Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2021) 45: 2455 – 2465
© TÜBİTAK
doi:10.3906/mat-2002-107

A semisymmetric metric connection on almost contact $B-$metric manifolds

Şenay BULUT*
Department of Mathematics, Faculty of Science, Eskişehir Technical University, Eskişehir, Turkey

Received: 25.02.2020 • **Accepted/Published Online:** 11.09.2021 • **Final Version:** 29.11.2021

Abstract: The object of the present paper is to study a semisymmetric metric connection on an almost contact $B-$metric manifold. We deduce a relation between the Levi–Civita connection and the semisymmetric metric connection on the considered manifold. We determined the class of the torsion tensor corresponding to the semisymmetric connection. We study Ricci-like solitons on almost contact $B-$metric manifolds with the semisymmetric connection. Finally, we give some examples to considered manifolds with the semisymmetric connection.

Key words: Semisymmetric metric connection, Almost contact $B-$metric manifold, Ricci-like soliton

1. Introduction

In this paper we undertake a study of semisymmetric metric connection on an almost contact $B-$metric manifold. Firstly, the concept of semisymmetric connection in a Riemannian manifold was introduced by Yano in [17]. Later, in [14] a semisymmetric metric connection in an almost contact manifold was defined and the properties of curvature tensors was studied in [11]. Some properties of semisymmetric connections have been examined on manifolds equipped with special structures [2, 4, 13, 17]. Kenmotsu manifolds with this connection were investigated in [1, 12]. In [16] the existence of a new connection on a Riemannian manifold is proved. In particular case, this connection corresponds to a semisymmetric metric connection.

The decomposition of the space of torsion tensors on almost contact $B-$metric manifolds is studied in [6]. The class of these manifolds equipped with the semisymmetric connection is described. If the manifold is Sasaki-like, then we attain some identities and theorems.

The concept of Ricci solitons was firstly described in Riemannian geometry. But, recently, Ricci solitons and $\eta-$Ricci solitons have been studied ([7, 12, 15]). Motivated by the study in [7] we investigate Ricci-like solitons on almost contact $B-$metric manifolds endowed with the semisymmetric metric connection.

The organization of the present paper is as follows: In Section 2 we give some necessary facts about Sasaki-like and almost contact $B-$metric manifolds. In Section 3 we determine the class of torsion tensor with respect to the connection and achieve a relation among curvature quantities of the semisymmetric metric connection and Levi–Civita connection. We prove that Sasaki-like manifold admitting a Ricci-like soliton also admits a Ricci-like soliton with respect to the considered connection. Finally, we find some geometric characteristics of examples of 3-dimensional $B-$metric manifolds equipped with the considered connection.

*Correspondence: math@tubitak.gov.tr
2010 *AMS Mathematics Subject Classification:* 53C05; 53C15; 53D10.

2. Almost contact $B-$metric manifolds

Let M be a $(2n+1)-$dimensional $C^\infty-$manifold and suppose that there exists in M a vector-valued linear function φ, a vector field ξ and a $1-$form η such that

$$\eta(\xi) = 1, \qquad \varphi^2 x = -x + \eta(x)\xi, \tag{2.1}$$

for any vector field x. Then (M, φ, ξ, η) is called an almost contact manifold.

By virtue of the relation (2.1), the following relations hold

$$\varphi\xi = 0, \qquad \eta \circ \varphi = 0. \tag{2.2}$$

In addition, if there exists a pseudo-Riemannian metric g of signature $(n+1, n)$ satisfying

$$g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y), \tag{2.3}$$

then $(M, \varphi, \xi, \eta, g)$ is said to be an almost contact $B-$metric manifold.

By using the contact $1-$form η on M, we determine the $2n-$dimensional contact distribution $\mathcal{H} = Ker\ \eta$, known as horizontal distribution. The sections of $\mathcal{H}$ are called the horizontal vector fields. Throughout this article we shall use X, Y, Z to denote elements of the smooth horizontal vector fields on M. x, y, z stand for arbitrary smooth vector fields on M.

The following equations are some consequences of (2.1), (2.2), (2.3)

$$g(x, \varphi y) = g(\varphi x, y), \qquad g(\xi, \xi) = 1, \qquad \eta(\nabla^g_x \xi) = 0 \tag{2.4}$$

where ∇^g is the Levi–Civita connection of g. The associated metric $\widetilde{g}$ of g is given by

$$\widetilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y).$$

A classification of almost contact $B-$metric manifolds is made in [6] by using the tensor field F of type $(0,3)$ determined by

$$F(x, y, z) = g((\nabla^g_x \varphi)y, z).$$

Moreover, F satisfies the following identities:

$$\begin{aligned} &F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi) \\ &F(x, \varphi y, \xi) = (\nabla^g_x \eta)y = g(\nabla^g_x \xi, y) \end{aligned} \tag{2.5}$$

See for more details [3, 8, 9].

2.1. Sasaki-like almost contact $B-$metric manifolds

Sasaki-like almost contact $B-$metric manifolds are determined by the conditions

$$\begin{aligned} &F(X, Y, Z) = F(\xi, Y, Z) = F(\xi, \xi, Z) = 0, \\ &F(X, Y, \xi) = -g(X, Y). \end{aligned} \tag{2.6}$$

The covariant derivative $\nabla^g \varphi$ satisfies the equality

$$(\nabla^g_x \varphi)y = -g(x, y)\xi - \eta(y)x + 2\eta(x)\eta(y)\xi = g(\varphi x, \varphi y)\xi + \eta(y)\varphi^2 x. \tag{2.7}$$

Hence, the following relations given in [5] hold:

$$\begin{array}{ll} \nabla^g_x \xi = -\varphi x, & R(x,y)\xi = \eta(y)x - \eta(x)y, \\ \nabla^g_\xi X = -\varphi X - [X,\xi], & R(\xi,y)\xi = \varphi^2 y, \\ (\nabla^g_x \eta)y = -g(x,\varphi y), & Ric(x,\xi) = 2n, \\ \nabla^g_\xi \xi = 0, & Ric(\xi,\xi) = 2n, \\ div\xi = 0. & \end{array} \tag{2.8}$$

where R and Ric stand for the curvature and the Ricci tensor, respectively.

3. A semisymmetric metric connection on almost contact $B-$metric manifolds

In this section we deal with a semisymmetric metric connection on an almost contact $B-$metric manifold. Let (M,φ,ξ,η,g) be an almost contact $B-$metric manifold with the Levi–Civita connection ∇^g and we define a semisymmetric connection $\widetilde{\nabla}$ on M by

$$\widetilde{\nabla}_x y = \nabla^g_x y + g(x,y)\xi - \eta(y)x. \tag{3.1}$$

By using above equation the torsion tensor T of M with respect to the connection $\widetilde{\nabla}$ is given by

$$T(x,y) = \widetilde{\nabla}_x y - \widetilde{\nabla}_y x - [x,y] = \eta(x)y - \eta(y)x. \tag{3.2}$$

Further, the semisymmetric connection $\widetilde{\nabla}$ satisfying the condition

$$(\widetilde{\nabla}_x g)(y,z) = 0 \tag{3.3}$$

for all $x,y,z \in \chi(M)$ is said to be a semisymmetric metric connection, otherwise it is called a semisymmetric nonmetric connection.

Now we shall show the existence of the semisymmetric metric connection $\widetilde{\nabla}$ on an almost contact $B-$metric manifold (M,φ,ξ,η,g).

Theorem 3.1 *There exists a unique linear connection $\widetilde{\nabla}$ satisfying (3.2) and (3.3) on an almost contact $B-$metric manifold (M,φ,ξ,η,g).*

Proof Suppose that $\widetilde{\nabla}$ is a linear connection on M given by

$$\widetilde{\nabla}_x y = \nabla^g_x y + H(x,y), \tag{3.4}$$

where H denotes a tensor field of type $(1,2)$. Now, we shall specify the tensor field H such that $\widetilde{\nabla}$ satisfies the conditions (3.2) and (3.3). By the definition of torsion tensor, the equation (3.4) leads to

$$T(x,y) = H(x,y) - H(y,x) \tag{3.5}$$

for all $x,y \in \chi(M)$. We have

$$g(T(x,y),z) = g(H(x,y),z) - g(H(y,x),z) \tag{3.6}$$

The equation (3.3) leads to

$$(\widetilde{\nabla}_x g)(y,z) = 0 = xg(y,z) - g(\widetilde{\nabla}_x y, z) - g(\widetilde{\nabla}_x z, y) = g(H(x,y),z) + g(H(x,z),y). \tag{3.7}$$

Then, we get

$$g(H(x,y),z) = -g(H(x,z),y). \tag{3.8}$$

From (3.6) and (3.8), we have

$$\begin{aligned} &g(T(x,y),z) + g(T(z,x),y) + g(T(z,y),x) \\ &= g(H(x,y),z) - g(H(y,x),z) + g(H(z,x),y) \\ &\quad -g(H(x,z),y) + g(H(z,y),x) - g(H(y,z),x) \\ &= 2g(H(x,y),z). \end{aligned} \tag{3.9}$$

By using Equations (3.2) and (3.9) we obtain

$$\begin{aligned} 2g(H(x,y),z) &= g(T(x,y),z) + g(T(z,x),y) + g(T(z,y),x) \\ &= g(\eta(x)y - \eta(y)x, z) + g(\eta(z)x - \eta(x)z, y) + g(\eta(z)y - \eta(y)z, x) \\ &= g(\eta(x)y - \eta(y)x, z) + 2\eta(z)g(x,y) - \eta(x)g(z,y) - \eta(y)g(z,x) \\ &= g(\eta(x)y - \eta(y)x, z) + 2g(z,\xi)g(x,y) - \eta(x)g(z,y) - \eta(y)g(z,x) \\ &= 2g(g(x,y)\xi - \eta(y)x, z) \end{aligned} \tag{3.10}$$

.

In consequence of (3.10), we get

$$H(x,y) = g(x,y)\xi - \eta(y)x. \tag{3.11}$$

In view of (3.10), we can get Equation (3.1). □

Let us consider the torsion tensor T of a semisymmetric metric connection $\widetilde{\nabla}$ on M. Then, the corresponding tensor of type $(0,3)$ is determined by

$$T(x,y,z) = g(T(x,y),z) = \eta(x)g(y,z) - \eta(y)g(x,z). \tag{3.12}$$

Especially, we have

$$T(\varphi x, \varphi y, z) = 0, \quad \text{and} \quad T(x,y,\xi) = 0. \tag{3.13}$$

Proposition 3.2 *The torsion tensor T of a semisymmetric metric connection on M belongs to the class $\mathcal{T}_9$.*

Proof Consider the vector space of all tensors of type $(0,3)$ over $T_p(M)$ in the following way:

$$\mathcal{T} = \{T(x,y,z) \in \mathbb{R} \mid T(x,y,z) = -T(y,x,z), x,y,z \in T_pM\}. \tag{3.14}$$

By using the operator $p_1 : \mathcal{T} \to \mathcal{T}$ by

$$p_1(T)(x,y,z) = -T(\varphi^2 x, \varphi^2 y, \varphi^2 z) = -g(T(\varphi^2 x, \varphi^2 y), \varphi^2 z),$$

the orthogonal decomposition of $\mathcal{T}$ by the image and the kernel of p_1 is obtained as follows:

$$W_1 = im(p_1) = \{T \in \mathcal{T} \mid p_1(T) = T\} \text{ and } W_1^{\perp} = ker(p_1) = \{T \in \mathcal{T} \mid p_1(T) = 0\}.$$

Since $p_1(T) = 0$ by (3.13), we have $T \in ker(p_1) = W_1^{\perp}$. Let us define the operator $p_2 : W_1^{\perp} \to W_1^{\perp}$ by

$$p_2(T)(x, y, z) = \eta(z)T(\varphi^2 x, \varphi^2 y, \xi) = \eta(z)g(T(\varphi^2 x, \varphi^2 y), \xi).$$

Since the operator has the property $p_2 \circ p_2 = p_2$, the decomposition of $W_1^{\perp}$ is given by

$$W_2 = im(p_2) = \{T \in W_1^{\perp} \mid p_2(T) = T\} \text{ and } W_2^{\perp} = ker(p_2) = \{T \in W_1^{\perp} \mid p_2(T) = 0\}.$$

Again from (3.13) we obtain $p_2(T) = 0$, namely, $T \in ker(p_2) = W_2^{\perp}$. Let us define the operator $p_3 : W_2^{\perp} \to W_2^{\perp}$ by

$$p_3(T)(x, y, z) = \eta(x)T(\xi, \varphi^2 y, \varphi^2 z) + \eta(y)T(\varphi^2 x, \xi, \varphi^2 z).$$

By making use of (2.3), (3.2) and (3.12), the above equality turns into

$$\begin{aligned} p_3(T)(x, y, z) &= \eta(x)T(\xi, \varphi^2 y, \varphi^2 z) + \eta(y)T(\varphi^2 x, \xi, \varphi^2 z) \\ &= \eta(x)g(T(\xi, \varphi^2 y), \varphi^2 z) + \eta(y)g(T(\varphi^2 x, \xi), \varphi^2 z) \\ &= \eta(x)g(\varphi^2 y, \varphi^2 z) - \eta(y)g(\varphi^2 x, \varphi^2 z) \\ &= -\eta(x)g(\varphi y, \varphi z) + \eta(y)g(\varphi x, \varphi z) \\ &= -\eta(x)(-g(y, z) + \eta(y)\eta(z)) + \eta(y)(-g(x, z) + \eta(x)\eta(z)) \\ &= \eta(x)g(y, z) - \eta(y)g(x, z) \\ &= g(T(x, y), z) = T(x, y, z). \end{aligned} \tag{3.15}$$

Hence, $p_3(T) = T$, i.e. $T \in im(p_3) = W_3$. The operators

$$\begin{aligned} L_{3,0}(T)(x, y, z) &= \eta(x)T(\xi, \varphi y, \varphi z) - \eta(y)T(\xi, \varphi x, \varphi z) \\ L_{3,1}(T)(x, y, z) &= \eta(x)T(\xi, \varphi^2 z, \varphi^2 y) - \eta(y)T(\xi, \varphi^2 z, \varphi^2 x) \end{aligned} \tag{3.16}$$

are involutive isometries on W_3. By virtue of (3.12) we obtain $L_{3,0}(T) = -T$, that is, $T \in W_3^-$ and $L_{3,1}(T) = T$, that is, $T \in W_{3,1}$ where

$$W_3^- = \{T \in W_3 \mid L_{3,0}(T) = -T\} \text{ and } W_{3,1} = \{T \in W_3^- \mid L_{3,1}(T) = T\}.$$

$t(x) = g^{ij}T(x, e_i, e_j)$ and $t^*(x) = g^{ij}T(x, e_i, \varphi e_j)$ are torsion forms of T with respect to the basis $\{\xi, e_1, \ldots, e_{2n}\}$. By the direct calculation we get $t \neq 0$, $t^* = 0$. Therefore, $T \in W_{3,1,1} = \mathcal{T}_9$, where

$$W_{3,1,1} = \{T \in W_{3,1} \mid t \neq 0, t^* = 0\}.$$

□

Lemma 3.3 *The relations between the curvature quantities of the semisymmetric metric connection* $\widetilde{\nabla}$ *and Levi–Civita connection* ∇^g *are given by the following formulas:*

1. *Curvature transformation:*

$$\begin{aligned} \widetilde{R}(x, y)z = {} & R(x, y)z + g(y, z)\nabla^g_x\xi - g(x, z)\nabla^g_y\xi + [g(\nabla^g_y\xi, z) + g(\varphi y, \varphi z)]x \\ & -[g(\nabla^g_x\xi, z) + g(\varphi x, \varphi z)]y + g(T(x, y), z)\xi. \end{aligned} \tag{3.17}$$

2. *Ricci curvature:*

$$\widetilde{Ric}(x, y) = Ric(x, y) + [div(\xi) + (1 - 2n)]g(x, y) + (2n - 1)[\eta(x)\eta(y) + g(\nabla^g_x\xi, y)]. \tag{3.18}$$

3. *Scalar curvature:*

$$\widetilde{s} = s^g + 4ndiv(\xi) + 2n(1-2n). \tag{3.19}$$

Proof

1. The curvature tensor $\widetilde{R}$ of type $(1,3)$ is defined by

$$\widetilde{R}(x,y)z = \widetilde{\nabla}_x\widetilde{\nabla}_y z - \widetilde{\nabla}_y\widetilde{\nabla}_x z - \widetilde{\nabla}_{[x,y]}z. \tag{3.20}$$

where $x,y,z \in \chi(M)$. Using (2.3), (3.1), (3.2), (3.3) in (3.20) we obtain the relation (3.17) by a routine computation.

2. The Ricci curvature tensor $\widetilde{Ric}$ with respect to $\widetilde{\nabla}$ is defined by

$$\widetilde{Ric}(x,y) = \sum_{i=0}^{2n} \varepsilon_i g(\widetilde{R}(e_i,x)y,e_i). \tag{3.21}$$

with regard to the basis $\{\xi, e_1, \ldots, e_{2n}\}$. The divergence of the vector field ξ is given by

$$div\xi = \sum_{i=0}^{2n} \varepsilon_i g(\nabla^g_{e_i}\xi, e_i). \tag{3.22}$$

By making use of (2.3), (3.1), (3.2), (3.3), (3.17) in (3.21) we get (3.18).

3. The scalar curvature tensor $\widetilde{s}$ with respect to $\widetilde{\nabla}$ is defined by

$$\widetilde{s} = \sum_{i=0}^{2n} \widetilde{Ric}(e_i,e_i).$$

By the way of the relations (2.3), (3.1), (3.2), (3.3), (3.18), (3.22) it can be easily computed the identity (3.19).

□

Note that we have the following relations with respect to the basis $\{\xi, e_1, \ldots, e_{2n}\}$ for (M,φ,ξ,η,g) as a consequence of the Lemma (3.3):

$$(\widetilde{\nabla}_x\eta)y = (\nabla^g_x\eta)y + g(\varphi x, \varphi y) \tag{3.23}$$

$$(\widetilde{\nabla}_x\varphi)y = (\nabla^g_x\varphi)y + g(x,\varphi y)\xi + \eta(y)\varphi x \tag{3.24}$$

$$\widetilde{\nabla}_\xi y = \nabla^g_\xi y \tag{3.25}$$

$$\widetilde{\nabla}_x\xi = \nabla^g_x\xi + \varphi^2 x \tag{3.26}$$

$$\widetilde{R}(x,y)\xi = R(x,y)\xi + \eta(y)\nabla^g_x\xi - \eta(x)\nabla^g_y\xi + \eta(\nabla^g_y\xi)x \tag{3.27}$$

$$\widetilde{Ric}(\xi,\xi) = Ric(\xi,\xi) + div(\xi) + \eta(\nabla^g_\xi\xi) \tag{3.28}$$

Theorem 3.4 *In a Sasaki-like almost contact $B-$ metric manifold with a semisymmetric metric connection we have the following identities:*

1. $(\widetilde{\nabla}_x\eta)y = g(\varphi x - x, \varphi y)$

2. $(\widetilde{\nabla}_x\varphi)y = g(x + \varphi x, \varphi y)\xi + \eta(y)\varphi(x + \varphi x)$

3. $\widetilde{\nabla}_x\xi = -\varphi x + \varphi^2 x$

4. $\widetilde{\nabla}_\xi x = \nabla^g_\xi x = -\varphi x + [\xi, x]$

5. $\widetilde{Ric}(\xi, \xi) = 2n$

6. $\widetilde{s} = s + 2n(1 - 2n)$

7. $\widetilde{Ric}(x, y) = Ric(x, y) + (2n - 1)(-g(\varphi x, y) + \eta(x)\eta(y)) + (1 - 2n)g(x, y)$

Proof

1. Using the relations (2.8) and (3.23) it follows that

$$\begin{aligned}(\widetilde{\nabla}_x\eta)y &= (\nabla^g_x\eta)y + g(\varphi x, \varphi y)\\ &= -g(x, \varphi y) + g(\varphi x, \varphi y)\\ &= g(\varphi x - x, \varphi y)\end{aligned}$$

2. The equations (2.7) and (3.24) imply that

$$\begin{aligned}(\widetilde{\nabla}_x\varphi)y &= (\nabla^g_x\varphi)y + g(x, \varphi y)\xi + \eta(y)\varphi x\\ &= g(\varphi x, \varphi y)\xi + \eta(y)\varphi^2 x + g(x, \varphi y)\xi + \eta(y)\varphi x\\ &= g(\varphi x + x, \varphi y)\xi + \eta(y)\varphi(\varphi x + x)\end{aligned}$$

3. It comes directly from Equation (2.8).

4. It can be easily verified by using the identities (2.8) and (3.1).

5. For any Sasaki-like almost contact $B-$metric manifold we know that div $\xi = 0$ and $(\nabla^g_\xi\xi) = 0$. Hence, from (3.28) and (2.8) the given identity is verified.

6. Since div $\xi = 0$, Equation (3.19) yields the desired identity.

7. It is obtained by using (3.18) and (2.8).

□

The manifold $(M, \varphi, \xi, \eta, g)$ is said to be Einstein-like if its Ricci tensor Ric satisfies the following condition:

$$Ric = \lambda g + \mu\widetilde{g} + \nu\eta\otimes\eta \tag{3.29}$$

where λ, μ, ν are constants.

Theorem 3.5 *If (M,φ,ξ,η,g) is a Einstein-like Sasaki-like almost contact $B-$metric manifold, then it is a Einstein-like almost contact $B-$metric manifold with respect to the semisymmetric metric connection $\widetilde{\widetilde{\nabla}}$.*

Proof Since (M,φ,ξ,η,g) is a Einstein-like, its Ricci tensor satisfies (3.29). Hence, there exist the constants (λ,μ,ν) such that

$$Ric(x,y)=\lambda g(x,y)+\mu\widetilde{g}(x,y)+\nu\eta(x)\eta(y). \tag{3.30}$$

Using the Theorem (3.4)(7) and (3.30) we derive that

$$\begin{aligned}\widetilde{Ric}(x,y) &= Ric(x,y)+(2n-1)(-g(\varphi x,y)+\eta(x)\eta(y))+(1-2n)g(x,y)\\ &= \lambda g(x,y)+\mu\widetilde{g}(x,y)+\nu\eta(x)\eta(y)+(2n-1)(-g(\varphi x,y)+\eta(x)\eta(y))+(1-2n)g(x,y)\\ &= (\lambda+1-2n)g(x,y)+(\mu-2n+1)\widetilde{g}(x,y)+(\nu+4n-2)\eta(x)\eta(y).\end{aligned}$$

Therefore, M is a Einstein-like with respect to $\widetilde{\widetilde{\nabla}}$ with the constants $(\lambda+1-2n,\mu-2n+1,\nu+4n-2)$. □

The manifold M is said to be an almost contact $B-$metric manifold with a torse-forming Reeb vector field ξ provided that $\nabla^g_x\xi=fx+\alpha(x)\xi$, where f is a smooth function and α is a one-form on M. If ξ is a torse-forming vector field on M, then the $1-$from α must be $-f\eta$ because of $\eta(\nabla^g_x\xi)=0$. Hence, we derive the following equivalent identities:

$$\nabla^g_x\xi=-f\varphi^2x \text{ and } (\nabla^g_x)y=-fg(\varphi x,\varphi y). \tag{3.31}$$

Taking into account the relations (3.31) we state the following:

Proposition 3.6 *If the Reeb vector field ξ of (M,φ,ξ,η,g) is a torse-forming with respect to the Levi–Civita connection ∇^g, then it is a torse-forming with respect to the semisymmetric connection $\widetilde{\widetilde{\nabla}}$.*

(M,φ,ξ,η,g) is a Ricci-like soliton if its Ricci tensor Ric satisfies the following condition with the constants (λ,μ,ν):

$$\frac{1}{2}\mathcal{L}_\xi g+Ric+\lambda g+\mu\widetilde{g}+\nu\eta\otimes\eta=0, \tag{3.32}$$

where $\mathcal{L}_\xi g$ is a Lie derivative defined by

$$(\mathcal{L}_\xi g)(x,y)=g(\nabla^g_x\xi,y)+g(x,\nabla^g_y\xi).$$

A generalization of the Ricci soliton and the $\eta-$Ricci soliton was introduced in [7].

Theorem 3.7 *If a Sasaki-like almost contact $B-$metric manifold M admits a Ricci-like soliton with constants (λ,μ,ν), then M admits a Ricci-like soliton with the constants $(\lambda+2n,\mu+2n-1,\nu-4n+1)$ with respect to the semisymmetric connection $\widetilde{\widetilde{\nabla}}$.*

Proof The manifold M satisfies Equation (3.32). Taking into account Theorem (3.4) and the equalities of $\nabla^g_x\xi$ and $\widetilde{\widetilde{\nabla}}_x\xi$ by (2.8), (3.26) the main assertion

$$\frac{1}{2}\widetilde{\mathcal{L}}_\xi g+\widetilde{Ric}+(\lambda+2n)g+(\mu+2n-1)\widetilde{g}+(\nu-4n+1)\eta\otimes\eta=0 \tag{3.33}$$

is valid. □

4. Examples

4.1. Example

Let us consider a $3-$dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 | z \neq 0\}$ where (x, y, z) are standard coordinates in $\mathbb{R}^3$. A linearly independent global frame field on M is $\{\xi = e_0, e_1, e_2\}$ defined by

$$e_0 = \frac{\partial}{\partial z}, \qquad e_1 = e^z \frac{\partial}{\partial y}, \qquad e_2 = e^z(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}). \tag{4.1}$$

Let g be a semi-Riemannian metric defined by $g(e_0, e_0) = g(e_1, e_1) = -g(e_2, e_2) = 1$. Define the $1-$form η by $\eta(x) = g(x, e_0)$ for any vector field $x \in \chi(M)$. Let φ be the $(1,1)-$tensor field defined by $\varphi(e_0) = 0$, $\varphi(e_1) = e_2$, $\varphi(e_2) = -e_1$. The linearity property of φ and g yields

$$\begin{aligned} &\eta(e_0) = 1, \\ &\varphi^2 x = -x + \eta(x)e_0, \\ &g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y), \end{aligned} \tag{4.2}$$

for any vector fields x, y. Hence, (φ, ξ, η, g) is an almost contact $B-$metric structure on M. But, the considered manifold is not Sasaki-like.

Now we have

$$[e_1, e_2] = 0, \qquad [e_0, e_1] = e_1, \qquad [e_0, e_2] = e_2. \tag{4.3}$$

By using Koszul's formula we obtain the following nonzero components of the Levi–Civita connection ∇^g.

$$\nabla^g_{e_1} e_1 = e_0, \qquad \nabla^g_{e_1} e_0 = -e_1, \qquad \nabla^g_{e_2} e_2 = -e_0, \qquad \nabla^g_{e_2} e_0 = -e_2. \tag{4.4}$$

The nonzero components of the semisymmetric connection $\widetilde{\nabla}$ are calculated by

$$\widetilde{\nabla}_{e_1} e_1 = 2e_0, \qquad \widetilde{\nabla}_{e_1} e_0 = -2e_1, \qquad \widetilde{\nabla}_{e_2} e_2 = -2e_0, \qquad \widetilde{\nabla}_{e_2} e_0 = -2e_2. \tag{4.5}$$

With the help of above results the nonzero components of curvature tensors R and $\widetilde{R}$ can be easily obtained by

$$\begin{aligned} &R(e_1, e_2)e_1 = e_2, \quad R(e_2, e_0)e_2 = -e_3, \quad R(e_1, e_0)e_1 = e_0, \\ &R(e_1, e_2)e_2 = e_1, \quad R(e_2, e_0)e_0 = -e_2, \quad R(e_1, e_0)e_0 = -e_1, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &\widetilde{R}(e_1, e_2)e_1 = 4e_2, \quad \widetilde{R}(e_2, e_0)e_2 = -2e_3, \quad \widetilde{R}(e_1, e_0)e_1 = 2e_0, \\ &\widetilde{R}(e_1, e_2)e_2 = 4e_1, \quad \widetilde{R}(e_2, e_0)e_0 = -2e_2, \quad \widetilde{R}(e_1, e_0)e_0 = -2e_1. \end{aligned} \tag{4.7}$$

All components of Ricci curvatures Ric and $\widetilde{Ric}$ with respect to ∇ and $\widetilde{\nabla}$, respectively are zero. Therefore, the scalar curvatures $Scal$ and $\widetilde{Scal}$ are zero.

4.2. Example

Let us consider the real connected $3-$dimensional Lie group L with a global basis $\{\xi = e_0, e_1, e_2\}$ of the left invariant vector fields on L such that the commutators of its associated Lie algebra are defined as follows:

$$[\xi, e_1] = e_2, \qquad [\xi, e_2] = -e_1. \tag{4.8}$$

Hence, L is endowed with an almost contact $B-$metric structure by

$$\begin{aligned} &g(e_0,e_0)=g(e_1,e_1)=-g(e_2,e_2)=1, \quad g(e_i,e_j)=0 \text{ for } i\neq j,\\ &\varphi(e_1)=e_2, \qquad \varphi(e_2)=-e_1. \end{aligned} \tag{4.9}$$

This example is given in [10]. The nonzero components of the Levi–Civita connection ∇ and the semisymmetric connection $\widetilde{\nabla}$ are respectively calculated by

$$\nabla_{e_1}e_0=-e_2, \qquad \nabla_{e_2}e_0=e_1, \qquad \nabla_{e_1}e_2=\nabla_{e_2}e_1=-e_0, \tag{4.10}$$

$$\begin{aligned} &\widetilde{\nabla}_{e_1}e_0=-e_1-e_2, \quad \widetilde{\nabla}_{e_2}e_0=e_1-e_2, \quad \widetilde{\nabla}_{e_1}e_2=\widetilde{\nabla}_{e_2}e_1=-e_0,\\ &\widetilde{\nabla}_{e_1}e_1=e_0, \qquad\qquad \widetilde{\nabla}_{e_2}e_2=-e_0. \end{aligned} \tag{4.11}$$

It can be easily proved that the constructed manifold L is a Sasaki-like. Taking into account (4.8), (4.9), (4.10) and (4.11), we compute the nonzero components of curvature tensors R and $\widetilde{R}$ with regard to ∇ and $\widetilde{\nabla}$ as follows:

$$\begin{aligned} &R_{010}=-e_1, \quad R_{020}=-e_2, \quad R_{011}=e_0, \quad R_{121}=e_2,\\ &R_{022}=-e_0, \quad R_{122}=e_1, \end{aligned} \tag{4.12}$$

$$\begin{aligned} &\widetilde{R}_{010}=-e_1+e_2, \quad \widetilde{R}_{020}=-e_1-e_2, \quad \widetilde{R}_{011}=e_0, \quad \widetilde{R}_{121}=2e_2,\\ &\widetilde{R}_{021}=e_0, \qquad\qquad \widetilde{R}_{012}=e_0, \qquad\qquad \widetilde{R}_{022}=-e_0, \quad \widetilde{R}_{122}=2e_1. \end{aligned} \tag{4.13}$$

The nonzero components of Ricci tensors Ric and $\widetilde{Ric}$ are determined by the following equalities:

$$\begin{aligned} &Ric_{00}=2,\\ &\widetilde{Ric}_{00}=2, \qquad\qquad \widetilde{Ric}_{11}=-1, \quad \widetilde{Ric}_{22}=1,\\ &\widetilde{Ric}_{12}=1=\widetilde{Ric}_{21} \end{aligned} \tag{4.14}$$

Scalar curvatures $Scal$ and $\widetilde{Scal}$ are given by $Scal=2$ and $\widetilde{Scal}=0$. Since we have the condition $Ric=2\eta\otimes\eta$, L is Einstein-like with constants $(0,0,2)$. The Ricci tensor $\widetilde{Ric}$ corresponding to the semisymmetric metric connection $\widetilde{\nabla}$ satisfies the relation

$$\widetilde{Ric}=-g-\widetilde{g}+4\eta\otimes\eta.$$

Hence, L is an Einstein-like with regard to the semisymmetric metric connection $\widetilde{\nabla}$ with constants $(-1,-1,4)$. Moreover, L is a Ricci-like soliton with potential vector field ξ since Ric has the following relation

$$\frac{1}{2}(\mathcal{L}_\xi g)(x,y)+Ric(x,y)+\widetilde{g}-3\eta(x)\eta(y)=0,$$

where $\mathcal{L}_\xi g$ is the Lie derivative of g along ξ. It can be easily proved that ξ is not a torse-forming vector field.

Let us consider the Lie derivative of g along ξ corresponding to the connection $\widetilde{\nabla}$ defined by

$$\widetilde{\mathcal{L}}_\xi g(x,y)=g(\widetilde{\nabla}_x\xi,y)+g(x,\widetilde{\nabla}_y\xi).$$

The following equality is valid:

$$\frac{1}{2}\widetilde{\mathcal{L}}_\xi g+\widetilde{Ric}+2g+2\widetilde{g}-6\eta\otimes\eta=0.$$

Therefore, L admits a Ricci-like soliton with the potential ξ with respect to the connection $\widetilde{\nabla}$ by constants $(2,2,-6)$. It can be easily checked that Theorem (3.7) is confirmed.

Acknowledgment

The author would like to thank anonymous referees for their valuable comments which helped to improve the manuscript.

References

[1] Chaubey SK, Yadav SK. Study of Kenmotsu manifolds with semi-symmetric connection. Universal Journal of Mathematics and Applications 2018; 1 (2): 89-97.

[2] De UC, Pathok G. On a semi-symmetric metric connection in a Kenmotsu manifold. Bulletin of the Calcutta Mathematical Society 2002; 94: 319-324.

[3] Ganchev G, Mihova V, Gribachev K. Almost contact manifolds with $B-$metric. Mathematica Balkanica, New Series 1993; 7: 261-276.

[4] Imai T. Notes on semi-symmetric metric connections. Tensor (N.S.) 1972; 24: 293-296.

[5] Ivanov S, Manev H, Manev M. Sasaki-like almost contact complex Riemannian manifolds. Journal of Geometry and Physics 2016; 107: 136-148.

[6] Manev M, Ivanova M. A classification of the torsion tensors on an almost contact manifolds with $B-$metric. Central European Journal of Mathematics 2014; 12 (10): 1416-1432.

[7] Manev M. Ricci-like solitons on almost contact $B-$metric manifolds. Journal of Geometry and Physics 2020; 154: 103734.

[8] Manev M, Ivanova M. Canonical-type connection on almost contact manifolds with $B-$metric. Annals of Global Analysis and Geometry 2013; 43: 397-408.

[9] Manev M. Contactly conformal transformations of general type of almost contact manifold with $B-$metric. Applications. Mathematica Balkanica, New Series 1997; 11: 347-357.

[10] Manev H, Mekerov D. Lie groups as 3-dimensional almost contact $B-$metric manifolds. Journal of Geometry 2015; 106: 229-242.

[11] Mishra RS, Pandey SN. Semi-symmetric metric connections in an almost contact manifold. Indian Journal of Pure and Applied Mathematics 1978; 9 (6): 570-580.

[12] Nagaraja HG, Premalatha CR. Ricci solitons in Kenmotsu manifolds. Journal of Mathematical Analysis 2012; 3 (2): 18-24.

[13] Pravanović M, Pušić N. On manifolds admitting some semi-symmetric metric connection. Indian Journal of Mathematics 1995; 37: 37-67.

[14] Sharfuddin A, Husain SI. Semi-symmetric metric connections in almost contact manifolds. Tensor (N.S.) 1976; 30: 133-139.

[15] Sharma R. Certain results on K-contact and $(\kappa, \mu)-$contact manifolds. Journal of Geometry 2008; 89 (1-2): 138-147.

[16] Tripathi MM. A new connection in a Riemannian manifold. International Electronic Journal of Geometry 2008; 1: 15-24.

[17] Yano K. On semi-symmetric metric connection. Revue Roumaine de Mathématique Pures et Appliquées 1970; 15: 1579-1586.