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# Scattering and characteristic functions of a dissipative operator generated by a system of equations

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**Abstract:** In this paper, we consider a system of first-order equations with the same eigenvalue parameter together with dissipative boundary conditions. Applying Lax-Phillips scattering theory and Sz.-Nagy-Foiaş model operator theory we prove a completeness theorem.

**Key words:** Boundary-value problem, dissipative operator, completeness theorem

## 1. Introduction

In this paper we will investigate some spectral properties of a dissipative operator generated by a system of equations that contains the continuous analogous of orthogonal polynomials on the unit-circle. However, firstly, we shall share some background information on orthogonal polynomials and related differential equations. For the basic theory and fundamental results we refer to [1], [5], [6], [7], [8], [10], [17], [18].

The theory of orthogonal polynomials has two directions: orthogonal polynomials on the real-line and orthogonal polynomials on the unit-circle. Orthogonal polynomials on the real-line are mainly related with finding suitable coefficients

$$\xi_{k,l} = \int \lambda^{k+l} d\boldsymbol{\mu},$$

where $k, l$ are some integers with $0 \leq k, l \leq n-1$, $\boldsymbol{\mu}$ is a measure on the real-line such that

$$y_n(\lambda) = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} \xi_{0,0} & \xi_{1,0} & \cdots & \xi_{n,0} \\ \vdots & \vdots & & \vdots \\ \xi_{0,n-1} & \xi_{1,n-1} & \cdots & \xi_{n,n-1} \\ 1 & \lambda & \cdots & \lambda^n \end{vmatrix} \tag{1.1}$$

and

$$D_n := \det(\xi_{r,s})_{r,s=0}^{n}. \tag{1.2}$$

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Here $D_n$ is the Hankel determinant. Polynomials given by (1.1) together with (1.2) satisfy the following equations

$$\int y_n(\lambda) y_m(\lambda) d\boldsymbol{\mu} = \delta_{nm}, \tag{1.3}$$

where $\delta_{nm}$ is the Kronecker delta and

$$y_n(\lambda) = \sum_{k=0}^{n} \eta_k \lambda^k, \ \eta_n > 0, \ n = 0, 1, 2, .... \tag{1.4}$$

One of the interesting results on such polynomials is that they satisfy a three-term recurrence relation

$$\frac{\eta_n}{\eta_{n+1}} y_{n+1} + \left( \int \lambda y_n^2 d\mu \right) y_n + \frac{\eta_{n-1}}{\eta_n} y_{n-1} = \lambda y_n$$

which can be embedded into the following

$$c_n y_{n+1} = (\lambda a_n + b_n) y_n - c_{n-1} y_{n-1}, \ n = 0, 1, 2, .... \tag{1.5}$$

Eq. (1.5) is a kind of discrete version of the following second-order Sturm-Liouville differential equation [6]

$$(c(x) y')' = (\lambda a(x) + b(x)) y, \tag{1.6}$$

where $a(x)$, $b(x)$ are some functions on a certain interval. Beside this relation between Eq. (1.5) satisfied by orthogonal polynomials on the real-line and second-order differential equation (1.6) there exists a huge number of works on the equations related with (1.5) and (1.6) (for example, see [6]).

The theory of orthogonal polynomials on the unit-circle which is the other direction of the theory of orthogonal polynomials is related with finding suitable coefficients

$$\xi_{k,l} = \int \lambda^{k-l} d\boldsymbol{\mu}, \tag{1.7}$$

where $0 \leq k, l \leq n-1$, $\boldsymbol{\mu}$ is now a measure on the unit-circle. Corresponding polynomials (1.1) constructed by (1.7) with (1.2) satisfy (1.3) and are of the form (1.4). We shall note that (1.2) is now known as Toeplitz determinant. The polynomials $y_n(\lambda)$ constructed by (1.1) with replacing $(D_n D_{n-1})^{-1/2}$ by $D_{n-1}^{-1}$ satisfy the relations

$$\begin{aligned} y_{n+1} &= \lambda y_n - \eta_n y_n^*, \\ y_{n+1}^* &= y_n^* - \lambda \overline{\eta}_n y_n, \end{aligned} \tag{1.8}$$

where $y_n^*(\lambda) = \lambda^n \overline{y}_n(\lambda^{-1})$ and $n = 0, 1, 2, ....$ Relations (1.8) can be embedded into the following relations

$$\begin{aligned} y_{n+1} &= \lambda a_n y_n + b_n z_n, \\ z_{n+1} &= \lambda \overline{b}_n y_n + a_n z_n, \end{aligned} \tag{1.9}$$

where $n = 0, 1, 2, ....$ Continuous version of the relations (1.8) or (1.9) with $a_n = 1$ is the following

$$\begin{aligned} y' &= i\mu y + b(x) z, \\ z' &= \overline{b(x)} y, \end{aligned} \tag{1.10}$$

where $b(x)$ is a continuous function on a certain interval. The system of Eq.s (1.10) has been firstly introduced by Krein with the conditions $y(0) = 1,\ z(0) = 1$ using some properties of integral equations containing Hermitian kernels [11]. However, much more direct approach has been given by Atkinson [6].

A generalization of the Krein system (1.10) has been introduced by Sakhnovich [16] with the aid of the following canonical system

$$Y' = i\mu JH(x)Y,$$

where $Y$ is a $2r \times 1$ vector, $J$ and $H(x)$ are $2r \times 2r$ matrices with $H(x) \geq 0$ and

$$J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Here $I$ is the $r \times r$ identity matrix and $JH(x)$ is linearly similar to the following matrix

$$\begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix},\ W = diag\left\{w_1, ..., w_r\right\},\ w_k = const. > 0.$$

Then Sakhnovich introduced the following system of equations

$$\begin{aligned} Y_1' &= i\mu W Y_1 + P(x)Y_1 + Q^*(x)Y_2, \\ Y_2' &= Q(x)Y_1, \end{aligned} \tag{1.11}$$

where $Y_1, Y_2, P, Q$ are $r \times r$ matrices such that $P = -P^*$. For $r = 1$, (1.11) gives

$$\begin{aligned} y_1' &= i\mu w y_1 + p(x)y_1 + \overline{q(x)}y_2,\ p = -\overline{p}, \\ y_2' &= q(x)y_1, \end{aligned} \tag{1.12}$$

and now (1.12) contains (1.10) with $w = 1$ and $p(x) \equiv 0$.

In this paper, we will consider much more general version of (1.12) as can be seen in (2.1). We aim to introduce some spectral properties of the system (2.1) together with well-defined boundary conditions. Indeed, we will construct an operator related with the problem and we will apply Pavlov's method [15] which uses the equivalence of Lax-Phillips scattering function [12] and Sz.-Nagy-Foiaş characteristic function [13]. In the literature, there has been a huge number of works that Pavlov's method has been applied (for example, see, [2]-[4], [19], [20]). However, we should note that, in this paper, we will apply this method for the first time to the operator generated by the system of equations which is a general version of the continuous analogous of the orthogonal polynomials on the unit circle. We shall also note that this problem has been investigated in [21] with the aid of the coordinate-free approach.

## 2. System of equations

Throughout the paper we will consider the following system of equations

$$\begin{aligned} y_1' &= i\lambda w(x)y_1 + ip(x)y_1 + q(x)y_2, \\ y_2' &= -i\lambda r(x)y_2 + \overline{q(x)}y_1 + is(x)y_2, \end{aligned} \tag{2.1}$$

where all the coefficients are defined on the interval $[a,b]$ such that $w>0$, $r>0$ and continuous functions on $[a,b]$, $p,s$ are real-valued and $q$ is a complex-valued function on $[a,b]$ and they are all continuous functions on $[a,b]$.

We can introduce the Hilbert space $H$ corresponding to the system of equations (2.1) with the inner product

$$(Y,Z)=\int_a^b (y_1\overline{z}_1 w+y_2\overline{z}_2 r)\,dx,$$

where

$$Y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, Z=\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\in H.$$

System of equations (2.1) can also be represented as the following first-order equation

$$Y'=[-i\lambda A+B]\,Y, \tag{2.2}$$

where

$$A=\begin{bmatrix} -w & 0 \\ 0 & r \end{bmatrix},\ B=\begin{bmatrix} ip & q \\ \overline{q} & is \end{bmatrix}.$$

Using this representation (2.2) and the assumptions on the coefficients of the equations in (2.1) we can introduce the following.

**Theorem 2.1.** *The system of equations (2.1) has a unique pair of solutions $y_1(x,\lambda)$, $y_2(x,\lambda)$ satisfying the initial conditions*

$$y_1(c,\lambda)=c_1,\ y_2(c,\lambda)=c_2,$$

*where $c\in[a,b]$ and $c_1,c_2$ are arbitrary complex numbers. Moreover, $y_1(.,\lambda)$ and $y_2(.,\lambda)$ are entire functions of $\lambda$.*

Let us adopt the notation

$$[Y,Z]:=-iy_1z_1+iy_2z_2, \tag{2.3}$$

where $Y,Z\in H$. Then we can introduce the following formula.

**Theorem 2.2.** *Let $Y(x,\lambda)$ and $Z(x,\mu)$ be the solutions of (2.2) corresponding with the parameters $\lambda$ and $\mu$, respectively. Then*

$$(\lambda-\overline{\mu})\int_a^b \left\{y_1(x,\lambda)\overline{z_1(x,\mu)}w(x)+y_2(x,\lambda)\overline{z_2(x,\mu)}r(x)\right\}dx=[Y(x,\lambda),\overline{Z(x,\mu)}]\mid_a^b,$$

*where $[Y,\overline{Z}]\mid_a^b:=[Y,\overline{Z}](b)-[Y,\overline{Z}](a)$.*

**Proof** The proof follows from the representation (2.3) and the equations in (2.1). In fact, for $Y(x,\lambda)$ and $Z(x,\mu)$ we have the following

$$\begin{aligned}
&\tfrac{\partial}{\partial x}[Y,\overline{Z}] = -i(i\lambda w y_1 + ipy_1 + qy_2)\overline{z}_1 - iy_1(-i\overline{\mu}w\overline{z}_1 - ip\overline{z}_1 + \overline{q}\overline{z}_2)\\
&+i(-i\lambda r y_2 + \overline{q}y_1 + isy_2)\overline{z}_2 + iy_2(i\overline{\mu}r\overline{z}_2 + q\overline{z}_1 - is\overline{z}_2)\\
&= (\lambda-\overline{\mu})y_1\overline{z}_1 w + (\lambda-\overline{\mu})y_2\overline{z}_2 r.
\end{aligned} \tag{2.4}$$

Integration of (2.4) over $[a,b]$ completes the proof. □

Now consider the subspace $S$ of $H$ such that the pair of functions $Y=\begin{bmatrix} y_1\\ y_2\end{bmatrix}$ in $S$ satisfy the following

$$\begin{bmatrix} \frac{-i}{w}\{y_1' - ipy_1 - qy_2\} \\ \frac{i}{r}\{y_2' - \overline{q}y_1 - isy_2\} \end{bmatrix} \in H.$$

We define the operator $L$ on the subspace $S$ with the following rule

$$LY = \begin{bmatrix} \frac{-i}{w}\{y_1' - ipy_1 - qy_2\} \\ \frac{i}{r}\{y_2' - \overline{q}y_1 - isy_2\} \end{bmatrix}, \; Y = \begin{bmatrix} y_1\\ y_2\end{bmatrix} \in S.$$

Now we consider a subspace $S_0$ of $S$ with the pair of functions $Y=\begin{bmatrix} y_1\\ y_2\end{bmatrix}$ satisfying the conditions

$$y_1(a) = y_2(a) = y_1(b) = y_2(b) = 0.$$

Then the restriction $L_0$ of $L$ to the subspace $S_0$ will be a symmetric, closed operator and $L_0^* = L$. These results can be obtained using similar steps introduced in [14].

We note that the deficiency indices of $L_0$ are $(2,2)$. For the details on deficiency indices of a symmetric, closed operator we refer to [14].

**Definition 2.3.** *Let $A$ be an operator with domain $D(A)$ and $A^*$ be its adjoint operator with domain $D(A^*)$ in the Hilbert space $K$ and $A$ has an equal deficiency indices. The triple $(M,\Gamma_1,\Gamma_2)$, where $M$ is a Hilbert space, is called boundary value space of $A$ if the following is satisfied*

(1) $(A^*f,g) - (f,A^*g) = (\Gamma_1 f,\Gamma_2 g) - (\Gamma_2 f,\Gamma_1 g)$ *for all* $f,g \in D(A^*)$,

(2) *for any* $F_1,F_2 \in M$ *there exists a function* $f \in D(A^*)$ *such that* $\Gamma_1 f = F_1$, $\Gamma_2 f = F_2$, *where* $\Gamma_{1,2} : D(A^*) \to M$ *are linear mappings.*

We define the following mappings

$$\Gamma_1 Y = \begin{bmatrix} \frac{i}{2}y_1(b) - \frac{1}{2}y_2(b) \\ \frac{i}{2}y_1(a) - \frac{1}{2}y_2(a) \end{bmatrix}, \; \Gamma_2 Y = \begin{bmatrix} -y_1(b) + iy_2(b) \\ y_1(a) - iy_2(a) \end{bmatrix}, \; Y = \begin{bmatrix} y_1\\ y_2\end{bmatrix} \in S.$$

It is known that for any symmetric operator with deficiency indices $(n,n)$ $(n \le \infty)$ there exists a boundary value space $(M,\Gamma_1,\Gamma_2)$ with $\dim M = n$ [9]. Then we can introduce the following.

**Lemma 2.4.** $(\mathbb{C}^2,\Gamma_1,\Gamma_2)$ *is a space of boundary values of* $L_0$.

**Proof** For $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in S$ we obtain that

$$\begin{aligned}
&(\Gamma_1 Y, \Gamma_2 Z) - (\Gamma_2 Y, \Gamma_1 Z) = \left(\tfrac{i}{2} y_1(b) - \tfrac{1}{2} y_2(b)\right)\left(-\overline{z}_1(b) - i\overline{z}_2(b)\right) \\
&+ \left(\tfrac{i}{2} y_1(a) - \tfrac{1}{2} y_2(a)\right)\left(\overline{z}_1(a) + i\overline{z}_2(a)\right) - \left(-y_1(b) + iy_2(b)\right)\left(-\tfrac{i}{2}\overline{z}_1(b) - \tfrac{1}{2}\overline{z}_2(b)\right) \\
&- \left(y_1(a) - iy_2(a)\right)\left(-\tfrac{i}{2}\overline{z}_1(a) - \tfrac{1}{2}\overline{z}_2(a)\right) = -iy_1(b)\overline{z}_1(b) + iy_2(b)\overline{z}_2(b) \\
&+ iy_1(a)\overline{z}_1(a) - iy_2(a)\overline{z}_2(a) = [Y, \overline{Z}] \mid_a^b .
\end{aligned} \tag{2.5}$$

On the other side from Theorem 2.2 we get for $Y, Z \in S_0^*$, where $S_0^*$ is the domain of the adjoint operator $L_0^*$ of $L$, that

$$(L_0^* Y, Z) - (Y, L_0^* Z) = [Y, \overline{Z}] \mid_a^b . \tag{2.6}$$

Using (2.5) and (2.6) we prove the first statement. For the proof of the second statement, a similar argument of Naimark's patching lemma can be applied [14]. Therefore we complete the proof. □

Using Lemma 2.4 we can introduce the following abstract maximal dissipative boundary condition with the help of Theorem 1.6 in [9], Chap. 3.

**Theorem 2.5.** *Let $C$ be a contraction on $\mathbb{C}^2$ and $I$ denote the $2 \times 2$ identity matrix. Then the condition*

$$(C - I)\Gamma_1 Y + i(C + I)\Gamma_2 Y = 0$$

*describes the maximal dissipative extension of $L_0$, where $Y \in S$.*

Then we have the following.

**Corollary 2.6.** *For $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in S$ following boundary conditions*

$$\begin{aligned}
&(ih_1 + 1)y_1(a) - (h_1 + i)y_2(a) = 0, \\
&(ih_2 - 1)y_1(b) + (-h_2 + i)y_2(b) = 0,
\end{aligned} \tag{2.7}$$

*where $h_1$ is a real number and $h_2$ is a complex number with $\Im h_2 > 0$ describe the maximal dissipative extension of $L_0$.*

Our aim is to investigate the spectral properties of the problem generated by (2.1) and (2.7). For this purpose we shall construct a suitable operator related with this problem.

Let $D$ be a subspace of $H$ consisting of all pair of functions $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in S$ satisfying the conditions (2.7). We define the operator $\mathcal{L}$ on $D$ with the rule

$$\mathcal{L}Y = LY, \ Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in D.$$

Note that $\mathcal{L}$ is a maximal dissipative operator on $H$.

**Theorem 2.7.** *There is no subspace of $D$ such that the restriction of $\mathcal{L}$ to this subspace is selfadjoint. In other words, $\mathcal{L}$ is simple.*

**Proof** We prove the theorem by assuming the contrary. So suppose that $D_1$ is a subspace of $D$ such that the restriction $\mathcal{L}_1$ of $\mathcal{L}$ to $D_1$ is selfadjoint. So for $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in D_1$ one gets

$$\begin{aligned} 0 &= (\mathcal{L}Y, Y) - (Y, \mathcal{L}Y) = -iy_1(b)\overline{y_1(b)} + iy_2(b)\overline{y_2(b)} \\ &= -i\left[\frac{-h_2+i}{ih_2-1}\frac{-\overline{h}_2-i}{-i\overline{h}_2-1} - 1\right]|y_2(b)|^2 = i\frac{4\Im h_2}{|ih_2-1|^2}|y_2(b)|^2 . \end{aligned}$$

So $y_1(b) = y_2(b) = 0$ and hence $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \equiv 0$. This completes the proof. □

## 3. Selfadjoint dilation

In this section we will construct an unitary operator family which will be used in the next section to construct a scattering function related with the operator $\mathcal{L}$.

Let us construct the following Hilbert space

$$\mathbf{H} = L^2((-\infty,0);\mathbb{C}) \oplus H \oplus L^2((0,\infty);\mathbb{C}),$$

where $L^2((-\infty,0);\mathbb{C})$ and $L^2((0,\infty);\mathbb{C})$ are Hilbert spaces with integrable square functions with ranges in $\mathbb{C}$ on $(-\infty,0)$ and $(0,\infty)$, respectively. We consider a subspace $\mathbf{D}$ in $\mathbf{H}$ with vector-functions $\mathbf{y} = \langle f_-, Y, f_+\rangle \in \mathbf{H}$ such that

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in D,\ f_- \in W_2^1((-\infty,0);\mathbb{C}),\ f_+ \in W_2^1((0,\infty);\mathbb{C})$$

and

$$\begin{aligned} f_-(0) &= \frac{1}{2\sqrt{\Im h_2}}\left[(-1+ih_2)y_1(b) + (i-h_2)y_2(b)\right], \\ f_+(0) &= \frac{1}{2\sqrt{\Im h_2}}\left[(-1+i\overline{h}_2)y_1(b) + (i-\overline{h}_2)y_2(b)\right], \end{aligned}$$

where $W_2^1$ denotes the Sobolev space.

On the subspace $\mathbf{D}$ we define the operator $\mathbf{L}$ with the following rule

$$\mathbf{L}\mathbf{y} = \left\langle i\frac{df_-}{dr}, \mathcal{L}Y, i\frac{df_+}{ds}\right\rangle,\ \mathbf{y} \in \mathbf{D}.$$

Then we have the following.

**Theorem 3.1.** $\mathbf{L}$ *is selfadjoint on* $\mathbf{H}$.

**Proof** First of all we shall show that $\mathbf{L}$ is symmetric. For this purpose we consider two arbitrary vector-functions $\mathbf{y} = \langle f_-, Y, f_+\rangle$, $\mathbf{z} = \langle g_-, Z, g_+\rangle \in \mathbf{D}$. Then we get

$$(\mathbf{L}\mathbf{y}, \mathbf{z}) - (\mathbf{y}, \mathbf{L}\mathbf{z}) = i\,(f_-(0), g_-(0)) - i\,(f_+(0), g_+(0)) - iy_1(b)\overline{z}_1(b) + iy_2(b)\overline{z}_2(b). \tag{3.1}$$

On the other side, a direct calculation shows that

$$i\left(f_{-}(0), g_{-}(0)\right) - i\left(f_{+}(0), g_{+}(0)\right) = iy_1(b)\overline{z}_1(b) - iy_2(b)\overline{z}_2(b). \tag{3.2}$$

Therefore from (3.1) and (3.2) we can infer that $\mathbf{L}$ is symmetric.

For $\mathbf{y} = \langle f_{-}, 0, f_{+}\rangle \in \mathbf{D}$ such that $f_{-}(0) = f_{+}(0) = 0$ and an arbitrary vector-function $\mathbf{z} = \langle g_{-}, Z, g_{+}\rangle \in \mathbf{D}$ we get that

$$(\mathbf{Ly}, \mathbf{z}) = \left(\left\langle i\frac{df_{-}}{dr}, 0, i\frac{df_{+}}{ds}\right\rangle, \langle g_{-}, Z, g_{+}\rangle\right) = \left(\langle f_{-}, 0, f_{+}\rangle, \left\langle i\frac{dg_{-}}{dr}, Z^{*}, i\frac{dg_{+}}{ds}\right\rangle\right). \tag{3.3}$$

Therefore adjoint operator $\mathbf{L}^{*}$ of $\mathbf{L}$ should be of the following form

$$\mathbf{L}^{*}\mathbf{z} = \left\langle i\frac{dg_{-}}{dr}, Z^{*}, i\frac{dg_{+}}{ds}\right\rangle, \; Z^{*} \in D.$$

Now letting $\mathbf{y} = \langle 0, Y, 0\rangle \in \mathbf{D}$ we obtain that

$$\begin{aligned}(\mathbf{Ly}, \mathbf{z}) &= \left(\langle 0, \mathcal{L}Y, 0\rangle, \langle g_{-}, Z, g_{+}\rangle\right) = \left(\langle 0, Y, 0\rangle, \left\langle i\tfrac{dg_{-}}{dr}, \mathcal{L}Z, i\tfrac{dg_{+}}{ds}\right\rangle\right) \\ &-iy_1(b)\overline{z}_1(b) + iy_2(b)\overline{z}_2(b).\end{aligned} \tag{3.4}$$

Comparing (3.4) with (3.3) and (3.1) we infer that

$$i\left(f_{-}(0), g_{-}(0)\right) - i\left(f_{+}(0), g_{+}(0)\right) - iy_1(b)\overline{z}_1(b) + iy_2(b)\overline{z}_2(b) = 0$$

or

$$\left(f_{+}(0), g_{+}(0)\right) - \left(f_{-}(0), g_{-}(0)\right) = -y_1(b)\overline{z}_1(b) + y_2(b)\overline{z}_2(b). \tag{3.5}$$

Using the boundary conditions we obtain from (3.5) that

$$\begin{aligned}&\tfrac{1}{2\sqrt{Imh_2}}\left[\left(1 - i\overline{h}_2\right)\overline{g}_{+}(0) + \left(-1 + ih_2\right)\overline{g}_{-}(0)\right] = \overline{z}_1(b), \\ &\tfrac{1}{2\sqrt{Imh_2}}\left[\left(i - \overline{h}_2\right)\overline{g}_{+}(0) - \left(i - h_2\right)\overline{g}_{-}(0)\right] = \overline{z}_2(b).\end{aligned} \tag{3.6}$$

Solving the equations in (3.6) in $\overline{g}_{+}(0)$ and $\overline{g}_{-}(0)$ we get that

$$\begin{aligned}&\tfrac{1}{2\sqrt{Imh_2}}\left[-4iImh_2\overline{g}_{-}(0)\right] = (i - \overline{h}_2)\overline{z}_1(b) + (-1 + i\overline{h}_2)\overline{z}_2(b), \\ &\tfrac{1}{2\sqrt{Imh_2}}\left[-4iImh_2\overline{g}_{+}(0)\right] = (i - h_2)\overline{z}_1(b) + (-1 + ih_2)\overline{z}_2(b),\end{aligned}$$

or equivalently

$$\begin{aligned}g_{-}(0) &= \tfrac{1}{2\sqrt{Imh_2}}\left[(-1 + ih_2)z_1(b) + (i - h_2)z_2(b)\right], \\ g_{+}(0) &= \tfrac{1}{2\sqrt{Imh_2}}\left[(-1 + i\overline{h}_2)z_1(b) + (i - \overline{h}_2)z_2(b)\right].\end{aligned} \tag{3.7}$$

Therefore $\mathbf{L}^{*} \subseteq \mathbf{L}$ and this completes the proof. □

Now consider the projections

$$\begin{array}{llll} P_1: & \mathbf{H} & \to & H, \\ & \langle f_{-}, Y, f_{+}\rangle & \to & Y, \end{array} \qquad \begin{array}{llll} P_2: & H & \to & \mathbf{H}, \\ & Y & \to & \langle 0, Y, 0\rangle. \end{array}$$

It is known that $U(t) := \exp(i\mathbf{L}t),\ t \in \mathbb{R},$ constitutes an unitary group and $Z(t) := P_1U(t)P_2$ is a strongly continuous semigroup of completely nonunitary contractions. The generator of $Z(t)$ is defined by

$$GY = \lim_{t\to 0^+} \frac{1}{it}\left(Z(t)Y - Y\right)$$

with the domain consisting of all vectors such that this limit exists. Note that $G$ is a maximal dissipative operator on $H$ and $\mathbf{L}$ is the selfadjoint dilation of $G$.

**Theorem 3.2.** $\mathbf{L}$ *is a selfadjoint dilation of* $\mathcal{L}$.

**Proof** We will prove this assertion by showing that $\mathcal{L} = G$.

Firstly, we obtain for $\lambda$ with $Im\lambda < 0$ that

$$P_1(\mathbf{L} - \lambda I)^{-1}P_2 = -iP_1\int_0^\infty U(t)e^{-i\lambda t}dtP_2 = -i\int_0^\infty Z(t)e^{-i\lambda t}dt = (G - \lambda I)^{-1}. \tag{3.8}$$

On the other side the equation

$$\left(\mathbf{L} - \lambda I\right)\mathbf{z} = \langle 0, Y, 0\rangle = P_2Y,$$

where $\mathbf{z} = \langle f_-, Z, f_+\rangle \in \mathbf{D},\ Y \in H,$ implies that

$$\begin{aligned} &f_-(r) = f_-(0)e^{-i\lambda r},\\ &f_+(s) = f_+(0)e^{-i\lambda s},\\ &\mathcal{L}Z - \lambda Z = Y, \end{aligned}$$

and for $Im\lambda < 0$ that

$$(\mathbf{L} - \lambda I)^{-1}P_2Y = \left\langle 0, (\mathcal{L} - \lambda I)^{-1}Y, \frac{e^{-i\lambda s}}{2\sqrt{Imh_2}}\left[(-1 + i\overline{h}_2)y_1(b) + (i - \overline{h}_2)y_2(b)\right]\right\rangle \tag{3.9}$$

as $f_- \in L^2((-\infty, 0); \mathbb{C})$. Applying $P_1$ to (3.9) we get

$$P_1(\mathbf{L} - \lambda I)^{-1}P_2Y = (\mathcal{L} - \lambda I)^{-1}Y. \tag{3.10}$$

Then (3.8) and (3.10) complete the proof. □

## 4. Scattering function

In this section we will construct a scattering function related with the unitary operator $U(t)$.

Firstly, we shall consider the pair of solutions $\varphi_1(x, \lambda),\ \varphi_2(x, \lambda)$ of (2.1) satisfying the initial conditions

$$\varphi_1(a, \lambda) = h_1 + i,\ \varphi_2(a, \lambda) = ih_1 + 1.$$

Clearly, the roots of the function $\Delta_{h_2}(\lambda)$ defined by

$$\Delta_{h_2}(\lambda) = (ih_2 - 1)\varphi_1(b, \lambda) + (-h_2 + i)\varphi_2(b, \lambda)$$

coincide with the eigenvalues of the operator $\mathcal{L}$. Using Theorem 2.1 and Theorem 2.7 we can introduce the following.

**Theorem 4.1.** *All eigenvalues of $\mathcal{L}$ belong to the open upper-half plane with possible limiting point infinity.*

Now we shall construct the following function

$$S(\lambda) = \frac{\Delta_{h_2}(\lambda)}{\Delta_{\overline{h}_2}(\lambda)}.$$

Then we obtain the following.

**Lemma 4.2.** *For $Im\lambda = 0$ we have*

$$|S(\lambda)| = 1,$$

*and for $Im\lambda > 0$ we have*

$$|S(\lambda)| < 1.$$

**Proof** These results follow from the following equation

$$1 - S^* S = \frac{8\ Imh_2\ Im\lambda \int\limits_a^b \left\{ |\varphi_1|^2 w + |\varphi_2|^2 r \right\} dx}{\left|\Delta_{\overline{h}_2}(\lambda)\right|^2}.$$

□

For the aim of construction of the scattering function we need to set suitable subspaces of $\mathbf{H}$. Therefore, we set the corresponding subspaces $D_-$ and $D_+$ as follows:

$$D_- = \left\langle L^2((-\infty, 0); \mathbb{C}), 0, 0 \right\rangle,\ D_+ = \left\langle 0, 0, L^2((0, \infty); \mathbb{C}) \right\rangle.$$

**Theorem 4.3.** *Following properties hold*

(1) $U(t)D_- \subset D_-,\ t \leq 0;\ U(t)D_+ \subset D_+,\ t \geq 0,$

(2) $\bigcap_{t \leq 0} U(t)D_- = \bigcap_{t \geq 0} U(t)D_+ = \{0\},$

(3) $\overline{\bigcup_{t \geq 0} U(t)D_-} = \overline{\bigcup_{t \leq 0} U(t)D_+} = \mathbf{H},$

(4) $D_- \perp D_+.$

**Proof** First of all, we shall show that $U(t)D_+ \subset D_+$ for $t \geq 0$. Let $\mathbf{y} = \langle 0, 0, f \rangle \in D_+$. For $Im\lambda < 0$ we have the following

$$(\mathbf{L} - \lambda I)^{-1}\mathbf{y} = \left\langle 0, 0, -ie^{-i\lambda s} \int\limits_0^s e^{i\lambda t} f(t) dt \right\rangle.$$

Therefore $(\mathbf{L}-\lambda I)^{-1}\mathbf{y}\in D_+$. Considering an arbitrary element $\mathbf{z}\perp D_+$ we get

$$0=\big((\mathbf{L}-\lambda I)^{-1}\mathbf{y},\mathbf{z}\big)=-i\int\limits_0^\infty e^{-i\lambda t}\left(U(t)\mathbf{y},\mathbf{z}\right)dt.$$

Therefore $(U(t)\mathbf{y},\mathbf{z})=0$ for all $t\geq 0$. Consequently, for all $t\geq 0$ we get $U(t)D_+\subset D_+$.

Other assertion $U(t)D_-\subset D_-$, $t\leq 0$, can be proved similarly and this completes the proof of (1).

For the proof of (2) we shall consider the following projections

$$\begin{array}{lllllll}\rho_1: & \mathbf{H} & \rightarrow & L^2((0,\infty);\mathbb{C}), & \rho_2: & L^2((0,\infty);\mathbb{C}) & \rightarrow & \mathbf{H}, \\ & \langle f_-,Y,f_+\rangle & \rightarrow & f_+, & & f & \rightarrow & \langle 0,0,f\rangle\,.\end{array}$$

Then we can construct the semigroup of isometries $U_1(t)=\rho_1 U(t)\rho_2$, $t\geq 0$, with the generator $G_1$ such that

$$G_1f=\rho_1\mathbf{L}\rho_2 f=\rho_1\mathbf{L}\left\langle 0,0,f\right\rangle=\rho_1\left\langle 0,0,i\frac{df}{ds}\right\rangle=i\frac{df}{ds},$$

where $f\in W_2^1((0,\infty);\mathbb{C})$ and $f(0)=0$. On the other side, the operator $id/ds$ with the boundary condition $f(0)=0$ can be represented as the generator of the semigroup $\widetilde{U}_1(t)$ of the one-sided shift in $L^2((0,\infty);\mathbb{C})$. Since a semigroup is uniquely determined by its generator we have $U_1(t)=\widetilde{U}_1(t)$ and therefore we obtain

$$\bigcap_{t\geq 0}U_1(t)D_+=\left\langle 0,0,\bigcap_{t\geq 0}\widetilde{U}_1(t)L^2((0,\infty);\mathbb{C})\right\rangle=\{0\}\,.$$

Therefore this implies the first part of the proof of (2). Other part can be proved similarly.

For the proof of (3) we shall consider the following spaces

$$\mathbf{H}_1=\overline{\bigcup_{t\geq 0}U(t)D_-},\ \mathbf{H}_2=\overline{\bigcup_{t\leq 0}U(t)D_+}.$$

Then we can infer that

$$\mathbf{H}=\mathbf{H}_1\oplus\mathbf{H}_2$$

as otherwise the restriction of $U(t)$ to the subspace $\mathbf{H}\ominus(\mathbf{H}_1\oplus\mathbf{H}_2)$ were unitary and $\mathcal{L}$ would have a selfadjoint part there. Now our aim is to show that $\mathbf{H}=\mathbf{H}_1=\mathbf{H}_2$. So we shall consider the vectors

$$V_1=\left\langle e^{-i\lambda r},\frac{1}{\Delta_{h_2}(\lambda)}\varphi(x,\lambda),S^*(\lambda)e^{-i\lambda s}\right\rangle, \tag{4.1}$$

and

$$V_2=\left\langle S(\lambda)e^{-i\lambda r},\frac{1}{\Delta_{\overline{h}_2}(\lambda)}\varphi(x,\lambda),e^{-i\lambda s}\right\rangle. \tag{4.2}$$

For the vector-functions $\mathbf{y}=\langle f_-,Y,f_+\rangle\in\mathbf{H}$ we define the Fourier transforms as follows:

$$F_1\mathbf{y}=\frac{1}{\sqrt{2\pi}}\left(\mathbf{y},V_1\right)$$

and

$$F_2\mathbf{y} = \frac{1}{\sqrt{2\pi}}\left(\mathbf{y}, V_2\right),$$

where $f_-, Y, f_+$ are compactly supported, smooth functions.

An immediate consequence of these definitions is the following result for $\mathbf{y} = \langle f_-, 0, 0\rangle \in D_-$

$$F_1\mathbf{y} = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{0} f_-(r)e^{i\lambda t}dt \in H_-^2,$$

and the result for $\mathbf{y} = \langle 0, 0, f_+\rangle \in D_+$

$$F_1\mathbf{y} = S(\lambda)\frac{1}{\sqrt{2\pi}}\int\limits_{0}^{\infty} f_+(s)e^{i\lambda s}dt \in S(\lambda)H_+^2,$$

where $H_\pm^2$ denotes the Hardy class in $L^2(\mathbb{R};\mathbb{C})$ consisting of all vector-valued functions analytically extendible to the upper (lower) half-planes.

Let $\widetilde{\mathbf{H}}_1$ be a dense set in $\mathbf{H}_1$ with the vectors $\mathbf{y}$ that is compactly supported in $D_-$ such that $\mathbf{y} \in \widetilde{\mathbf{H}}_1$ if $\mathbf{y} = U(-T)\mathbf{y}_0$ such that $\mathbf{y}_0 = \langle f_-, 0, 0\rangle$ and $f_- \in C_0^\infty(-\infty, 0)$, $T = T_{\mathbf{y}} > 0$. For $\mathbf{y}, \mathbf{z} \in \widetilde{\mathbf{H}}_1$ we obtain that $U(-T)\mathbf{y}_0, U(-T)\mathbf{z}_0 \in D_-$, where $T > T_{\mathbf{y}}$ and $T > T_{\mathbf{z}}$ and first components of these elements belong to $C_0^\infty(-\infty, 0)$. Therefore we get

$$\begin{aligned}&(\mathbf{y}, \mathbf{z}) = (U(-T)\mathbf{y}_0, U(-T)\mathbf{z}_0) = (F_1U(-T)\mathbf{y}_0, F_1U(-T)\mathbf{z}_0)_{L^2}\\ &= \left(e^{-i\lambda T}U(-T)\mathbf{y}_0, e^{-i\lambda T}U(-T)\mathbf{z}_0\right)_{L^2} = (F_1\mathbf{y}, F_1\mathbf{z})_{L^2}.\end{aligned}$$

Therefore we have

$$F_1\mathbf{H}_1 = \overline{\bigcup_{t\geq 0} F_1U(t)D_-} = \overline{\bigcup_{t\geq 0} e^{-i\lambda t}H_-^2} = L^2(\mathbb{R}).$$

With a similar argument one can show that $F_1\mathbf{H}_2 = L^2(\mathbb{R})$. Therefore we have

$$\mathbf{H} = \mathbf{H}_1 = \mathbf{H}_2$$

and this completes the proof of (3).

(4) is trivial and hence the proof is completed. □

Using (4.1) and (4.2) together with Lemma 4.2 we have for real $\lambda$ that

$$V_1 = S^*(\lambda)V_2$$

and hence

$$S(\lambda)F_1 = F_2.$$

Therefore we can introduce the following.

**Theorem 4.4.** *$S(\lambda)$ is the scattering function associated with the unitary group $\{U(t)\}_{t\in\mathbb{R}}$.*

## 5. Characteristic function

In Section 4, we have seen that Fourier transformation $F_1$ matches the subspaces $D_-$ and $D_+$ with $H^2_-$ and $S(\lambda)H^2_+$, respectively. Therefore, $\mathbf{H}$ can also be represented as follows:

$$\mathbf{H} = H^2_+ \ominus S(\lambda)H^2_+. \tag{5.1}$$

On the other side, $F_1$ suggests us to handle $U(t)\mathbf{y}$ as $e^{i\lambda t}F_1\mathbf{y}$. Therefore, semigroup of contractions $Z(t)\mathbf{y} = P_1U(t)\mathbf{y}$ can be considered as

$$Z(t)\mathbf{y} = P_1\left[e^{i\lambda t}F_1\mathbf{y}\right],\ t \geq 0. \tag{5.2}$$

However, now, (5.1) and (5.2) allow us to consider $S(\lambda)$ as the characteristic function of the maximal dissipative operator which is the generator of $Z(t)$. Using the information that the characteristic functions of unitary equivalent dissipative operators coincide we obtain the following result.

**Theorem 5.1.** *$S(\lambda)$ is the characteristic function of $\mathcal{L}$.*

Since $\varphi_1(b,\lambda)$ and $\varphi_2(b,\lambda)$ are entire functions of $\lambda$ the poles of $S(\lambda)$ are discrete. Therefore Lemma 4.2 allows us to introduce the following.

**Lemma 5.2.** *$S(\lambda)$ is an inner function on the closed upper-half plane $Im\lambda \geq 0$.*

Using the maximal dissipative operator $\mathcal{L}$ we can construct a contraction with the domain the whole $\mathbf{H}$ as follows:

$$C = (\mathcal{L} - iI)(\mathcal{L} + iI)^{-1}.$$

Then we obtain the following.

**Theorem 5.3.** *$C$ is a completely nonunitary contraction on $\mathbf{H}$ and $1$ cannot be an eigenvalue of $C$.*

**Proof** Let $Y \in D$ and for $F \in H$ we set $Y = (\mathcal{L} + iI)^{-1}F$. Since $\mathcal{L}$ is simple in $H$, we get

$$\|(\mathcal{L} - iI)\,Y\|^2 < \|(\mathcal{L} + iI)\,Y\|^2$$

or

$$\|CF\|^2 < \|F\|^2 \tag{5.3}$$

and (5.3) completes the proof of the first part of the theorem.

Second part follows from the fact that $\mathcal{L}$ is a maximal dissipative operator [13] and this completes the proof. □

**Theorem 5.4.** *$S(\lambda)$ is a Blaschke product on the closed upper-half plane $Im\lambda \geq 0$.*

**Proof** From Lemma 5.2 we can write the following equality

$$S(\lambda) = B(\lambda)e^{i\lambda\theta}, \tag{5.4}$$

where $B(\lambda)$ is the Blaschke product, $\theta > 0$ and $Im\lambda \geq 0$. Then for $\lambda = i\zeta$, $\zeta > 0$, we get from (5.4) that

$$S(i\zeta) = B(\lambda)e^{-\zeta\theta},$$

and as $\zeta \to \infty$ we have that $S(i\zeta) \to 0$. This implies that $\lambda_\infty := \lim_{\zeta\to\infty}(i\zeta)$ is an eigenvalue of $\mathcal{L}$ and hence

$$\mu_\infty := \lim_{\zeta\to\infty}\left(\frac{i\zeta - i}{i\zeta + i}\right) = 1$$

is an eigenvalue of $C$. However, this contradicts with Theorem 5.3 and we complete the proof. □

Consequently we have the following result.

**Theorem 5.5.** *Roots functions of* $\mathcal{L}$ *span* $H$.

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