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Finite topological type of complete Finsler gradient shrinking Ricci solitons

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Abstract: In the present work it is shown that on a Finslerian space, a forward complete gradient shrinking Ricci soliton has finite topological type, provided either the Ricci scalar is bounded above or the Ricci scalar is bounded from below and injectivity radius is bounded away from zero.

Key words: Finsler metric, Ricci soliton, finite topological type, Ricci flow

1. Introduction

The Ricci flow, which evolves a Riemannian metric by its Ricci curvature is a natural analog of the heat equation for metrics. In Hamilton's celebrated paper [13], it is shown that there is a unique solution to the Ricci flow for an arbitrary smooth Riemannian metric on a closed manifold over a sufficiently short time. Ricci soliton emerges as the limit of the solutions of the Ricci flow. The concept of Ricci solitons was introduced by Hamilton in 1988, as a solution to the Ricci flow which it moves only by a one-parameter group of diffeomorphism and scaling, see [14]. They are natural generalizations of Einstein metrics called also Ricci solitons and are subject to a great interest in geometry and physics especially in relation to string theory. Thus it is important to understand the geometry/topology of Ricci solitons and their classification.

J. Lott has shown that the fundamental group of a closed manifold M is finite for any gradient shrinking Ricci soliton, see [16]. By a result of A. Derdzinski, every compact shrinking Ricci soliton has only finitely many free homotopy classes of closed curves in M that are in a bijective correspondence with the conjugacy classes in the fundamental group of M, see [11]. M. F. López and E. G. Río have proved that a compact shrinking Ricci soliton has finite fundamental group, see [15]. Moreover, Wylie has shown that a complete shrinking Ricci soliton has finite fundamental group, see [18]. An extension of similar results is obtained for Yamabe solitons by the first present author in a joint work with Bidabad, see [5]. Also, it is shown that a complete gradient shrinking Ricci soliton has finite topological type if its scalar curvature is bounded by Fang, Man and Zhang, see [12].

The concept of Ricci flow on Finsler manifolds is introduced first by D. Bao, [3], choosing the Ricci tensor defined by Akbar-Zadeh, see [1]. The existence and uniqueness of solutions to the Ricci flow on Finsler surfaces and deformation of hh-Cartan are shown by Bidabad and Sedaghat, see [9]. Next, the first present author in a joint work proved that a family of Finslerian metrics $g(t)$ which are solutions to the Finslerian Ricci flow converge in C^∞ to a smooth limit Finslerian metric, as t approaches the finite time T, see [20]. Also, it is

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shown that on a compact Finsler manifold of constant flag curvature of initial non-negative scalar curvature, the scalar curvature remains non-negative by Ricci flow and blows up in a short time in a joint work with the first author, see [7].

Ricci solitons Finsler spaces as a generalization of Einstein spaces are considered by the present authors and it is shown that if there is a Ricci soliton on a compact Finsler manifold then there exists a solution to the Finsler Ricci flow equation and vice versa, see [8]. Next, a Bonnet–Myers type theorem was studied and it is proved that on a Finsler space, a forward complete shrinking Ricci soliton space is compact if and only if the corresponding vector field is bounded. Moreover, it is proved that a compact shrinking Ricci soliton Finsler space has finite fundamental group and hence the first de Rham cohomology group vanishes, see [19]. Also the results on shrinking Ricci soliton Finsler spaces previously obtained in compact case by the present authors, see [19], are extended for geodesically complete spaces, see [6]. Some similar results are obtained for complete Finslerian Yamabe solitons, see [4, 21].

The following theorem applies to a more general class of Finsler manifolds than Ricci solitons.

Theorem 1.1 *Let (M, F_0) be a forward complete Finsler manifold satisfying*

$$2Ric_{jk} + \mathcal{L}_{\hat{V}} g_{jk} \geq 2\lambda g_{jk}, \tag{1.1}$$

where $V = grad\rho$ and $\lambda > 0$. Then M has finite topological type if either (i) the Ricci scalar $\mathcal{R}ic$ is bounded above or (ii) there exists a real $\delta > 0$ such that the injectivity radius $i(M) \geq \delta$ and $\mathcal{R}ic + \frac{1}{\delta} \geq 0$.

Corollary 1.2 *A forward complete gradient shrinking Ricci soliton has finite topological type provided either (i) or (ii) is satisfied.*

In particular, it follows that a forward or backward complete shrinking Ricci soliton Finsler space with constant flag curvature has finite topological type.

2. Preliminaries and terminologies

Let M be a real n-dimensional differentiable manifold. We denote by TM its tangent bundle and by $\pi : TM_0 \longrightarrow M$ the bundle of nonzero tangent vectors. A Finsler structure on M is a function $F : TM \longrightarrow [0, \infty)$ with the following properties:
(i) Regularity: F is C^∞ on the entire slit tangent bundle $TM_0 = TM \backslash 0$.
(ii) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$.
(iii) Strong convexity: The $n \times n$ Hessian matrix $g_{ij} = ([\frac{1}{2}F^2]_{y^i y^j})$ is positive definite at every point of TM_0.
A Finsler manifold (M, F) is a pair consisting of a differentiable manifold M and a Finsler structure F. The formal Christoffel symbols of second kind and the spray coefficients are denoted respectively by

$$\gamma^i_{jk} := g^{is}\frac{1}{2}(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j}), \tag{2.1}$$

where $g_{ij} = [\frac{1}{2}F^2]_{y^i y^j}$, and

$$G^i := \frac{1}{2}\gamma^i_{jk} y^j y^k. \tag{2.2}$$

We consider also the reduced curvature tensor R^i_k which is a connection free quantity and is expressed in terms of partial derivatives of the spray coefficients G^i:

$$R^i_k := \frac{1}{F^2}\left(2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}\right). \tag{2.3}$$

In the general Finslerian setting, one of the remarkable definitions Ricci tensors is introduced by H. Akbar-Zadeh as $Ric_{jk} := [\frac{1}{2}F^2 \mathcal{R}ic]_{y^j y^k}$, where $\mathcal{R}ic = R^i_i$ and R^i_k is defined by (2.3).

Let $V = v^i(x)\frac{\partial}{\partial x_i}$ be a vector field on M. If $\{\varphi_t\}$ is the local one-parameter group of M generated by V, then it induces an infinitesimal point transformation on M defined by $\varphi^\star_t(x^i) := \bar{x}^i$, where $\bar{x}^i = x^i + v^i(x)t$. This is naturally extended to the point transformation $\tilde{\varphi}_t$ on the tangent bundle TM defined by $\tilde{\varphi}_t{}^\star := (\bar{x}^i, \bar{y}^i)$, where

$$\bar{x}^i = x^i + v^i(x)t, \quad \bar{y}^i = y^i + \frac{\partial v^i}{\partial x^m} y^m t. \tag{2.4}$$

It can be shown that, $\{\tilde{\varphi}_t\}$ induces a vector field $\hat{V} = v^i(x)\frac{\partial}{\partial x^i} + y^j \frac{\partial v^i}{\partial x^j}\frac{\partial}{\partial y^i}$ on TM called the complete lift of V. The one-parameter group associated to the complete lift $\hat{V}$ is given by $\tilde{\varphi}_t(x, y) = (\varphi_t(x), y^i \frac{\partial \varphi_t}{\partial x^i})$. The Lie derivative of an arbitrary geometric object $\Upsilon^I(x, y)$ on TM, where I is a mixed multi index, with respect to the complete lift $\hat{V}$ of a vector field V on M, is defined by

$$\mathcal{L}_{\hat{V}} \Upsilon^I = \lim_{t \to 0} \frac{\tilde{\varphi}_t{}^\star(\Upsilon^I) - \Upsilon^I}{t} = \frac{d}{dt}\tilde{\varphi}_t{}^\star(\Upsilon^I), \tag{2.5}$$

where $\tilde{\varphi}_t{}^\star(\Upsilon^I)$ is the deformation of $\Upsilon^I(x, y)$ under the extended point transformation (2.4). Whenever the geometric object Υ^I is a tensor field, $\tilde{\varphi}_t{}^\star(\Upsilon^I)$ coincides with the classical notation of pullback of $\Upsilon^I(x, y)$. The Lie derivative of the Finsler metric tensor g_{jk} is given in the following tensorial form by

$$\mathcal{L}_{\hat{V}} g_{jk} = \nabla_j V_k + \nabla_k V_j + 2(\nabla_0 V^l) C_{ljk}, \tag{2.6}$$

where $\hat{V}$ is the complete lift of a vector field V on M, ∇ is the Cartan connection, $\nabla_0 = y^p \nabla_p$ and $\nabla_p = \nabla_{\frac{\delta}{\delta x^p}}$ and C_{ljk} is the Cartan tensor, see [10].

Let (M, F) be a Finsler manifold and $\gamma : [a, b] \longrightarrow M$ a piecewise C^∞ curve on M, with velocity $\frac{d\gamma}{dt} = \frac{d\gamma^i}{dt}\frac{\partial}{\partial x^i} \in T_{\gamma(t)}M$. Its integral length $L(\gamma)$ is defined by $L(\gamma) = \int_a^b F(\gamma, \frac{d\gamma}{dt})dt$. For $p, q \in M$, denote by $\Gamma(p, q)$ the collection of all piecewise smooth curves $\gamma : [a, b] \longrightarrow M$ with $\gamma(a) = p$ and $\gamma(b) = q$. Define the Finslerian distance function $d : M \times M \longrightarrow [0, \infty)$ by $d(p, q) := \inf_{\gamma \in \Gamma(p,q)} L(\gamma)$. Note that in general this distance function does not have the symmetric property. A Finsler manifold (M, F) is said to be forward (resp. backward) geodesically complete if every geodesic $\gamma(t)$, $a \leq t < b$ (resp. $a < t \leq b$), parametrized to have constant Finslerian speed, can be extended to a geodesic defined on $a \leq t < \infty$ (resp. $\infty < t \leq b$). Also, in general, a forward geodesically completeness is not equivalent to backward geodesically completeness (see [2], p. 168). According to the Hopf–Rinow theorem, on a forward (or backward) geodesically complete Finsler space, every two points $p, q \in M$ can be joined by a minimal geodesic. The forward and backward metric balls $\mathcal{B}^+_p(r)$

and $\mathcal{B}_p^-(r)$ are defined by $\mathcal{B}_p^+(r) := \{x \in M \mid d(p,x) < r\}$ and $\mathcal{B}_p^-(r) := \{x \in M \mid d(x,p) < r\}$, respectively. Note that a manifold M has finite topological type if M is homeomorphic to the interior of a compact manifold with boundary.

3. The finite topological type of Ricci soliton Finsler spaces

An estimation for the integral of Ricci scalar

Fix a point x in the Finsler manifold (M,F). Let $\sigma_y(t)$ be a unit speed geodesic that passes through x at time $t=0$, with initial velocity $y \in S_xM$, where $S_xM := \{y \in T_xM \mid F(x,y)=1\}$. Define a function $i_x : S_xM \longrightarrow \mathbb{R}$ as follows:

$$i_x(y) := \sup\{r > 0 \mid t = d(x, \sigma_y(t)), \forall x \in [0,r]\}.$$

The forward injectivity radius at x is defined by $i(x) := \inf\{i_x(y) \mid y \in S_xM\}$. The injectivity radius $i(M)$ of (M,F) is defined as $i(M) := \inf\{i(x) \mid x \in M\}$.

Lemma 3.1 *Let (M,F) be a complete Finsler manifold, $p,q \in M$ such that $r := d(p,q)$ and γ is a minimal geodesic from p to q parameterized by the arc length s. If there exists $\delta > 0$ such that $i(M) \geq \delta$ and $\mathcal{R}ic + \frac{1}{\delta} \geq 0$, then*

$$\int_0^r \mathcal{R}ic(\gamma,\gamma')\, ds \leqslant \frac{18}{\delta}(n-1) + \frac{2}{3} \tag{3.1}$$

Proof Let $\varphi : [0,r] \longrightarrow [0,1]$ be a piecewise smooth function with $\varphi(0) = \varphi(r) = 0$. By the second variation of arc length in the Finslerian setting, we have (see [6] for more details)

$$0 \leq \int_0^r \Big((n-1)(\frac{d\varphi}{ds})^2 - \varphi^2(s)\mathcal{R}ic(\gamma,\gamma')\Big)ds. \tag{3.2}$$

Consider the piecewise smooth function $\varphi : [0,r] \longrightarrow [0,1]$, where

$$\varphi(s) = \begin{cases} \frac{3}{\delta}s & 0 \leqslant s \leqslant \frac{\delta}{3}, \\ 1 & \frac{\delta}{3} \leqslant s \leqslant r - \frac{\delta}{3}, \\ \frac{3}{\delta}(r-s) & r - \frac{\delta}{3} \leqslant s \leqslant r. \end{cases}$$

By decomposition (3.2) we have

$$\begin{aligned} 0 \leqslant & \int_0^{\frac{\delta}{3}} (n-1)(\frac{d\varphi}{ds})^2 ds + \int_{\frac{\delta}{3}}^{r-\frac{\delta}{3}} (n-1)(\frac{d\varphi}{ds})^2 ds + \int_{r-\frac{\delta}{3}}^{r} (n-1)(\frac{d\varphi}{ds})^2 ds - \int_0^{\frac{\delta}{3}} \varphi^2 \mathcal{R}ic(\gamma,\gamma')\, ds \\ & - \int_{\frac{\delta}{3}}^{r-\frac{\delta}{3}} \varphi^2 \mathcal{R}ic(\gamma,\gamma')\, ds - \int_{r-\frac{\delta}{3}}^{r} \varphi^2 \mathcal{R}ic(\gamma,\gamma')\, ds \\ \leqslant & \frac{6}{\delta}(n-1) - \int_0^{\frac{\delta}{3}} \varphi^2 \mathcal{R}ic(\gamma,\gamma')\, ds - \int_{\frac{\delta}{3}}^{r-\frac{\delta}{3}} \mathcal{R}ic(\gamma,\gamma')\, ds - \int_{r-\frac{\delta}{3}}^{r} \varphi^2 \mathcal{R}ic(\gamma,\gamma') ds. \end{aligned} \tag{3.3}$$

Adding $\int_0^r \mathcal{R}ic(\gamma,\gamma')ds$ to the both sides of (3.3) yields

$$\int_0^r \mathcal{R}ic(\gamma,\gamma')\, ds \leqslant \frac{6}{\delta}(n-1) + \int_0^{\frac{\delta}{3}} (1-\varphi^2(s))\mathcal{R}ic(\gamma,\gamma')\, ds + \int_{r-\frac{\delta}{3}}^{r} (1-\varphi^2(s))\mathcal{R}ic(\gamma,\gamma')\, ds. \tag{3.4}$$

On the other hand, since $\mathcal{R}ic + \frac{1}{\delta} \geqslant 0$ and $0 \leqslant \varphi(s) \leqslant 1$, we have

$$\begin{aligned}\int_0^{\frac{\delta}{3}} (1-\varphi^2(s))\mathcal{R}ic(\gamma,\gamma')\ ds &= \int_0^{\frac{\delta}{3}} (1-\varphi^2(s))(\mathcal{R}ic(\gamma,\gamma') + \frac{1}{\delta} - \frac{1}{\delta})\ ds \\ &= \int_0^{\frac{\delta}{3}} (1-\varphi^2(s))(\mathcal{R}ic(\gamma,\gamma') + \frac{1}{\delta})ds - \frac{1}{\delta}\int_0^{\frac{\delta}{3}} (1-\varphi^2(s))ds \\ &\leqslant \int_0^{\frac{\delta}{3}} (\mathcal{R}ic(\gamma,\gamma') + \frac{1}{\delta})ds - \frac{1}{\delta}\int_0^{\frac{\delta}{3}} (1-\varphi^2(s))ds = \int_0^{\frac{\delta}{3}} \mathcal{R}ic(\gamma,\gamma')ds + \frac{1}{\delta}\int_0^{\frac{\delta}{3}} \varphi^2(s)ds \\ &= \int_0^{\frac{\delta}{3}} \mathcal{R}ic(\gamma,\gamma')ds + \frac{1}{\delta}\int_0^{\frac{\delta}{3}} (\frac{3}{\delta}s)^2 ds = \int_0^{\frac{\delta}{3}} \mathcal{R}ic(\gamma,\gamma')ds + \frac{9}{\delta^3}\int_0^{\frac{\delta}{3}} s^2 ds \\ &= \int_0^{\frac{\delta}{3}} \mathcal{R}ic(\gamma,\gamma')ds + \frac{1}{9},\end{aligned}$$

and consequently we get

$$\int_0^{\frac{\delta}{3}} (1-\varphi^2(s))\mathcal{R}ic(\gamma,\gamma')\ ds \leqslant \int_0^{\frac{\delta}{3}} \mathcal{R}ic(\gamma,\gamma')ds + \frac{1}{9}. \tag{3.5}$$

Similarly, one can get

$$\int_{r-\frac{\delta}{3}}^{r} (1-\varphi^2(s))\mathcal{R}ic(\gamma,\gamma')\ ds \leqslant \int_{r-\frac{\delta}{3}}^{r} \mathcal{R}ic(\gamma,\gamma')ds + \frac{1}{9}. \tag{3.6}$$

Replacing (3.5) and (3.6) in (3.4) we conclude that

$$\int_0^{r} \mathcal{R}ic(\gamma,\gamma')\ ds \leqslant \frac{6}{\delta}(n-1) + \int_0^{\frac{\delta}{3}} \mathcal{R}ic(\gamma,\gamma')\ ds + \int_{r-\frac{\delta}{3}}^{r} \mathcal{R}ic(\gamma,\gamma')\ ds + \frac{2}{9}. \tag{3.7}$$

Note that $i(M) \geqslant \delta$. Let $\sigma : [0,\delta] \longrightarrow M$ be a minimal geodesic such that $\gamma(s) = \sigma(s+\frac{\delta}{3})$ for $s \in [0,\frac{\delta}{3}]$. By estimation (3.2) for the minimal geodesic σ, we get

$$0 \leq \int_0^{\delta} \Big((n-1)(\frac{d\varphi}{ds})^2 - \varphi^2(s)\mathcal{R}ic(\sigma,\sigma')\Big)ds = \frac{6}{\delta}(n-1) - \int_0^{\delta} \varphi^2(s)\mathcal{R}ic(\sigma,\sigma')ds.$$

By using $\mathcal{R}ic + \frac{1}{\delta} \geqslant 0$ and last formula, we have

$$\begin{aligned}\int_0^{\frac{\delta}{3}} \mathcal{R}ic(\gamma,\gamma')\ ds = \int_{\frac{\delta}{3}}^{\frac{2\delta}{3}} \mathcal{R}ic(\sigma,\sigma')\ ds &\leqslant \frac{6}{\delta}(n-1) - \int_0^{\frac{\delta}{3}} \varphi^2(s)\mathcal{R}ic(\sigma,\sigma')ds - \int_{\frac{2\delta}{3}}^{\delta} \varphi^2(s)\mathcal{R}ic(\sigma,\sigma')ds \\ &\leqslant \frac{6}{\delta}(n-1) + \frac{2}{9}.\end{aligned} \tag{3.8}$$

Also, by similar argument, we have

$$\int_{r-\frac{\delta}{3}}^{r} \mathcal{R}ic(\gamma,\gamma')\ ds \leqslant \frac{6}{\delta}(n-1) + \frac{2}{9}. \tag{3.9}$$

Replacing (3.8) and (3.9) in (3.7), we conclude

$$\int_0^r \mathcal{R}ic(\gamma, \gamma')\, ds \leqslant \frac{18}{\delta}(n-1) + \frac{2}{3},$$

as we have claimed. □

Gradient Ricci solitons on Finsler spaces

Let $\rho : M \to \mathbb{R}$ be a real differentiable function on the Finsler manifold (M, g). We consider here the vector field $grad\rho(p) \in T_pM$, defined by $grad\rho := \rho^i(x)\frac{\partial}{\partial x^i}$, where $\rho^i(x) = g_{ij}(x, grad\rho(x))\frac{\partial \rho}{\partial x^j}$ as the gradient of ρ at point $p \in M$. Equivalently

$$g_{grad\rho(p)}(X, grad\rho(p)) = d\rho_p(X), \qquad \forall X \in T_pM.$$

Let (M, F_0) be a Finsler manifold and $V = v^i(x)\frac{\partial}{\partial x^i}$ a vector field on M. We call the triple (M, F_0, V) a Finslerian quasi-Einstein or a Ricci soliton if g_{jk} the Hessian related to the Finsler structure F_0 satisfies

$$2Ric_{jk} + \mathcal{L}_{\hat{V}} g_{jk} = 2\lambda g_{jk}, \tag{3.10}$$

where $\hat{V}$ is complete lift of V and $\lambda \in \mathbb{R}$. A Finslerian Ricci soliton is said to be shrinking, steady or expanding if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. If the vector field V is gradient of a potential function f, then (M, F_0, V) is said to be gradient Ricci soliton. The Ricci soliton is said to be forward complete (resp. compact) if (M, F_0) is forward complete (resp. compact). Note that according to the Hopf–Rinow theorem, two notions forward complete and forward geodesically complete are equivalent. For a vector field $X = X^i(x)\frac{\partial}{\partial x^i}$ on M define $\|X\|_x = \max_{y \in S_xM} \sqrt{g_{ij}(x, y)X^iX^j}$, where $x \in M$, (see [2], p. 321). Since S_xM is compact $\|X\|_x$ is well defined.

Proof of Theorem 1.1

Let p and q be two points in M joining by a minimal geodesic γ parametrized by the arc length t, $\gamma : [0, \infty) \longrightarrow M$ and $r := d(p, q)$. Let $V = v^i(x)\frac{\partial}{\partial x_i}$ be a vector field on M. It is well known that the Lie derivative of a Finsler metric tensor g_{jk} is given in the following tensorial form by

$$\mathcal{L}_{\hat{V}} g_{jk} = \nabla_j V_k + \nabla_k V_j + 2(\nabla_0 V^l)C_{ljk}, \tag{3.11}$$

where $\hat{V}$ is the complete lift of a vector field V on M, ∇ is the Cartan connection, $\nabla_0 = y^p\nabla_p$ and $\nabla_p = \nabla_{\frac{\delta}{\delta x^p}}$. Now by using $V = \nabla f$ and (3.11) we have along γ

$$\gamma'^j\gamma'^k \mathcal{L}_{\hat{V}} g_{jk} = \gamma'^j\gamma'^k\big(\nabla_j\nabla_k f + \nabla_k\nabla_j f + 2(\nabla_0\nabla^l f)C_{ljk}\big). \tag{3.12}$$

Along γ, we have $\gamma'^j\gamma'^k(\nabla_0\nabla^l f)C_{ljk}(\gamma(t), \gamma'(t)) = 0$. Hence (3.12) reduces to

$$\gamma'^j\gamma'^k \mathcal{L}_{\hat{V}} g_{jk} = 2\gamma'^j\gamma'^k\nabla_j\nabla_k f = 2\nabla_{\hat{\gamma}'}\nabla_{\hat{\gamma}'} f. \tag{3.13}$$

Contracting (1.1) with $\gamma'^j\gamma'^k$ gives along γ

$$\mathcal{R}ic(\gamma, \gamma') + \nabla_{\hat{\gamma}'}\nabla_{\hat{\gamma}'} f \geq \lambda. \tag{3.14}$$

On the other hand, by compatibility of metric in Cartan connection we have along the geodesic γ

$$\nabla_{\hat{\gamma}'}\nabla_{\hat{\gamma}'}f = \nabla_{\hat{\gamma}'}({\gamma'}^k\nabla_k f) = \frac{d}{dt}({\gamma'}^k\nabla_k f) = \frac{d}{dt}(<\gamma', \nabla f>), \tag{3.15}$$

where $\hat{\gamma}' = {\gamma'}^j\frac{\delta}{\delta x^j}$. Therefore we have

$$\int_0^r \nabla_{\hat{\gamma}'}\nabla_{\hat{\gamma}'}f \ ds =< \gamma', \nabla f > (q) - < \gamma', \nabla f > (p).$$

By means of (3.14) we have

$$<\gamma', \nabla f> (q) \geq <\gamma', \nabla f> (p) + \int_0^r \big(\lambda - \mathcal{R}ic(\gamma, \gamma')\big) \ ds = <\gamma', \nabla f> (p) + \lambda r - \int_0^r \mathcal{R}ic(\gamma, \gamma') \ ds.$$

By means of Cauchy–Schwarz inequality we have

$$\|\nabla f\|_q \geq \lambda r - \|\nabla f\|_p - \int_0^r \mathcal{R}ic(\gamma, \gamma') \ ds. \tag{3.16}$$

There are two situations:

(a) $|\mathcal{R}ic| \leq C$ for some constant $C < \infty$. By Lemma 3.1 of [6], we have $\int_0^r \mathcal{R}ic(\gamma, \gamma') \ ds \leq 2(n-1) + 2C$.

(b) there exists a real $\delta > 0$ such that the injectivity radius $i(M) \geq \delta$ and $\mathcal{R}ic + \frac{1}{\delta} \geq 0$. Applying Lemma (3.1) we have $\int_0^r \mathcal{R}ic(\gamma, \gamma') \ ds \leqslant \frac{18}{\delta}(n-1) + \frac{2}{3}$.

Hence, we conclude that the integral $\int_0^r \mathcal{R}ic(\gamma, \gamma') \ ds$ is bounded above by some constant Λ. Therefore (3.16) leads to

$$\|\nabla f\|_q \geq \lambda r - \|\nabla f\|_p - \Lambda.$$

Therefore $\|\nabla f\|_q$ has a linear growth in $r = d(p, q)$. Obviously, $f^{-1}\big((-\infty, a]\big)$ is compact for any $a < \infty$ and so f is a proper function. Also, one can easily check that f has no critical points outside of a compact set. In fact, it's enough to consider a compact set $\bar{\mathcal{B}}_p^+(\frac{2(\|\nabla f\|_p + \Lambda)}{\lambda})$. The deformation lemma (Isotopy Lemma) of Morse theory leads to M has finite topological type.

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