linear combination.*

Proof We let the function $f(z)$ of the form (2.3) and $g(z)$ defined by

$$g(z) = z - \frac{p(1-2^{-r})(c)}{\left(\frac{5}{2} - 2^{-r}\right)(a)} z^2 - \sum_{n=2}^{\infty} b_n z^n$$

where $b_n \geq o$ and $p \in [0,1]$. Let us assume $f(z)$ and $g(z)$ are in the class $\mathcal{TS}_p(2^{-r},2^{-1})$. In the case, it is sufficient to prove for the function $f(z)$ given by

$$h(z) = \delta f(z) + (1-\delta)g(z), \delta \in [0,1]$$

is also in the class $\mathcal{TS}_p(2^{-r},2^{-1})$. On the other hand,

$$h(z) = z - \frac{p(1-2^{-r})(c)}{\left(\frac{5}{2}-2^{-r}\right)(a)}z^2 - \sum_{n=2}^{\infty}\Big\{\delta a_n + (1-\delta)b_n\Big\}z^n, a_n \geq 0, b_n \geq 0, p \in [0,1].$$

We now apply Theorem (2.5), to get $h(z) \in \mathcal{TS}_p(2^{-r},2^{-1})$ which completes the proof. □

Theorem 2.8 *Let function $f_i(z)$ be in the class $\mathcal{TS}_p(2^{-r},2^{-1})$ and defined by*

$$h(z) = z - \frac{p(1-2^{-r})(c)}{\left(\frac{5}{2}-2^{-r}\right)(a)}z^2 - \sum_{n=3}^{\infty} a_{n,i}z^n, a_{n,i} \geq 0, i \in \{1,2,3,\dots\}. \tag{2.5}$$

Then the function $H(z)$ defined by

$$H(z) = \sum_{i=1}^{\infty} \eta_i f_i(z) \tag{2.6}$$

is also in the class $\mathcal{TS}_p(2^{-r},2^{-1})$, where

$$\sum_{i=1}^{\infty} \eta_i = 1. \tag{2.7}$$

Proof Now, if we combine (2.5) and (2.6) equations, also by (2.7), we see that

$$H(z) = z - \frac{p(1-2^{-r})(c)}{\left(\frac{5}{2}-2^{-r}\right)(a)}z^2 - \sum_{n=3}^{\infty}\Big\{\sum_{i=1}^{k}\delta_i a_{n,i}\Big\}z^n.$$

On the other hand, since $f_i(z) \in \mathcal{TS}_p(2^{-r},2^{-1})$ for every $i \in \{1,2,3,\dots\}$ and in view of Theorem (2.5) implies that

$$\sum_{n=3}^{\infty}\left[n\left(\frac{3}{2}\right) - \left(2^{-r}+\frac{1}{2}\right)\right]\frac{(a)_{n-1}}{(c)_{n-1}}a_{n,i} \leq (1-p)(1-2^{-r}). \tag{2.8}$$

Thus,

$$\sum_{n=3}^{\infty}\left[n\left(\frac{3}{2}\right) - \left(2^{-r}+\frac{1}{2}\right)\right]\frac{(a)_{n-1}}{(c)_{n-1}}\left(\sum_{i=1}^{k}\delta_i a_{n,i}\right)$$

$$= \sum_{i=1}^{k}\delta_i\left(\sum_{n=3}^{\infty}\left[n\left(\frac{3}{2}\right) - \left(2^{-r}+\frac{1}{2}\right)\right]\frac{(a)_{n-1}}{(c)_{n-1}}a_{n,i}\right)$$

$$\leq (1-b)(1-2^{-r}).$$

This completes the proof of our Theorem. □

References

[1] Alexander JW. *Functions which map the interior of the unit circle upon simple regions.* Annals of Mathematics 1915; 17 (2):12-22.

[2] Andrews GE, Askey R, Roy R. *Special functions* . Cambridge, UK: Cambridge Univ. Press, 1999.

[3] Branges LD. *A proof of the Biberbach conjecture.* Acta Mathematica 1985; 154:137-152.

[4] Carlson BC, Shaffer DB. *Starlike and prestarlike hypergeometric functions* . SIAM Journal on Mathematical Analysis 1984; 15: 737-745.

[5] Duren PL. *Coefficient of meromorphic schlicht functions.* Proceedings of the American Mathematical Society 1971; 28: 169-172.

[6] Duren PL. *Univalent Functions.* Grundlehren der Mathematischen Wissenschaften, Band 259, New York, NY, USA: Springer-Verlag, 1983.

[7] Goodman AW. *On uniformly starlike functions.* Journal of Mathematical Analysis and Applications 1991; 155 (2): 364-370.

[8] Goodman AW. *On uniformly convex functions* . Annales Polonici Mathematici 1991; 56 (1): 87-92.

[9] Koebe P. *Ueber die Uniformisierung beliebiger analytischer Kurven.* Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse 1907; 191-210 (in German).

[10] Ma W, Minda D. *Uniformly convex functions.* In Annales Polonici Mathematici 1992; 57 (2): 165-175.

[11] Mateljevic M. *Quasiconformal and Quasiregular harmonic analogues of Koebe's Theorem and Applications.*Annales Academiae Scientiarum Fennicae Mathematica 2007; 32: 301–315.

[12] Murugusundaramoorthy G, Magesh N. *A new subclass of uniformly convex functions and a corresponding subclass of starlike functions with fixed second coefficient*, Journal of Inequalities in Pure and Applied Mathematics 2004; 5 (4):85.

[13] Pochhammer LA. *Zur theorie der Euler'schen integrale*, Mathematische Annalen 1890; 35 (4): 495-526 (in German).

[14] Ponnusamy S. *Hypergeometric transforms of functions with derivative in a half plane* . Journal of Computational and Applied Mathematics 1998; 96: 35-49.

[15] Riemann B. *Grundlagen für eine allgemeine Theorie der Functionen einer veranderlichen complexen Grösse*,(Göttingen, 1851), [141] pp. 3-48 (in German).

[16] Robertson MS. *Univalent functions starlike with respect to a boundary poin.* Journal of Mathematical Analysis and Applications 1981; 81: 327–345.

[17] Robertson MS. *Certain classes of starlike functions.* Michigan Mathematical Journal 1954; 76 (1): 755-758.

[18] Rønning F. *Uniformly convex functions and a corresponding class of starlike functions* . Proceedings of the American Mathematical Society 1993; 118 (1): 189-196. doi: 10.2307/2160026

[19] Ruscheweyh S. *New criteria for Univalent Functions.* Proceedings of the American Mathematical Society 1975; 49: 109-115.

[20] Sheil-Small T. *Hadamard product of schlicht functions and the Polya-Schoenberg conjecture.* Commentarii Mathematici Helvetici 1973; 48 (1): 119-135.

[21] Silverman H, Silvia EM. *Fixed coefficients for subclasses of starlike functions.* Houston Journal of Mathematics 1981; 7 (1): 129-136.

[22] Silverman H. *Univalent functions with negative coefficients.* Proceedings of the American Mathematical Society 1975; 51 (1): 109-116.

[23] Slater LJ. *Generalized Hypergeometric Functions.* Cambridge, UK: Cambridge University Press, 1966.

[24] Study E. *Konforme Abbildung Einfachzusammenhangender Bereiche*. B. C. Teubner, Leipzig und Berlin 1913 (in German).

[25] Temme NM. *Special functions: An Introduction to the Classical Functions of Mathematical Physics*. New York, NY, USA: Wiley, 1996.

[26] Vidunas R. *Transformations of Gauss hypergeometric functions* . Journal of Computational and Applied Mathematics 2005; 178 :473-487, arxiv: math.CA/0310436

[27] Weierstrass K. *Zur Theorie der eindeutigen analytischen Functionen*. Annales Scientifiques de l'Ecole Normale Superieure 1876; 8: 115-150 (in German).