Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 2731 – 2745
© TÜBİTAK
doi:10.3906/mat-2104-24

On the extension of Hermite–Hadamard type inequalities for coordinated convex mappings

Mehmet Zeki SARIKAYA*, **Dilşatnur KILIÇER**
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

Received: 05.04.2021 • **Accepted/Published Online:** 31.10.2021 • **Final Version:** 29.11.2021

Abstract: In this paper, we obtain an important inequalities for coordinated convex functions and as a result of these inequalities we give the extension of Hermite–Hadamard type inequalities for Riemann–Liouville fractional integral and logarithmic integral. The inequalities obtained in this study provide generalizations of some result given in earlier works.

Key words: Hermite–Hadamard's inequalities, fractional integral, coordinated convex, integral inequalities

1. Introduction

The Hermite–Hadamard inequality discovered by C. Hermite and J. Hadamard see, e.g., [12], [25, p.137]) is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite–Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite–Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied (see, for example, [3], [8]-[15], [16], [23], [24], [32], [37], [38], [42]).

A formal defination for coordinated convex function may be stated as follows: Let us now consider a bidemensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x,y) + (1-t)f(z,w)$$

holds, for all $(x,y), (z,w) \in \Delta$ and $t \in [0,1]$. A function $f : \Delta \to \mathbb{R}$ is said to be on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b]$ and $y \in [c,d]$ (see [11]).

*Correspondence: sarikayamz@gmail.com
2010 *AMS Mathematics Subject Classification:* 26D07, 26D10, 26D15, 26B15, 26B25

Definition 1.1 *A function $f : \Delta \to \mathbb{R}$ will be called coordinated canvex on Δ, for all $t, s \in [0,1]$ and $(x,y),(u,w) \in \Delta$, if the following inequality holds:*

$$\begin{aligned} & f(tx+(1-t)\,y, su+(1-s)\,w) \\ \leq \quad & tsf(x,u)+s(1-t)f(y,u)+t(1-s)f(x,w)+(1-t)(1-s)f(y,w). \end{aligned} \tag{1.2}$$

Clearly, every convex function is coordinated convex. Furthermore, there exists coordinated convex function which is not convex (see, [11]). For several recent results concerning Hermite–Hadamard's inequality for some convex function on the coordinates on a rectangle from the plane $\mathbb{R}^2$, we refer the reader to ([1], [2],[4], [11], [17], [18], [19]-[22], [29],[33], [35], [36], [39], [40]).

Now, we give the definitions Riemann–Liouville fractional integrals for two variable functions:

Definition 1.2 *[29] Let $f \in L_1([a,b]\times[c,d])$. The Riemann–Liouville fractional integrals $J^{\alpha,\beta}_{a+,c+}$, $J^{\alpha,\beta}_{a+,d-}$, $J^{\alpha,\beta}_{b-,c+}$ and $J^{\alpha,\beta}_{b-,d-}$ are defined by*

$$J^{\alpha,\beta}_{a+,c+}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1}f(t,s)dsdt, \quad x>a,\ y>c,$$

$$J^{\alpha,\beta}_{a+,d-}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1}f(t,s)dsdt, \quad x>a,\ y<d,$$

$$J^{\alpha,\beta}_{b-,c+}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1}f(t,s)dsdt, \quad x<b,\ y>c,$$

and

$$J^{\alpha,\beta}_{b-,d-}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}f(t,s)dsdt, \quad x<b,\ y<d.$$

Similar the above definitons, we can give the following integrals:

$$J^{\alpha}_{a+}f(x,c)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t,c)dt, \quad x>a$$

$$J^{\alpha}_{a+}f(x,d)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t,d)dt, \quad x>a,$$

$$J^{\beta}_{c+}f(a,y)=\frac{1}{\Gamma(\beta)}\int_c^y (y-s)^{\beta-1}f(a,s)ds, \quad y>c,$$

and

$$J_{d-}^{\beta} f(b,y) = \frac{1}{\Gamma(\beta)} \int\limits_{y}^{d} (s-y)^{\beta-1} f(b,s)ds, \quad y < d.$$

One can find some recent Hermite–Hadamard type inequalities for function of one and two variables via Riemann–Liouville fractional integrals in ([1], [5], [6], [7], [26], [27], [28], [30], [31], [34], [36], [39], [41], [43]-[45]).

The aim of this paper is to establish an important inequalities for coordinated convex functions and as a result of these inequalities we give the extension of Hermite–Hadamard type inequalities for Riemann–Liouville fractional integral and logarithmic integral. The results presented in this paper provide extensions of those given in [11] and [29].

In [42], Zabandan gave the following important inequalities associated with the Hermite–Hadamard inequalities and he gave a few inequalities regarding the special cases of these inequalities.

Theorem 1.3 *Let $f:[a,b] \to R$ be a positive convex function on $[a,b]$ and $h:[0,1] \to R$ be a positive function such that $h \in L([0,1])$. Then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2I_h(b-a)} \int\limits_{a}^{b} \left[h\left(\frac{x-a}{b-a}\right) + h\left(\frac{b-x}{b-a}\right)\right] f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

where $I_h = \int\limits_0^1 h(t)\,dt$.

2. Main results

Throughout this section, we will use the following sympols

$$H_1(x) = h_1\left(\frac{b-x}{b-a}\right) + h_1\left(\frac{x-a}{b-a}\right),$$

$$H_2(y) = h_2\left(\frac{d-y}{d-c}\right) + h_2\left(\frac{y-c}{d-c}\right)$$

and

$$I_{h_1} = \int\limits_0^1 h_1(t)\,dt \text{ and } I_{h_2} = \int\limits_0^1 h_2(s)\,ds.$$

In this part, we will give the following inequalities by using convex functions of 2-variables on the coordinates.

Theorem 2.1 *Let $f:\Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is coordinated convex on $\Delta := [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $0 \leq a < b,\ 0 \leq c < d$ and $f \in L_1(\Delta)$ and $h_1, h_2 : [0,1] \to R$ be two positive functions such that $h_1, h_2 \in L([0,1])$. Then, one has the inequalities:*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{4I_{h_1}I_{h_2}(b-a)(d-c)} \int_a^b \int_c^d H_1(x)\,H_2(y) f(x,y)\,dydx \tag{2.1}$$

$$\leq \quad \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

Proof According to (1.2) with $x = ta + (1-t)b$, $y = (1-t)a + tb$, $u = sc + (1-s)d$, $w = (1-s)c + sd$ and $t = s = \frac{1}{2}$, we find that

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq\ & \frac{1}{4}\left[f(ta + (1-t)b, sc + (1-s_1)d) + f(ta + (1-t)b, (1-s)c + sd)\right. \\ & \left. + f((1-t)a + tb, sc + (1-s)d) + f((1-t)a + tb, (1-s)c + sd)\right]. \end{aligned} \tag{2.2}$$

Thus, multiplying both sides of (2.2) by $h_1(t)\, h_2(s)$, then by integrating with respect to (t, s) on $[0,1] \times [0,1]$, we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_0^1 \int_0^1 h_1(t)\, h_2(s)\, dsdt \\ \leq\ & \frac{1}{4}\left[\int_0^1 \int_0^1 h_1(t)\, h_2(s) \left[f(ta + (1-t)b, sc + (1-s)d) + f(ta + (1-t)b, (1-s)c + sd)\right] dsdt \right. \\ & \left. + \int_0^1 \int_0^1 h_1(t)\, h_2(s) \left[f((1-t)a + tb, sc + (1-s)d) + f((1-t)a + tb, (1-s)c + sd)\right] dsdt \right]. \end{aligned}$$

Using the change of the variable, we get

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_0^1 \int_0^1 h_1(t)\, h_2(s)\, dsdt \\ \leq\ & \frac{1}{4(b-a)(d-c)} \left\{ \int_a^b \int_c^d h_1\left(\frac{b-x}{b-a}\right) h_2\left(\frac{d-y}{d-c}\right) f(x,y)\, dydx \right. \\ & + \int_a^b \int_c^d h_1\left(\frac{b-x}{b-a}\right) h_2\left(\frac{y-c}{d-c}\right) f(x,y)\, dydx \\ & + \int_a^b \int_c^d h_1\left(\frac{x-a}{b-a}\right) h_2\left(\frac{d-y}{d-c}\right) f(x,y)\, dydx \\ & \left. + \int_a^b \int_c^d h_1\left(\frac{x-a}{b-a}\right) h_2\left(\frac{y-c}{d-c}\right) f(x,y)\, dydx \right\} \end{aligned}$$

which the first inequality is proved. For the proof of the second inequality in (2.1), we first note that if f is a coordinated convex on Δ, then, by using (1.2) with $x = a$, $y = b$, $u = c$ and $w = d$, it yields

$$f(ta + (1-t)b, sc + (1-s)d) \leq tsf(a,c) + s(1-t)f(b,c) + t(1-s)f(a,d) + (1-t)(1-s)f(b,d)$$

$$f(ta + (1-t)b, (1-s)c + sd) \leq t(1-s)f(a,c) + (1-t)(1-s)f(b,c) + tsf(a,d) + (1-t)sf(b,d)$$

$$f\left((1-t)a+tb, sc+(1-s)d\right) \leq (1-t)sf(a,c)+stf(b,c)+(1-t)(1-s)f(a,d)+t(1-s)f(b,d)$$

and

$$f\left((1-t)a+tb, (1-s)c+sd\right) \leq (1-t)(1-s)f(a,c)+t(1-s)f(b,c)+(1-t)sf(a,d)+tsf(b,d).$$

By adding these inequalities, we have

$$\begin{aligned} & f\left(ta+(1-t)b, sc+(1-s)d\right)+f\left(ta+(1-t)b, (1-s)c+sd\right) \\ & +f\left((1-t)a+tb, sc+(1-s)d\right)+f\left((1-t)a+tb, (1-s)c+sd\right) \\ \leq\ & f(a,c)+f(b,c)+f(a,d)+f(b,d). \end{aligned} \tag{2.3}$$

Then, multiplying both sides of (2.3) by $h_1(t)\,h_2(s)$ and integrating with respect to (t,s) over $[0,1]\times[0,1]$, we get

$$\begin{aligned} & \int_0^1\int_0^1 h_1(t)\,h_2(s)\left[f\left(ta+(1-t)b, sc+(1-s)d\right)+f\left(ta+(1-t)b, (1-s)c+sd\right)\right. \\ & \left.+f\left((1-t)a+tb, sc+(1-s)d\right)+f\left((1-t)a+tb, (1-s)c+sd\right)\right] dsdt \\ \leq\ & \left[f(a,c)+f(b,c)+f(a,d)+f(b,d)\right]\int_0^1\int_0^1 h_1(t)\,h_2(s)\,dsdt. \end{aligned}$$

Here, using the change of the variable we have

$$\begin{aligned} & \frac{1}{4(b-a)(d-c)}\left\{\int_a^b\int_c^d h_1\left(\frac{b-x}{b-a}\right)h_2\left(\frac{d-y}{d-c}\right)f(x,y)\,dydx\right. \\ & +\int_a^b\int_c^d h_1\left(\frac{b-x}{b-a}\right)h_2\left(\frac{y-c}{d-c}\right)f(x,y)\,dydx \\ & +\int_a^b\int_c^d h_1\left(\frac{x-a}{b-a}\right)h_2\left(\frac{d-y}{d-c}\right)f(x,y)\,dydx \\ & \left.+\int_a^b\int_c^d h_1\left(\frac{x-a}{b-a}\right)h_2\left(\frac{y-c}{d-c}\right)f(x,y)\,dydx\right\} \\ \leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_0^1\int_0^1 h_1(t)\,h_2(s)\,dsdt. \end{aligned}$$

The proof is completed. □

Remark 2.2 *If in Theorem 2.1,*

i) we choose $h_1(t) = t,\ h_2(s) = s$ *on* $[0,1]$, *then the inequalities (2.1) become the inequalities*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}, \tag{2.4}$$

which are proved by Sarikaya and Yaldiz in [28].

ii) we choose $h_1(t) = t^{\alpha-1}\,(\alpha>0),\ h_2(s) = s^{\beta-1}\,(\beta>0)$ *on* $[0,1]$, *then the inequalities (2.1) become the fractional integral inequalities*

$$\begin{aligned} & f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ \le\ & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[J^{\alpha,\beta}_{a+,c+}f(b,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{b-,d-}f(a,c)\right] \\ \le\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \end{aligned}$$

which are proved by Sarikaya in [29].

Corollary 2.3 *Under assumption of Theorem 2.1 with* $h_1(t) = (-\ln t)^{\alpha-1}\,(\alpha>0),\ h_2(s) = (-\ln s)^{\beta-1}\,(\beta>0)$ *on* $[0,1]$, *we get the following logarithmic integral inequalities*

$$\begin{aligned} & f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ \le\ & \frac{1}{4\Gamma(\alpha)\Gamma(\beta)(b-a)(d-c)} \\ & \times\int_a^b\int_c^d\left[\left[\ln\left(\frac{b-a}{b-x}\right)\right]^{\alpha-1}+\left[\ln\left(\frac{b-a}{x-a}\right)\right]^{\alpha-1}\right] \\ & \times\left[\left[\ln\left(\frac{d-c}{d-y}\right)\right]^{\beta-1}+\left[\ln\left(\frac{d-c}{y-c}\right)\right]^{\beta-1}\right]f(x,y)\,dydx \\ \le\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{2.5}$$

Corollary 2.4 *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *is coordinated convex on* $\Delta := [a,b]\times[c,d]$ *in* $\mathbb{R}^2$ *with* $0\le a<b,\ 0\le c<d$ *and* $f\in L_1(\Delta)$. *Then, one has the inequalities:*

$$\begin{aligned} & f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ \le\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \end{aligned} \tag{2.6}$$

$$
\begin{aligned}
&\leq \frac{1}{4(b-a)(d-c)} \\
&\quad\times \int_a^b \int_c^d \left[\left[\ln\left(\frac{b-a}{b-x}\right)\right] + \left[\ln\left(\frac{b-a}{x-a}\right)\right]\right] \\
&\quad\times \left[\left[\ln\left(\frac{d-c}{d-y}\right)\right] + \left[\ln\left(\frac{d-c}{y-c}\right)\right]\right] f(x,y)\,dydx \\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}
$$

Proof From (2.4), we have

$$
\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \qquad (2.7) \\
\leq\ & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dydx \\
=\ & \frac{1}{(b-a)(d-c)} \left\{ \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(x,y)\,dydx + \int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d f(x,y)\,dydx \right. \\
& \left. + \int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} f(x,y)\,dydx + \int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d f(x,y)\,dydx \right\}.
\end{aligned}
$$

By change of variable $x = \frac{a+t}{2}$ and $y = \frac{c+s}{2}$ in the first integral of right side of (2.7), using inequaliy (2.4) and from Fubini Theorem, we get

$$
\begin{aligned}
\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(x,y)\,dydx &= \frac{1}{4}\int_a^b \int_c^d f\left(\frac{a+t}{2}, \frac{c+s}{2}\right) dsdt \\
&\leq \frac{1}{4}\int_a^b \int_c^d \left(\frac{1}{(t-a)(s-c)} \int_a^t \int_c^s f(x,y)\,dydx\right) dsdt \\
&= \frac{1}{4}\int_a^b \int_c^d \left(\int_x^b \int_y^d \frac{1}{(t-a)(s-c)} dsdt\right) f(x,y)\,dydx \\
&= \frac{1}{4}\int_a^b \int_c^d \left(\ln\frac{b-a}{x-a}\right)\left(\ln\frac{d-c}{y-c}\right) f(x,y)\,dydx.
\end{aligned}
$$

By similar way, using change of variable $x = \frac{a+t}{2}$ and $y = \frac{d+s}{2}$, $x = \frac{b+t}{2}$ and $y = \frac{c+s}{2}$, $x = \frac{b+t}{2}$ and $y = \frac{d+s}{2}$ in the other integrals of right side of (2.7), respectively, we have

$$
\int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d f(x,y)\,dydx = \frac{1}{4}\int_a^b \int_c^d f\left(\frac{a+t}{2}, \frac{d+s}{2}\right) dsdt
$$

$$\begin{aligned} &\leq \frac{1}{4}\int_a^b\int_c^d\left(\frac{1}{(t-a)(d-s)}\int_a^t\int_s^d f(x,y)\,dydx\right)dsdt \\ &= \frac{1}{4}\int_a^b\int_c^d\left(\ln\frac{b-a}{x-a}\right)\left(\ln\frac{d-c}{d-y}\right)f(x,y)\,dydx, \end{aligned}$$

$$\begin{aligned} \int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}} f(x,y)\,dydx &= \frac{1}{4}\int_a^b\int_c^d f\left(\frac{b+t}{2},\frac{c+s}{2}\right)dsdt \\ &\leq \frac{1}{4}\int_a^b\int_c^d\left(\frac{1}{(b-t)(s-c)}\int_t^b\int_c^s f(x,y)\,dydx\right)dsdt \\ &= \frac{1}{4}\int_a^b\int_c^d\left(\ln\frac{b-a}{b-x}\right)\left(\ln\frac{d-c}{y-c}\right)f(x,y)\,dydx, \end{aligned}$$

$$\begin{aligned} \int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d f(x,y)\,dydx &= \frac{1}{4}\int_a^b\int_c^d f\left(\frac{b+t}{2},\frac{d+s}{2}\right)dsdt \\ &\leq \frac{1}{4}\int_a^b\int_c^d\left(\frac{1}{(b-t)(d-s)}\int_t^b\int_s^d f(x,y)\,dydx\right)dsdt \\ &= \frac{1}{4}\int_a^b\int_c^d\left(\ln\frac{b-a}{b-x}\right)\left(\ln\frac{d-c}{d-y}\right)f(x,y)\,dydx. \end{aligned}$$

By above calculated integrals, writing instead of the (2.7), and using the last inequality of (2.5) for $\alpha=\beta=2$, we get

$$\begin{aligned} & f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ \leq\; & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\ \leq\; & \frac{1}{4(b-a)(d-c)}\left\{\int_a^b\int_c^d\left(\ln\frac{b-a}{x-a}\right)\left(\ln\frac{d-c}{y-c}\right)f(x,y)\,dydx\right. \\ & +\int_a^b\int_c^d\left(\ln\frac{b-a}{x-a}\right)\left(\ln\frac{d-c}{d-y}\right)f(x,y)\,dydx \\ & +\int_a^b\int_c^d\left(\ln\frac{b-a}{b-x}\right)\left(\ln\frac{d-c}{y-c}\right)f(x,y)\,dydx \end{aligned}$$

$$+\int_a^b\int_c^d\left(\ln\frac{b-a}{b-x}\right)\left(\ln\frac{d-c}{d-y}\right)f(x,y)\,dydx\Bigg\}$$

$$\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}$$

which is proved the inequalities (2.6). □

Theorem 2.5 *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ is coordinated convex on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $0\leq a<b,\ 0\leq c<d$ and $f\in L_1(\Delta)$ and $h_1,h_2:[0,1]\to R$ be two positive functions such that $h_1,h_2\in L([0,1])$. Then, one has the inequalities:*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{2.8}$$

$$\leq \frac{1}{4I_{h_1}(b-a)}\int_a^b H_1(x)\,f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{4I_{h_2}(d-c)}\int_c^d H_2(y)f\left(\frac{a+b}{2},y\right)dy$$

$$\leq \frac{1}{4I_{h_1}I_{h_2}(b-a)(d-c)}\int_a^b\int_c^d H_1(x)\,H_2(y)f(x,y)\,dydx$$

$$\leq \frac{1}{8I_{h_1}(b-a)}\left[\int_a^b H_1(x)\,f(x,c)\,dx+\int_a^b H_1(x)\,f(x,d)\,dx\right]$$

$$+\frac{1}{8I_{h_2}(d-c)}\left[\int_c^d H_2(y)f(a,y)\,dy+\int_c^d H_2(y)f(b,y)\,dy\right]$$

$$\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

Proof Since $f:\Delta\to\mathbb{R}$ is convex on the coordinates, it follows that the mapping $g_x:[c,d]\to\mathbb{R}$, $g_x(y)=f(x,y)$, is convex on $[c,d]$ for all $x\in[a,b]$. Then by using inequalities (1.3), we can write

$$g_x\left(\frac{c+d}{2}\right)\leq\frac{1}{2I_{h_2}(d-c)}\int_c^d H_2(y)g_x(y)dy\leq\frac{g_x(c)+g_x(d)}{2},\quad x\in[a,b].$$

That is,

$$f\left(x,\frac{c+d}{2}\right)\leq\frac{1}{2I_{h_2}(d-c)}\int_c^d H_2(y)f(x,y)\,dy$$

$$\leq\frac{f(x,c)+f(x,d)}{2} \tag{2.9}$$

for all $x \in [a,b]$. Then, multiplying both sides of (2.9) by $\frac{1}{2I_{h_1}(b-a)} H_1(x)$, and integrating with respect to x over $[a,b]$, we have

$$\frac{1}{2I_{h_1}(b-a)} \int_a^b H_1(x) f\left(x, \frac{c+d}{2}\right) dx \tag{2.10}$$

$$\leq \frac{1}{4I_{h_1} I_{h_2}(b-a)(d-c)} \int_a^b \int_c^d H_1(x) H_2(y) f(x,y)\, dy dx$$

$$\leq \frac{1}{4I_{h_1}(b-a)} \left[\int_a^b H_1(x) f(x,c)\, dx + \int_a^b H_1(x) f(x,d)\, dx \right].$$

By similar argument applied for the mapping $g_y : [a,b] \to \mathbb{R}$, $g_y(x) = f(x,y)$, we have

$$\frac{1}{2I_{h_2}(d-c)} \int_c^d H_2(y) f\left(\frac{a+b}{2}, y\right) dy \tag{2.11}$$

$$\leq \frac{1}{4I_{h_1} I_{h_2}(b-a)(d-c)} \int_a^b \int_c^d H_1(x) H_2(y) f(x,y)\, dy dx$$

$$\leq \frac{1}{4I_{h_2}(d-c)} \left[\int_c^d H_2(y) f(a,y)\, dy + \int_c^d H_2(y) f(b,y)\, dy \right].$$

Adding the inequalities (2.10) and (2.11), we have

$$\frac{1}{2I_{h_1}(b-a)} \int_a^b H_1(x) f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{2I_{h_2}(d-c)} \int_c^d H_2(y) f\left(\frac{a+b}{2}, y\right) dy$$

$$\leq \frac{1}{2I_{h_1} I_{h_2}(b-a)(d-c)} \int_a^b \int_c^d H_1(x) H_2(y) f(x,y)\, dy dx$$

$$\leq \frac{1}{4I_{h_1}(b-a)} \left[\int_a^b H_1(x) f(x,c)\, dx + \int_a^b H_1(x) f(x,d)\, dx \right]$$

$$+ \frac{1}{4I_{h_2}(d-c)} \left[\int_c^d H_2(y) f(a,y)\, dy + \int_c^d H_2(y) f(b,y)\, dy \right]$$

which give the second and the third inequalities in (2.8).

Now, by using the first ineqaulity in (1.3), we also have

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2I_{h_1}(b-a)} \int_a^b H_1(x) f\left(x, \frac{c+d}{2}\right) dx$$

and

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2I_{h_2}(d-c)} \int_c^d H_2(y) f\left(\frac{a+b}{2}, y\right) dy$$

by addition,

$$
\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq \quad & \frac{1}{4I_{h_1}(b-a)}\int_a^b H_1(x)\, f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{4I_{h_2}(d-c)}\int_c^d H_2(y) f\left(\frac{a+b}{2},y\right)dy
\end{aligned}
$$

which give the first inequality in (2.8).

Finaly, by using the second ineqaulity in (1.3) we can also state,

$$
\begin{aligned}
\frac{1}{2I_{h_1}(b-a)}\int_a^b H_1(x)\, f(x,c)\,dx &\leq \frac{f(a,c)+f(b,c)}{2} \\
\frac{1}{2I_{h_1}(b-a)}\int_a^b H_1(x)\, f(x,d)\,dx &\leq \frac{f(a,d)+f(b,d)}{2} \\
\frac{1}{2I_{h_2}(d-c)}\int_c^d H_2(y) f(a,y)\,dy &\leq \frac{f(a,c)+f(a,d)}{2}
\end{aligned}
$$

and

$$
\frac{1}{2I_{h_2}(d-c)}\int_c^d H_2(y) f(b,y)\,dy \leq \frac{f(b,c)+f(b,d)}{2}
$$

which give, by addition, the last inequality in (2.8). □

Remark 2.6 *If in Theorem 2.5 with* $h_1(t)=t,\ h_2(s)=s$ *on* $[0,1]$, *then the inequalities (2.8) become the inequalities*

$$
\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq \quad & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\
\leq \quad & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
\leq \quad & \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right. \\
& \left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\
\leq \quad & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}
$$

which are proved by Dragomirin [11].

Remark 2.7 *If in Theorem 2.5 with* $h_1(t) = t^{\alpha-1} (\alpha > 0)$, $h_2(s) = s^{\beta-1} (\beta > 0)$ *on* $[0,1]$, *then the inequalities (2.8) become the fractional integral inequalities*

$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\ & \frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b,\frac{c+d}{2}) + J_{b-}^{\alpha} f(a,\frac{c+d}{2})\right] \\
& +\frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J_{c+}^{\beta} f(\frac{a+b}{2},d) + J_{d-}^{\beta} f(\frac{a+b}{2},c)\right] \\
\leq\ & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[J_{a+,c+}^{\alpha,\beta} f(b,d) + J_{a+,d-}^{\alpha,\beta} f(b,c) + J_{b-,c+}^{\alpha,\beta} f(a,d) + J_{b-,d-}^{\alpha,\beta} f(a,c)\right] \\
\leq\ & \frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b,c) + J_{a+}^{\alpha} f(b,d) + J_{b-}^{\alpha} f(a,c) + J_{b-}^{\alpha} f(a,d)\right] \\
& +\frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J_{c+}^{\beta} f(a,d) + J_{c+}^{\beta} f(b,d) + J_{d-}^{\beta} f(a,c) + J_{d-}^{\beta} f(b,c)\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}
\end{aligned}$$

which are proved by Sarikaya in [29].

Corollary 2.8 *Under assumption of Theorem 2.5 with* $h_1(t) = (-\ln t)^{\alpha-1} (\alpha > 0)$, $h_2(s) = (-\ln s)^{\beta-1} (\beta > 0)$ *on* $[0,1]$, *we get the following logarithmic integral inequalities:*

$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\ & \frac{1}{4I_{h_1}(b-a)}\int_a^b \left[\left[\ln\left(\frac{b-a}{b-x}\right)\right]^{\alpha-1} + \left[\ln\left(\frac{b-a}{x-a}\right)\right]^{\alpha-1}\right] f\left(x,\frac{c+d}{2}\right)dx \\
& +\frac{1}{4I_{h_2}(d-c)}\int_c^d \left[\left[\ln\left(\frac{d-c}{d-y}\right)\right]^{\beta-1} + \left[\ln\left(\frac{d-c}{y-c}\right)\right]^{\beta-1}\right] f\left(\frac{a+b}{2},y\right)dy \\
\leq\ & \frac{1}{4I_{h_1}I_{h_2}(b-a)(d-c)}\int_a^b\int_c^d \left[\left[\ln\left(\frac{b-a}{b-x}\right)\right]^{\alpha-1} + \left[\ln\left(\frac{b-a}{x-a}\right)\right]^{\alpha-1}\right] \\
& \times\left[\left[\ln\left(\frac{d-c}{d-y}\right)\right]^{\beta-1} + \left[\ln\left(\frac{d-c}{y-c}\right)\right]^{\beta-1}\right] f(x,y)\,dydx
\end{aligned}$$

$$\leq \frac{1}{8I_{h_1}(b-a)}\int_a^b \left[\left[\ln\left(\frac{b-a}{b-x}\right)\right]^{\alpha-1}+\left[\ln\left(\frac{b-a}{x-a}\right)\right]^{\alpha-1}\right][f(x,c)+f(x,d)]\,dx$$

$$+\frac{1}{8I_{h_2}(d-c)}\int_c^d \left[\left[\ln\left(\frac{d-c}{d-y}\right)\right]^{\beta-1}+\left[\ln\left(\frac{d-c}{y-c}\right)\right]^{\beta-1}\right][f(a,y)+f(b,y)]\,dy$$

$$\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

References

[1] Akkurt A., Sarikaya MZ., Budak B., Yildirim H. On the Hadamard's type inequalities for co-ordinated convex functions via fractional integrals. Journal of King Saud University-Science 2017; 29: 380-387.

[2] Alomari M, Darus M. The Hadamards inequality for s-convex function of 2-variables on the coordinates. International Journal of Mathematical Analysis 2008; 2(13): 629-638.

[3] Azpeitia A.G. , Convex functions and the Hadamard inequality. Revista Colombiana de Matemáticas 1994; 28: 7-12.

[4] Bakula MK. An improvement of the Hermite-Hadamard inequality for functions convex on the coordinates. Australian Journal of Mathematical Analysis and Applications 2014; 11 (1): 1-7.

[5] Belarbi S., Dahmani Z. On some new fractional integral inequalities. Journal of Inequalities in Pure and Applied Mathematics (JIPAM) 2009; 10 (3): 1-12.

[6] Chen F. On Hermite-Hadamard type inequalities for s-convex functions on the coordinates via Riemann-Liouville fractional integrals. Journal of Applied Mathematics 2014; 2014: Article ID 248710, 1-8.

[7] Chen F. A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals. Italian Journal of Pure and Applied Mathematics 2014; 33: 299-306.

[8] Chen F. A note on Hermite-Hadamard inequalities for products of convex functions. Journal of Applied Mathematics 2013; 2013: Article ID 935020, 1-6.

[9] Chen F., Wu S. Several complementary inequalities to inequalities of Hermite-Hadamard type for s-convex functions. Journal of Nonlinear Sciences and Applications 2016; 9: 705-716.

[10] Deng J., Wang J. Fractional Hermite-Hadamard inequalities for (α, m)-logarithmically convex functions. Journal of Inequalities and Applications, 2013; 2013(1): 1-11.

[11] Dragomir SS. On Hadamards inequality for convex functions on the co-ordinates in a rectangle from the plane. Taiwanese Journal of Mathematics 2001; 4: 775-788.

[12] Dragomir SS.,Pearce CEM. Selected Topics on Hermite-Hadamard Inequalities and Applications. RGMIA Monographs, Victoria University, 2000.

[13] Dragomir SS. Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces. Proyecciones Journal of Mathematics 2015; 37 (4): 343-341.

[14] Dragomir SS. Two mappings in connection to Hadamard's inequalities. Journal of Mathematical Analysis and Applications 1992; 167: 49-56.

[15] Dragomir SS., Pecaric J., Persson LE. Some inequalities of Hadamard type. Soochow Journal of Mathematics 1995; 21: 335-341.

[16] Kırmacı US., Bakula MK., Özdemir ME., Pečarić J. Hadamard-tpye inequalities for s-convex functions. Applied Mathematics and Computation 2007; 193: 26-35.

[17] Latif MA., Alomari M. Hadamard-type inequalities for product two convex functions on the co-ordinates. International Mathematical Forum 2009; 4 (47): 2327-2338.

[18] Latif MA. Hermite-Hadamard type inequalities for GA-convex functions on the co-ordinates with applications. Proceedings of the Pakistan Academy of Science 2015; 52 (4): 367-379.

[19] Ozdemir ME, Yildiz C., Akdemir AO. On the co-ordinated convex functions. Applied Mathematics & Information Sciences 2014; 8 (3): 1085-1091.

[20] Ozdemir ME, Latif MA, Akdemir AO. On some Hadamard-type inequalities for product of two s-convex functions on the co-ordinates. Journal of Inequalities and Applications 2012; 21: 1-13.

[21] Ozdemir ME., Set E., Sarikaya MZ. New some Hadamard's type inequalities for co-ordinated m-convex and (α, m)-convex functions. Hacettepe Journal of Mathematics and Statistics 2011; 40 (2): 219-229.

[22] Ozdemir ME, Latif MA, Akdemir AO. On some Hadamard-type inequalities for product of two h-convex functions on the co-ordinates. Turkish Journal of Science 2016; 1: 41-58.

[23] Pachpatte BG. On some inequalities for convex functions. Research Group in Mathematical Inequalities and Applications 2003; 6 (E): 1-9.

[24] Pavic Z. Improvements of the Hermite-Hadamard inequality. Journal of Inequalities and Applications 2015; 2015 (1): 1-11.

[25] Pečarić JE., Proschan F., Tong YL. Convex Functions, Partial Orderings and Statistical Applications. Academic Press, Boston, 1992.

[26] Peng S., Wei W., Wang JR. On the Hermite-Hadamard inequalities for convex functions via Hadamard fractional integrals. Facta Universitatis, Series Mathematics and Informatics 2014; 29 (1): 55-75.

[27] Sarikaya MZ, Budak H. Generalized Hermite-Hadamard type integral inequalities for fractional integrals. Filomat 2016; 30 (5): 1315-1326.

[28] Sarikaya MZ,, Set E., Yaldiz H., Basak N. Hermite -Hadamard's inequalities for fractional integrals and related fractional inequalities. Mathematical and Computer Modelling 2013; 57: 2403-2407.

[29] Sarikaya MZ, *On the Hermite-Hadamard-type inequalities for co-ordinated convex function via fractional integrals.* Integral Transforms and Special Functions 2014; 25 (2): 134-147.

[30] Sarikaya MZ., Ogunmez H. On new inequalities via Riemann-Liouville fractional integration. Abstract and Applied Analysis 2012; Volume 2012: Article ID 428983, 10 pages.

[31] Set E., Ozdemir ME., Dragomir SS. On the Hermite-Hadamard inequality and other integral inequalities involving two functions. Journal of Inequalities and Applications 2010; 9: 1-9.

[32] Tseng KL., Hwang SR. New Hermite-Hadamard inequalities and their applications. Filomat 2016; 30 (14): 3667-3680.

[33] Wang DY., Tseng KL., Yang GS. Some Hadamard's inequalities for co-ordinated convex functions in a rectangle from the plane. Taiwanese Journal of Mathematics 2007; 11: 63-73.

[34] Wang JR., Li X., Zhu C. Refinements of Hermite-Hadamard type inequalities involving fractional integrals. Bulletin of the Belgian Mathematical Society - Simon Stevin 2013; 20: 655-666.

[35] Xi BY., Hua J., Qi F. Hermite-Hadamard type inequalities for extended s-convex functions on the co-ordinates in a rectangle. Journal of Applied Analysis. 2014; 20 (1): 1-17.

[36] Yaldiz H., Sarikaya MZ, Dahmani Z. On the Hermite-Hadamard-Fejer-type inequalities for co-ordinated convex functions via fractional integrals. International Journal of Optimization and Control: Theories & Applications (IJOCTA) 2017; 7 (2): 205-215.

[37] Yang GS., Tseng KL. On certain integral inequalities related to Hermite-Hadamard inequalities. Journal of Mathematical Analysis and Applications 1999; 239: 180-187.

[38] Yang GS., Hong MC. A note on Hadamard's inequality. Tamkang Journal of Mathematics 1997; 28: 33-37.

[39] Yıldırım ME., Akkurt A., Yıldırım H. Hermite-Hadamard type inequalities for co-ordinated $(\alpha_1, m_1) - (\alpha_2, m_2)$-convex functions via fractional integrals. Contemporary Analysis and Applied M athematics 2016; 4 (1): 48-63.

[40] Sarikaya MZ., Set E., Ozdemir ME., Dragomir SS. New some Hadamard's type inequalities for co-ordinated convex functions. Tamsui Oxford Journal of Information and Mathematical Sciences 2012; 28 (2): 137-152.

[41] Sarikaya MZ., Yaldiz H. On weighted Montogomery identities for Riemann-Liouville fractional integrals. Konuralp Journal of Mathematics 2013; 1 (1) 48-53.

[42] Zabandan G. An extension and refinement of Hermite-Hadamard inequality and related results. International Journal of Nonlinear Analysis and Applications 2020; 11 (2): 379-390.

[43] Mohammed PO. On new trapezoid type inequalities for h-convex functions via generalized fractional integral. Turkish Journal of Analysis and Number Theory 2018; 6: 125-128.

[44] Mohammed PO., Sarikaya MZ. On generalized fractional integral inequalities for twice differentiable convex functions. Journal of Computational and Applied Mathematics 2020; 372: 112740.

[45] Qi F., Mohammed PO., Yao JC., Yao YH. Generalized fractional integral inequalities of Hermite–Hadamard type for (α, m)-convex functions. Journal of Inequalities and Applications 2019; 2019 (1): 1-17.