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Hermitian–Toeplitz determinants for functions with bounded turning

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Abstract: There is a rich literature on estimation of second and third Hankel determinants for normalised analytic functions in geometric function theory. It is also, therefore, natural to explore the concept of the Hermitian–Toeplitz determinants for such functions. In this paper, the sharp lower and upper estimations for third-order Hermitian-Toeplitz determinant for functions with bounded turning of order α, are obtained.

Key words: Analytic functions, functions with bounded turning of order α, Hermitian–Toeplitz determinant

1. Introduction

Finding the sharp estimates on the coefficient's functionals has been one of the major research area of geometric function theory since the advent of the Bieberbach conjecture for normalised univalent functions and then theory of the univalent functions developed around this conjecture. Later, among the coefficient functionals, the major area have been the estimation of bound on the Fekete–Szegö functional(1933) and the Hankel determinants. Although the estimation of bound on Hankel determinant started during 1960's, since the Bieberbach conjecture was unsolved, not so many researchers took interest in investigating the bound on the Hankel determinants, except a few articles [17, 28]. In the last few years investigation of the Hankel determinant gained much attention and brief survey of those work until 2013 can be found in the introduction of the paper [24]. Much recent history of development in this direction can be found in [1, 3, 4, 7, 9, 13, 16, 26, 31]. Wide variety of applications of Toeplitz–plus–Hankel systems arise in linear filtering theory, discrete inverse scattering, and discretization of certain integral equations arising in mathematical physics [30]. The paper [6] gives the higher-order asymptotic formulas for the eigenvalues of large Hermitian–Toeplitz matrices with moderately smooth symbols which trace out a simple loop on the real line and related applications in physics. The determinant of Hermitian–Toeplitz matrices finds its applications in signal processing [29], see also [20].

The class of normalised analytic functions of the form $f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots$ defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is represented by the symbol $\mathcal{A}$. The collection of functions in $\mathcal{A}$ which are univalent also is denoted by $\mathcal{S}$. Recently, Ali et al. [2] introduced symmetric Toeplitz determinant $T_{q,n}(f)$

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defined by

$$T_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}$$

and estimated the bounds on $T_{2,n}(f), T_{3,1}(f), T_{3,2}(f)$ and $T_{2,3}(f)$ for certain subclasses of analytic functions. Actuated by above work, recently Cudna et al. [12] considered Hermitian–Toeplitz determinants with its entries as coefficients of a normalised analytic function as follows:

$$H^T_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ \bar{a}_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \bar{a}_{n+q-1} & \bar{a}_{n+q-2} & \cdots & a_n \end{vmatrix}, \quad \text{where } a_n \in \mathbb{C}.$$

Further, they investigated the sharp lower and upper bound for third order Hermitian–Toeplitz determinants for the classes of starlike and convex functions of order α $(0 \le \alpha < 1)$. It should be noted that the determinant $H^T_{q,n}(f)$ is rationally invariant. Moreover, if a_n are real, then $H^T_{q,n}(f)$ is Hermitian and therefore, the determinant $H^T_{q,n}(f)$ is a real number, see [12]. From the above definition, it is easy to verify that $H^T_{2,1}(f) = 1 - |a_2|^2$ and

$$H^T_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ \bar{a}_2 & 1 & a_2 \\ \bar{a}_3 & \bar{a}_2 & 1 \end{vmatrix} = 2\,\mathrm{Re}(a_2^2\bar{a}_3) - 2|a_2|^2 - |a_3|^2 + 1.$$

Kumar et al. [22] gave a generalisation to the results investigated in [12] by investigating those results for Janowski starlike and convex functions. Kumar [20] investigated lower and upper bounds on the second and third order Hermitian–Toeplitz determinants for certain subclasses of close-to-convex functions. Some more results in this direction may be found in [10, 21]. The book [5] edited by Bättcher and Grudsky contains a chapter on Toepltix matrices which describes the condition on a Toeplitz matrix with entries from a given sequence of complex numbers under which the Toeplitz matrix induces a bounded operator on space certain space. For more details on Toeplitz matrices one refer to [8, 23]. A review on Toplitz matrices is also available in [15].

Now we consider the class of analytic functions f whose derivative have positive real part of order α $(0 \le \alpha < 1)$ i.e. $\mathrm{Re}\, f'(z) > \alpha$ $(z \in \mathbb{D})$. The collection of such functions is denoted by $\mathcal{R}(\alpha)$. The class is important in the sense that the functions in this class are univalent. More precisely, the condition $\mathrm{Re}\, f'(z) > 0$ $(z \in \mathbb{D})$ gives an important sufficient condition for univalency of normalised analytic functions, see [14]. The $p-$valent analogue of this class was considered in [18, 19] and investigated the bounds on second and third Hankel determinants. A generalisation of their work was done by Cho et al. [11]. The work reported in the papers [12] and [2] inculcates us to estimate sharp lower and upper bounds on the third-order Hermitian–Toeplitz determinant for functions in the class $\mathcal{R}(\alpha)$.

Now we introduce the class of functions with positive real part which is going to be an important tool for calculating the bounds. Let $\mathcal{P}(\alpha)$ denote the class of analytic functions of the form $p(z) = 1 + p_1 z + p_2 z^2 +$

$p_3z^3+\cdots$ with $p(0)=1$ and $\operatorname{Re} p(z)>\alpha$ $(0\le\alpha<1)$. It is easy to see that $f\in\mathcal{R}(\alpha)$ if and only if $p\in\mathcal{P}(\alpha)$. Also, let $\mathcal{P}(0)=:\mathcal{P}$.

Lemma 1.1 [25] *If* $p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots\in\mathcal{P}$, *then*

$$2p_2=p_1^2+(4-p_1^2)\zeta,\quad p_1\ge 0 \tag{1.1}$$

for some $\zeta\in\mathbb{C}$ *such that* $|\zeta|\le 1$.

2. Third order Hermitian–Toeplitz determinant

It is well-known that for $f(z)=z+a_2z^2+a_3z^3+\cdots\in\mathcal{R}(\alpha)$, the sharp bound $|a_2|\le 1-\alpha$ holds. Now the sharp upper bound for $H_{2,1}^T(f)=1-|a_2|^2$ is naturally 1, however, the sharp lower bound is $1-(1-\alpha)^2$ i.e. $\alpha(2-\alpha)$. The equality in the upper bound holds for the function

$$\tilde{f}_0(z)=z. \tag{2.1}$$

Equality in lower bound holds in case of the function

$$\tilde{f}_1(z)=\int_0^z\left((1-\alpha)\frac{1-t}{1+t}+\alpha\right)dt=z-(1-\alpha)z^2+\frac{2(1-\alpha)}{3}z^3+\cdots. \tag{2.2}$$

We state this result as a theorem:

Theorem 2.1 *Let* $f(z)=z+a_2z^2+a_3z^3+\cdots\in\mathcal{R}(\alpha)$. *Then the following sharp estimations hold:*

$$(2-\alpha)\alpha\le H_{2,1}^T(f)\le 1.$$

Example 2.2 *For the function* $\tilde{f}_0(z)=z,\ z\in\mathbb{D}$, *we have* $a_2=a_3=a_4=0$ *and clearly* $f\in\mathcal{R}$. *Thus we have* $H_{2,1}^T(f)=1$. *The function* $\tilde{f}_1$ *is also an example in the class* $\mathcal{R}(\alpha)$ *which gives* $H_{2,1}^T(f)=\alpha(2-\alpha)$ *which is the lower bound in case of Theorem 2.1.*

We now investigate the sharp lower and upper bound on third order Hermitian–Toeplitz determinant.

Theorem 2.3 *Let* $f(z)=z+a_2z^2+a_3z^3+\cdots\in\mathcal{R}(\alpha)$. *Then the following sharp estimations hold:*

$$1\ge H_{3,1}^T(f)\ge\begin{cases}\frac{\left(2\alpha^2-7\alpha+1\right)^2}{8(3\alpha-1)}, & 0\le\alpha<\frac{1}{18};\\ -\frac{1}{9}(2\alpha+1)\left(6\alpha^2-10\alpha+1\right), & \frac{1}{18}\le\alpha<1.\end{cases}$$

Proof Since $f(z)=z+a_2z^2+a_3z^3+\cdots\in\mathcal{R}(\alpha)$, it follows that there exists $p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots\in\mathcal{P}$, such that

$$f'(z)=(1-\alpha)p(z)+\alpha. \tag{2.3}$$

Comparing the coefficients of similar power terms in (2.3), we have

$$a_2=\frac{(1-\alpha)p_1}{2}\quad\text{and}\quad a_3=\frac{(1-\alpha)p_2}{3}. \tag{2.4}$$

With the aid of (2.4), we have

$$\begin{aligned} H_{3,1}^{T}(f) &= 1+2\operatorname{Re}(a_2^2 a_3)-2|a_2|^2-|a_3|^2 \\ &= 1+\frac{1}{6}(1-\alpha)^3\operatorname{Re} p_1^2\bar{p}_2-\frac{1}{2}(1-\alpha)^2|p_1|^2-\frac{1}{9}(1-\alpha)^2|p_2|^2. \end{aligned}$$

It is well-known that the class $\mathcal{P}(\alpha)$ and $\mathcal{R}(\alpha)$ are rotationally invariant. For this reason, we shall limit ourselves to a consideration of nonnegative value of p_1. Also since $|p_1|\le 2$, we can very much assume $0\le p_1\le 2$ and let $p_1^2=:x$, and so $x\in[0,4]$. Also throughout this proof we also use the notation $y:=|\xi|\in[0,1]$ wherever be needed. Now replacing p_2 with its equivalent expression using (1.1) from Lemma 1.1 and simplifying, we get

$$\begin{aligned} H_{3,1}^{T}(f) =&1+\frac{(1-\alpha)^3}{12}p_1^4-\frac{p_1^2}{2}(1-\alpha)^2+\frac{(1-\alpha)^3}{12}p_1^2(4-p_1^2)\operatorname{Re}\zeta \\ &-\frac{(1-\alpha)^2}{36}\left(|p_1|^4+(4-p_1^2)|\zeta|^2+2p_1^2(4-p_1^2)\operatorname{Re}\zeta\right). \end{aligned} \tag{2.5}$$

In view of this fact and assumptions, we can write (2.5), as

$$\begin{aligned} H_{3,1}^{T}(f) =&1+\frac{(2-3\alpha)(1-\alpha)^2}{36}x^2-\frac{(1-\alpha)^2}{2}x-\frac{(1-\alpha)^2(4-x)^2}{36}|\zeta|^2+\frac{(1-\alpha)^2(1-3\alpha)(4-x)x}{36}\operatorname{Re}\zeta \\ =:& F(\alpha,x,|\zeta|,\operatorname{Re}\zeta). \end{aligned} \tag{2.6}$$

We now proceed further in the proof through several steps.

Case I(a): For $\alpha=0$, from (2.6) we have $F(0,x,|\zeta|,\operatorname{Re}\zeta)\le F(0,x,|\zeta|,|\zeta|)=:\Upsilon(x,y)$, where

$$\Upsilon(x,y)=1+\frac{x^2}{18}-\frac{x}{2}-\frac{(4-x)^2}{36}y^2+\frac{(4-x)x}{36}y,\quad (x,y)\in[0,1]\times[0,1]. \tag{2.7}$$

On the boundary line segments of the rectangular region $[0,4]\times[0,1]$, we have $\Upsilon(0,y)=1-4y/9\le 1$, $\Upsilon(4,y)=-1/9$, $y\in(0,1)$. Since $\Upsilon'(x,0)=(2x-9)/18<0$, it follows that

$$\Upsilon(x,0)=\frac{x^2}{18}-\frac{x}{2}+1\le\Upsilon(0,0)=1,\ \forall\, x\in(0,4).$$

Further when $y=1$, we find that $F(x,1)=\frac{5}{9}-\frac{x}{6}\le\frac{5}{9}$, $\forall\, x\in(0,4)$. It is matter of simple calculation now to verify that the function Υ has no critical point in the domain $(0,4)\times(0,1)$. Therefore, keeping the conclusions of the above discussion together, we arrive at

$$H_{3,1}^{T}(f)\le\max\left\{1,-\frac{1}{9},\frac{5}{9}\right\}=1. \tag{2.8}$$

Now we find the minimum in the case $\alpha=0$. From (2.7), we have

$$\begin{aligned} H_{3,1}^{T}(f) &= F(0,x,|\zeta|,\operatorname{Re}\zeta) \\ &\ge F(x,1,-1)=\frac{1}{18}\left(x^2-7x+10\right) \\ &\ge -\frac{1}{8}. \end{aligned}$$

Therefore,

$$H^T_{3,1}(f) \geq -\frac{1}{8}. \tag{2.9}$$

Thus, for $\alpha = 0$, from (2.8) and (2.9), we have

$$-\frac{1}{8} \leq |H^T_{3,1}(f)| \leq 1. \tag{2.10}$$

The upper bound is sharp in case of the function $\tilde{f}_0$ given in (2.1), whereas the equality in lower bound holds for the function f_2 given by

$$\tilde{f}_2(z) = \int_0^z \frac{1-t^2}{1-2\sqrt{7/8}\,t+t^2}dt = z + \frac{1}{2}\sqrt{\frac{7}{2}}z^2 + \frac{1}{2}z^3 + \frac{1}{8}\sqrt{\frac{7}{2}}z^4 + \frac{1}{20}z^5 + \cdots \quad (z \in \mathbb{D}). \tag{2.11}$$

Case I(b) For $\alpha = 1/3$, we have

$$\begin{aligned} F(1/3, x, |\zeta|, \operatorname{Re}\zeta) &= 1 + \frac{1}{81}x^2 - \frac{2}{9}x - \frac{1}{81}(4-x)^2|\zeta|^2 \\ &\leq \frac{1}{81}(x-9)^2 \\ &\leq 1. \end{aligned}$$

For minimum, consider

$$\begin{aligned} F(1/3, x, |\zeta|, \operatorname{Re}\zeta) &= 1 + \frac{1}{18}x^2 - \frac{2}{9}x - \frac{1}{81}(4-x)^2|\zeta|^2 \\ &\geq \frac{5}{81}(13-2x) \\ &\geq \frac{25}{81}. \end{aligned}$$

Thus, we have, for $\alpha = 1/3$:

$$\frac{25}{81} \leq H^T_{3,1}(f) \leq 1. \tag{2.12}$$

The lower bound equals in case of the function $\tilde{f}_1$ defined in (2.2). The upper bound is sharp in case of the function $\tilde{f}_0$ defined in (2.1).

Case (II): If $0 < \alpha < 1/3$, then $H^T_{3,1}(f) = F(\alpha, x, |\zeta|, \operatorname{Re}\zeta) \leq F(\alpha, x, |\zeta|, |\zeta|) = G(x, y)$, where

$$G(x,y) := 1 + \frac{1}{36}(2-3\alpha)(1-\alpha)^2x^2 + \frac{1}{36}(1-3\alpha)(1-\alpha)^2(4-x)xy - \frac{1}{36}(1-\alpha)^2(4-x)^2y^2 - \frac{1}{2}(1-\alpha)^2x.$$

On the boundary line segments of the rectangular region $[0,4] \times [0,1]$, we have

$$\begin{aligned} G(0,y) =& 1 - \frac{4}{9}(1-\alpha)^2y^2 \leq 1, \forall y \in [0,1] \\ G(4,y) =& \frac{1}{9}\left(-12\alpha^3 + 14\alpha^2 + 8\alpha - 1\right), \forall y \in [0,1] \\ G(x,0) =& \frac{1}{36}(2-3\alpha)(1-\alpha)^2x^2 - \frac{1}{2}(1-\alpha)^2x + 1. \end{aligned}$$

Now computation reveals that $G(x,0)$ has no maximum in $(0,4)$ and $G(4,0) = \left(-12\alpha^3 + 14\alpha^2 + 8\alpha - 1\right)/9$. Further, as before, it can be verified that the function

$$G(x,1) = -\frac{1}{18}(2\alpha+1)\left(3\alpha^2 x + \alpha(4-6x) + 3x - 10\right)$$

has no critical point inside $(0,4)$, and $G(0,1) = -(2\alpha-5)(2\alpha+1)/9$. Consider the case when the function G is defined inside the rectangular region $(x,y) \in (0,4)\times(0,1)$. It is a matter of routine calculation to see that the function G has no maximum inside this region. For $\alpha \in (0,1/3)$, on the basis of above discussion, we have

$$H_{3,1}^T(f) \le \max\left\{1, \frac{1}{9}\left(-12\alpha^3 + 14\alpha^2 + 8\alpha - 1\right)\right\} = 1.$$

The equality attained for the function defined by (2.2).

We now find the minimum of $H_{3,1}^T(f)$ for the case $0 < \alpha < 1/3$. In this case,we have $H_{3,1}^T(f) = F(\alpha, x, |\zeta|, \operatorname{Re}\zeta) \ge F(\alpha, x, |\zeta|, -|\zeta|) \ge F(\alpha, x, 1, -1) =: h(x)$. Here we see that

$$h'(x) = \frac{1}{18}(a-1)^2(6a-7) - \frac{1}{9}(a-1)^2(3a-1)x = 0$$

if and only if

$$x = x_2 = \frac{6\alpha - 7}{2(3\alpha - 1)}.$$

We now consider two subcases, namely (a) $0 < \alpha < 1/18$; and (b) $1/18 \le \alpha < 1/3$.

(a) It is easy to check that $x_2 \in (0,4)$ for $0 < \alpha < 1/18$. Further, the values of h at the critical point is given by

$$h(x_2) = \frac{\left(2\alpha^2 - 7\alpha + 1\right)^2}{8(3\alpha - 1)}, \quad 0 < \alpha < \frac{1}{18}.$$

Also at the end points of the line segment $0 \le x \le 4$, we have

$$h(0) = 1 - \frac{4}{9}(1-\alpha)^2 \quad \text{and} \quad h(4) = \frac{1}{9}\left(-12\alpha^3 + 14\alpha^2 + 8\alpha - 1\right).$$

From the above, for $0 < \alpha < 1/18$, we conclude that

$$H_{3,1}^T(f) \ge \min\{h(0), h(4), h(x_2)\} = h(x_2).$$

Thus, for $0 < \alpha < 1/18$, we have

$$\frac{\left(2\alpha^2 - 7\alpha + 1\right)^2}{8(3\alpha - 1)} \le H_{3,1}^T(f) \le 1. \tag{2.13}$$

The equality for the lower bound is attained for the function

$$\tilde{f}_5(z) = \int_0^z \left((1-\alpha)\frac{1-t^2}{1-u_0 t + t^2} + \alpha\right) dt, \quad \text{where } u_0 = \sqrt{\frac{6\alpha - 7}{2(3\alpha - 1)}}$$

$$= z + \frac{1-\alpha}{2\sqrt{2}}\sqrt{\frac{6\alpha - 7}{3\alpha - 1}}\, z^2 + \frac{(\alpha-1)(2\alpha+1)}{2(3\alpha - 1)}\, z^3 + \cdots .$$

(b) For $1/18 \le \alpha < 1/3$, we find that h has no critical point in $(0,4)$, hence the minimum will be attained at the end points. From the above, for $1/18 \le \alpha < 1/3$, we conclude that

$$\begin{aligned} H_{3,1}^{T}(f) &\ge \min\left\{1-\frac{4}{9}(1-\alpha)^2, \frac{1}{9}\left(-12\alpha^3+14\alpha^2+8\alpha-1\right)\right\} \\ &= \frac{1}{9}\left(-12\alpha^3+14\alpha^2+8\alpha-1\right). \end{aligned}$$

Thus, for $1/18 \le \alpha < 1/3$, we have

$$\frac{1}{9}\left(-12\alpha^3+14\alpha^2+8\alpha-1\right) \le H_{3,1}^{T}(f) \le 1. \tag{2.14}$$

The equality in the lower bound occurs for the function $\tilde{f}_1$ and that of upper bound for the function $\tilde{f}_0$.

Case (III): Now consider the case $1/3 < \alpha < 1$. As before, we have $H_{3,1}^{T}(f) = F(\alpha, x, |\zeta|, \operatorname{Re}\zeta) \le F(\alpha, x, |\zeta|, -|\zeta|) = H(x,y)$, where

$$H(x,y) := \frac{1}{36}(2-3\alpha)(1-\alpha)^2x^2 - \frac{1}{36}(1-\alpha)^2(4-x)^2y^2 - \frac{1}{36}(1-3\alpha)(1-\alpha)^2(4-x)xy - \frac{1}{2}(1-\alpha)^2x + 1.$$

On the boundary line segment of the rectangular region $[0,4]\times[0,1]$, we have

$$\begin{aligned} H(0,y) &= 1-\frac{4}{9}(1-\alpha)^2y^2 \le 1 \\ H(4,y) &= \frac{4}{9}(2-3\alpha)(1-\alpha)^2 - 2(1-\alpha)^2 + 1 \\ H(x,0) &= \frac{1}{36}(2-3\alpha)(1-\alpha)^2x^2 - \frac{1}{2}(1-\alpha)^2x + 1 \\ H(x,1) &= \frac{1}{36}(2-3\alpha)(1-\alpha)^2x^2 - \frac{1}{36}(1-\alpha)^2(4-x)^2 - \frac{1}{2}(1-\alpha)^2x - \frac{1}{36}(1-3\alpha)(1-\alpha)^2(4-x)x + 1. \end{aligned}$$

The function $H(x,0)$ is decreasing and for all $x \in (0,4)$ and $1/3 < \alpha < 1$ and so $G(x,0) \le G(0,0) = 1$. Moreover the function $H(x,1)$ has no critical point in $(0,4)$. A similar computations show that the function H has no critical point in $(x,y) \in (0,4)\times(0,1)$. Therefore, on the basis of above discussion, we see that

$$H_{3,1}^{T}(f) \le \max\left\{1, \frac{1}{9}\left(-12\alpha^3+14\alpha^2+8\alpha-1\right), 1-\frac{4}{9}(1-\alpha)^2\right\} = 1.$$

For $1/3 < \alpha < 1$ equality occurs in case of the function $\tilde{f}_0$ defined in (2.1).

We now proceed to find the minimum and for this we consider $H_{3,1}^{T}(f) = F(x, |\zeta|, \operatorname{Re}\zeta) \ge F(x,|\zeta|,|\zeta|) \ge F(x,1,1) = h_1(x)$. Now since $h_1'(x) = -(2\alpha+1)\left(3\alpha^2-6\alpha+3\right)/18 < 0 \quad (1/3 < \alpha < 1)$, it follows that the minimum attained at the end point $x=4$ and $h_1(4) = -\left(-12\alpha^3+14\alpha^2+8\alpha-1\right)/9$. Therefore, we have

$$H_{3,1}^{T}(f) \ge \frac{1}{9}\left(-12\alpha^3+14\alpha^2+8\alpha-1\right)$$

with equality for the function $\tilde{f}_1$ defined in (2.2). Thus, in this case, we have

$$\frac{1}{9}\left(-12\alpha^3+14\alpha^2+8\alpha-1\right) \le H_{3,1}^{T}(f) \le 1. \tag{2.15}$$

The estimations given in (2.10), (2.12), (2.13), (2.14) and (2.15) end the proof. □

Example 2.4 *For the function* $\tilde{f}_0(z) = z,\ z \in \mathbb{D}$*, we have* $a_2 = a_3 = a_4 = 0$ *and clearly* $f \in \mathcal{R}$*. Thus we have* $H^T_{3,1}(f) = 1$*, which is the upper bound as investigated in Theorem 2.3. The function* $\tilde{f}_2$ *given by* (2.11) *is also an example in the class* $\mathcal{R}$ *which gives* $H^T_{3,1}(f) = -1/8$*, which is the lower bound in case of Theorem 2.3 in the case when* $\alpha = 0$*.*

Conclusion

It should be noted that the class $\mathcal{R}(\alpha)$ is a subclass of $\mathcal{S}$, so it is interesting to compare the lower and upper bounds on Hermitian–Toeplitz determinants of second and third orders. Obradović and Tuneski [27] proved that $-3 \leq H^T_{2,1}(f) \leq 1$ and $-1 \leq H^T_{3,1}(f) \leq 8$. Here we find that the upper bound on $T_{2,1}(f)$ for both the classes are same whereas the lower bound for the class $\mathcal{R}(\alpha)$ is less than that of the class $\mathcal{S}$. If we compare the upper and lower bounds on $H^T_{3,1}(f)$, we find that the corresponding bounds in case of the class $\mathcal{R}(\alpha)$ lies entirely in the range [–1, 8] of the bound for the class $\mathcal{S}$. Thus, the bounds obtained in this paper very much follow the expected outcomes which confirm the correctness of technicality and procedures adopted in proving our results. It would be interesting to find the lower and upper bounds for $H^T_{4,1}(f)$ for the class $\mathcal{R}(\alpha)$.

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