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An exponential equation involving k-Fibonacci numbers

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Abstract: For $k \geq 2$, consider the k-Fibonacci sequence $(F_n^{(k)})_{n\geq 2-k}$ having initial conditions $0,\ldots,0,1$ (k terms) and each term afterwards is the sum of the preceding k terms. Some well-known sequences are special cases of this generalization. The Fibonacci sequence is a special case of $(F_n^{(k)})_{n\geq 2-k}$ with $k = 2$ and Tribonacci sequence is $(F_n^{(k)})_{n\geq 2-k}$ with $k = 3$. In this paper, we use Baker's method to show that 4, 16, 64, 208, 976, and 1936 are all k-Fibonacci numbers of the form $(3^a \pm 1)(3^b \pm 1)$, where a and b are nonnegative integers.

Key words: k-Fibonacci numbers, linear form in logarithms, reduction method

1. Introduction

The Fibonacci sequence plays a very important role in mathematics and has many interesting applications in other disciplines such as Mathematics, Statistics, Biology, Physics, Finance, Architecture, Computer Science, etc. We can see [13] for the history, properties, and rich applications of Fibonacci sequence. In the literature there exist a lot of generalizations of the Fibonacci sequence, see for example [9, 12]. In [20], G. Ozdemir and Y. Simsek introduced the Fibonacci type polynomials in two variables and generalized by G. Ozdemir, Y. Simsek and G. Milovanović in [21] to a higher order. Several interesting properties, as well as their connections with other polynomials and numbers of the Bernoulli, Euler, Apostol–Bernoulli, Apostol–Euler, Genocchi were obtained.

Let $k \geq 2$ be a positive integer. In this work, we consider a generalization of Fibonacci sequence, which is called the k-generalized Fibonacci sequence or, for simplicity, the k-Fibonacci sequence. The k-Fibonacci sequence $(F_n^{(k)})_{n\geq 2-k}$ is given by the recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for all } n \geq 2,$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. We note that $F_n^{(k)}$ is the *nth* k*-Fibonacci number*. This sequence generalizes the usual Fibonacci sequence. We obtain the Fibonacci

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sequence for $k = 2$. The generating function of n^{th} k-Fibonacci number is given by

$$\sum_{n=0}^{\infty} F_n^{(k)} x^n = \frac{x}{1 - x - x^2 - \cdots - x^k}.$$

For more details about this sequence, see [7, 10, 16, 17, 19, 24].

Finding the k-Fibonacci numbers of special forms attracts the attention of many researchers. For instance, in [5], the authors found all the k-Fibonacci numbers which are powers of 2. The problem of finding the repdigits in the k-Fibonacci sequence was treated in [4]. Recently, in [11] we showed that $F_6^{(4)}$ is the only k-Fibonacci number which is the product of two Fermat numbers.

In this paper, we investigate the problem of finding the k-Fibonacci numbers which are of the form $(3^a \pm 1)(3^b \pm 1)$, where a and b are nonnegative integers. Therefore, we will show the following result.

Theorem 1.1 *All the solutions of the Diophantine equation*

$$F_n^{(k)} = (3^a \pm 1)(3^b \pm 1) \tag{1.1}$$

in positive integers n, k, a*, and* b *with* $k \geq 2$ *and* $1 \leq a \leq b$*, are*

$$\begin{array}{ll} F_4^{(k)} = (3^1-1)^2(3^1-1),\ k \geq 3, & F_6^{(k)} = (3^1-1)(3^2-1) = (3^1+1)^2,\ k \geq 5, \\ F_8^{(k)} = (3^2-1)^2,\ k \geq 7, & F_{12}^{(6)} = (3^1+1)(3^5+1), \\ F_{10}^{(4)} = (3^2-1)(3^3-1), & F_{13}^{(6)} = (3^2-1)(3^5-1). \end{array}$$

To show Theorem 1.1, the strategy will be as follows. First, we rewrite Equation (1.1) in two different ways in order to obtain two different linear forms in logarithms of algebraic numbers which are both nonzero and small. Then, we apply a result due to Matveev [14] to bound n polynomially in terms of k. For the case $2 \leq k \leq 400$, we will use the reduction algorithm due to Baker-Davenport (version Dujella-Pethő [8]) to reduce the upper bounds to a suitable size that we can easily treat. When $k > 400$, we will use some estimates from [5] based on the fact that the dominant root of $F^{(k)}$ is exponentially close to 2, so one can replace this root by 2 in future calculations with linear forms in logarithms and will end up with absolute upper bounds for all the variables, which we will then reduce using again the reduction method.

2. The tools

This section is devoted to collect a few definitions, notations, proprieties and results which will be used in the rest of this work.

2.1. Linear forms in logarithms

For any nonzero algebraic number η of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^{d}\left(X - \eta^{(j)}\right)$, we denote by

$$h(\eta) = \frac{1}{d}\left(\log|a| + \sum_{j=1}^{d} \log\max\left(1, |\eta^{(j)}|\right)\right)$$

the usual absolute logarithmic height of η. In particular, if $\eta = p/q$ is a rational number with $\gcd(p,q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. The following properties of the logarithmic height function $h(\cdot)$, which will be used in the next sections without special reference, are known:

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{2.1}$$

$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \tag{2.2}$$

$$h(\eta^s) = |s|\, h(\eta), \quad (s \in \mathbb{Z}). \tag{2.3}$$

The first tool that we need is the following result due to Matveev [14]; also see Bugeaud, Mignotte and Siksek [6, Theorem 9.4].

Theorem 2.1 *Let $\eta_1, \ldots, \eta_s$ be real algebraic numbers and let $b_1, \ldots, b_s$ be nonzero integers. Let $d_{\mathbb{K}}$ be the degree of the number field $\mathbb{Q}(\eta_1, \ldots, \eta_s)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j = \max\{d_{\mathbb{K}} h(\eta_j), |\log \eta_j|, 0.16\} \quad \textit{for} \quad j = 1, \ldots, s.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_s|\}.$$

If $\eta_1^{b_1} \cdots \eta_s^{b_s} - 1 \neq 0$, then

$$|\eta_1^{b_1} \cdots \eta_s^{b_s} - 1| \geq \exp(-1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{K}}^2 (1 + \log d_{\mathbb{K}})(1 + \log B) A_1 \cdots A_s).$$

2.2. The reduction algorithm

Here, we present a variant of the reduction method of Baker and Davenport due to Dujella and Pethő [8]. The version presented here is also a slight variant.

Lemma 2.2 *Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, C, μ be some real numbers with $A > 0$ and $B > 1$. Let*

$$\varepsilon = ||\mu q|| - M \cdot ||\gamma q||,$$

where $||\cdot||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < u\gamma - v + \mu < AB^{-w}$$

in positive integers u, v and w with

$$u \leq M \quad \textit{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

The above lemma cannot be applied when μ is a linear combination of 1 and γ, since then $\varepsilon < 0$. In this case, we use the following nice property of continued fractions (see Theorem 8.2.4 and top of page 263 in [18])

Lemma 2.3 *Let p_i/q_i be the convergents of the continued fraction $[a_0, a_1, \ldots]$ of the irrational number γ. Let M be a positive integer and put $a_M := \max\{a_i | 0 \leq i \leq N+1\}$ where $N \in \mathbb{N}$ is such that $q_N \leq M < q_{N+1}$. If $x, y \in \mathbb{Z}$ with $x > 0$, then*

$$|x\gamma - y| > \frac{1}{(a_M + 2)x}, \quad \textit{for all } x < M.$$

2.3. Useful lemmas

We conclude this section by recalling two lemmas that we need in this work:

Lemma 2.4 *[22, Lemma 7] If $m \geq 1$, $T > (4m^2)^m$ and $T > y/(\log y)^m$. Then,*

$$y < 2^m T(\log T)^m.$$

Lemma 2.5 *[23, Lemma 2.2] Let $d, x \in \mathbb{R}$ and $0 < d < 1$. If $|x| < d$, then*

$$|\log(1+x)| < \frac{-\log(1-d)}{d}|x|.$$

2.4. On k-Fibonacci sequence

In this subsection, we recall some facts and properties of the k-Fibonacci sequence which will be used later. The characteristic polynomial of this sequence is

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1.$$

The polynomial $\Psi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and has just one root $\alpha(k)$ outside the unit circle (see, for example, [16], [17] and [24]). It is real and positive so it satisfies $\alpha(k) > 1$. The other roots are strictly inside the unit circle. Furthermore, in [24] Wolfram showed that

$$2(1-2^{-k}) < \alpha(k) < 2, \quad \text{for all } k \geq 2. \tag{2.4}$$

To simplify the notation, in general, we omit the dependence on k of $\alpha(k)$ and use α. For $s \geq 2$, let

$$f_s(x) := \frac{x-1}{2+(s+1)(x-2)}. \tag{2.5}$$

In [2], Bravo, Gomez and Luca proved that the inequalities

$$1/2 < f_k(\alpha) < 3/4 \quad \text{and} \quad \left|f_k(\alpha^{(i)})\right| < 1,\ 2 \leq i \leq k$$

hold, where $\alpha := \alpha^{(1)}, \ldots, \alpha^{(k)}$ are all the zeros of $\Psi_k(x)$. So, the number $f_k(\alpha)$ is not an algebraic integer. In addition, they proved that the logarithmic height of f satisfies

$$h(f_k(\alpha)) < \log(k+1) + \log 4, \quad \text{for all } k \geq 2. \tag{2.6}$$

With the above notation, Dresden and Du showed in [7] that we have

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha^{(i)}){\alpha^{(i)}}^{n-1} \quad \text{and} \quad \left|F_n^{(k)} - f_k(\alpha)\alpha^{n-1}\right| < \frac{1}{2}, \tag{2.7}$$

for all $n \geq 1$ and $k \geq 2$. Furthermore, for $n \geq 1$ and $k \geq 2$, it was shown in [5] that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}. \tag{2.8}$$

In [5], Bravo and Luca obtained the following inequality

$$F_n^{(k)} \leq 2^{n-2}, \quad \text{for all} \quad n \geq 2. \tag{2.9}$$

Moreover, if $n \geq k+2$, then the above inequality is strict.

3. Proof of Theorem 1.1

In this section, we will show Theorem 1.1 in four steps.

3.1. Setup

It is known that the first $k+1$ nonzero terms in $(F_n^{(k)})_{n\geq 2-k}$ are powers of two, namely

$$F_1^{(k)} = 1 \quad \text{and} \quad F_n^{(k)} = 2^{n-2}, \quad 2 \leq n \leq k+1.$$

So, in this case, Equation (1.1) turns into

$$2^{n-2} = (3^a \pm 1)(3^b \pm 1).$$

Hence $3^a \pm 1$ and $3^b \pm 1$ are both powers of 2. From [15], it is known that the only solution of the Catalan's equation $x^a - y^b = 1$ is $(x, y, a, b) = (3, 2, 2, 3)$. Thus, $3^a \pm 1$ and $3^b \pm 1$ belong to the set $\{2, 4, 8\}$. Therefore, in this case, the only solutions of Diophantine equation (1.1) are given by

$$4 = (3^1 - 1)^2, \quad 16 = (3^1 + 1)^2 = (3^1 - 1)(3^2 - 1) \quad \text{and} \quad 64 = (3^2 - 1)^2.$$

From now we suppose that $n \geq k+2$ and so $n \geq 4$.

Next we will give a relation between n and $a+b$. Using inequality (2.8), we get that

$$3^{a+b-1} \leq (3^a - 1)(3^b - 1) \leq (3^a \pm 1)(3^b \pm 1) = F_n^{(k)} \leq \alpha^{n-1}$$

and

$$\alpha^{n-2} \leq F_n^{(k)} = (3^a \pm 1)(3^b \pm 1) \leq 3^{a+b}\left(1 + \frac{1}{3^b} + \frac{1}{3^a} + \frac{1}{3^{a+b}}\right) \leq 3^{a+b+1}.$$

Hence, we obtain

$$(a+b-1)\frac{\log 3}{\log \alpha} + 1 \leq n \leq (a+b+1)\frac{\log 3}{\log \alpha} + 2.$$

Furthermore, using the fact that $3/2 < \alpha < 2$ for $k \geq 2$ (see (2.4)), we deduce that

$$1.58(a+b) - 0.59 < n < 2.71(a+b) + 3.71. \tag{3.1}$$

3.2. An inequality for n versus k

In this subsection, we will show the following lemma, that allows us to have an upper bound of n in relation to k.

Lemma 3.1 *If (a, b, k, n) is a solution in integers of equation (1.1) with $k \geq 2$ and $n \geq k+2$, then we have the following inequality*

$$n < 1.36 \times 10^{29} k^8 \log^5 k. \tag{3.2}$$

Proof We rewrite equation (1.1) as

$$3^{a+b} = F_n^{(k)} \mp 3^a \mp 3^b - 1. \tag{3.3}$$

Thus, from estimate (2.7), we obtain

$$\left|f_k(\alpha)\alpha^{n-1} - 3^{a+b}\right| = \left|f_k(\alpha)\alpha^{n-1} - F_n^{(k)} \pm 3^a \pm 3^b + 1\right| \leq \frac{1}{2} + 3^a + 3^b + 1.$$

Dividing both sides by 3^{a+b}, we get

$$|\Gamma_1| \leq \frac{1}{2 \cdot 3^{a+b}} + \frac{1}{3^b} + \frac{1}{3^a} + \frac{1}{3^{a+b}} < \frac{2.5}{3^a}, \tag{3.4}$$

where

$$\Gamma_1 := f_k(\alpha) \cdot \alpha^{n-1} \cdot 3^{-(a+b)} - 1. \tag{3.5}$$

If $\Gamma_1 = 0$, then we obtain

$$f_k(\alpha) = \alpha^{-(n-1)} \cdot 3^{a+b}.$$

Thus $f_k(\alpha)$ is an algebraic integer, which is not possible. Therefore $\Gamma_1 \neq 0$. We can apply Theorem 2.1 to Γ_1 given by (3.5). To do this, we consider

$$(\eta_1, b_1) := (f_k(\alpha), 1), \quad (\eta_2, b_2) := (\alpha, n-1), \quad (\eta_3, b_3) := (3, -a-b).$$

The algebraic numbers η_1, η_2, η_3 are elements of the field $\mathbb{K} := \mathbb{Q}(\alpha)$ and $d_{\mathbb{K}} = k$. The facts that

$$h(\eta_1) \leq \log(k+1) + \log 4 < 3.6 \log k, \quad \text{for all} \quad k \geq 2,$$

$$1/2 < f_k(\alpha) < 3/4, \quad h(\eta_2) = (\log \alpha)/k < (\log 2)/k \quad \text{and} \quad h(\eta_3) = \log 3$$

enable us to take

$$A_1 := 3.6k \log k, \quad A_2 := \log 2 \quad \text{and} \quad A_3 := k \log 3.$$

Finally, inequality (3.1) implies that we can take $B := n$. Therefore, inequality (3.4) and Theorem 2.1 tell us that

$$\exp\left(-3.93 \times 10^{11} k^4 \log k (1 + \log k)(1 + \log n)\right) < |\Gamma_1| < \frac{2.5}{3^a}. \tag{3.6}$$

By the facts $1 + \log k < 2.5 \log k$ and $1 + \log n < 1.8 \log n$ which hold for all $k \geq 2$ and $n \geq 4$ respectively, we obtain

$$a \log 3 < 1.78 \times 10^{12} k^4 \log^2 k \log n. \tag{3.7}$$

We go back to Equation (1.1) and we rewrite it as

$$\frac{F_n^{(k)}}{3^a \pm 1} \mp 1 = 3^b \tag{3.8}$$

and consequently

$$\left|\frac{f_k(\alpha)\alpha^{n-1}}{3^a \pm 1} - 3^b\right| = \left|\frac{f_k(\alpha)\alpha^{n-1} - F_n^{(k)}}{3^a \pm 1} \pm 1\right| \leq \frac{1}{2(3^a \pm 1)} + 1 < 1.25.$$

Dividing through 3^b, we obtain

$$\left|\Gamma_2^{(\pm)}\right| < \frac{1.25}{3^b}, \tag{3.9}$$

where

$$\Gamma_2^{(\pm)} := \frac{f_k(\alpha)}{3^a \pm 1} \cdot \alpha^{n-1} \cdot 3^{-b} - 1. \tag{3.10}$$

If $\Gamma_2^{(\pm)} = 0$, then we get

$$f_k(\alpha) = \alpha^{-(n-1)} \cdot 3^b \cdot (3^a \pm 1),$$

which is not possible since the right-hand side is an algebraic integer while the left-hand side is not. So, $\Gamma_2^{(\pm)} \neq 0$. Now, we will apply Theorem 2.1 to $\Gamma_2^{(\pm)}$ by taking

$$(\eta_1, b_1) := (f_k(\alpha)/(3^a \pm 1), 1), \quad (\eta_2, b_2) := (\alpha, n-1), \quad (\eta_3, b_3) := (3, -b).$$

Clearly, $\mathbb{K} := \mathbb{Q}(\alpha)$ contains η_1, η_2, η_3 and has the degree $d_{\mathbb{K}} = k$. As calculated before we take

$$A_2 := \log 2, \quad A_3 := k \log 3, \quad \text{and} \quad B := n.$$

We need to compute A_1. The estimates (2.6) and (3.7) together with the proprieties (2.1)–(2.3) imply that the inequalities

$$\begin{aligned} h(\eta_1) &\leq h(f_k(\alpha)) + h(3^a) + h(1) + \log 2 \\ &< \log(k+1) + \log 4 + a \log 3 + \log 2 \\ &< 1.79 \times 10^{12} k^4 \log^2 k \log n \end{aligned}$$

hold for all $k \geq 2$. Since

$$\eta_1 := \frac{f_k(\alpha)}{3^a \pm 1} < \frac{11}{8} \quad \text{and} \quad \eta_1^{-1} = \frac{3^a \pm 1}{f_k(\alpha)} < 2 \cdot 3^{a+1},$$

then by (3.7) we have

$$|\log \eta_1| < (a+1) \log 3 + \log 2 < 1.79 \times 10^{12} k^4 \log^2 k \log n.$$

Thus, we conclude that

$$\max\{k h(\eta_1), |\log \eta_1|, 0.16\} < 1.79 \times 10^{12} k^5 \log^2 k \log n := A_1.$$

Applying Theorem 2.1 and comparing the resulting inequality with (3.9), we get

$$b < 8 \times 10^{23} k^8 \log^3 k \log^2 n,$$

where we have used the facts $1 + \log k < 2.5 \log k$ and $1 + \log n < 1.8 \log n$, which hold for $k \geq 2$ and $n \geq 4$. By inequality (3.1), we get

$$n < 2.71(a+b) + 3.71 < 5.42b + 3.71 < 4.34 \times 10^{24} k^8 \log^3 k \log^2 n.$$

Hence, we obtain

$$\frac{n}{\log^2 n} < 4.34 \times 10^{24} k^8 \log^3 k. \tag{3.11}$$

Taking $m = 2$ and $T := 4.34 \times 10^{24} k^8 \log^3 k$ in Lemma 2.4 and as

$$56.73 + 8 \log k + 3 \log \log k < 88.3 \log k,$$

for all $k \geq 2$, we get

$$\begin{aligned} n &< 4(4.34 \times 10^{24} k^8 \log^3 k)(\log(4.34 \times 10^{24} k^8 \log^3 k))^2 \\ &< 1.736 \times 10^{25} k^8 \log^3 k(56.73 + 8\log k + 3\log\log k)^2 \\ &< 1.36 \times 10^{29} k^8 \log^5 k. \end{aligned}$$

This establishes (3.2) and finishes the proof of Lemma 3.1. □

3.3. The case $2 \leq k \leq 400$

In this subsection, we study the problem when $k \in [2, 400]$ by using Lemma 2.2. Consider

$$\Lambda_1 := \log(\Gamma_1 + 1) = (n-1)\log\alpha - (a+b)\log 3 + \log(f_k(\alpha)). \tag{3.12}$$

Since $a \geq 1$, then by (3.4), we have $|\Gamma_1| < 0.84$. Hence, applying Lemma 2.5 with $d = 0.84$, we get

$$|\Lambda_1| < \frac{-\log 0.16}{0.84} \cdot |\Gamma_1| < 5.46 \cdot 3^{-a}. \tag{3.13}$$

Replacing (3.12) into (3.13) and dividing through by $\log 3$, we obtain

$$\left|(n-1)\left(\frac{\log\alpha}{\log 3}\right) - (a+b) + \frac{\log(f_k(\alpha))}{\log 3}\right| < 5 \cdot 3^{-a}. \tag{3.14}$$

With the goal to apply Lemma 2.2 to (3.14), we take

$$\gamma := \frac{\log\alpha}{\log 3}, \quad \mu := \frac{\log(f_k(\alpha))}{\log 3}, \quad A := 5 \quad \text{and} \quad B := 3.$$

We have $\gamma \notin \mathbb{Q}$ since if we assume the contrary, then there exist coprime integers a and b such that $\gamma = a/b$, then we get that $\alpha^b = 3^a$. Let $\sigma \in Gal(\mathbb{K}/\mathbb{Q})$ such that $\sigma(\alpha) = \alpha_i$, for some $i \in \{2, \ldots, k\}$. Applying this to the above relation and taking absolute values we get $1 < 3^a = |\alpha_i| < 1$, which is a contradiction.

For each $k \in [2, 400]$, we find a good approximation of γ and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M_k$ and $\varepsilon = \varepsilon(k) = ||\mu q|| - M_k||\gamma q|| > 0$, where $M_k := \lfloor 1.36 \times 10^{29} k^8 \log^5 k \rfloor$, which is is an upper bound of $n-1$ from Lemma 3.1. After doing this, we use Lemma 2.2 on inequality (3.14). A computer program with Mathematica revealed that the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$ over all $k \in [1, 400]$ is $249.2840\ldots$, which is an upper bound of a by Lemma 2.2.

Now, we consider $1 \leq a \leq 249$ and

$$\Lambda_2^{(\pm)} := \log(\Gamma_2^{(\pm)} + 1) = (n-1)\log\alpha - b\log 3 + \log(f_k(\alpha)/(3^a \pm 1)). \tag{3.15}$$

Since $b \geq 1$, then by (3.9), we have $|\Gamma_2^{(\pm)}| < 0.42$. Thus, by Lemma 2.5 with $d = 0.42$ we deduce that

$$|\Lambda_2^{(\pm)}| < \frac{-\log 0.58}{0.42} \cdot |\Gamma_2^{(\pm)}| < 1.63 \cdot 3^{-b}. \tag{3.16}$$

Replacing (3.15) into (3.16) and dividing through by $\log 3$, we obtain

$$\left|(n-1)\left(\frac{\log\alpha}{\log 3}\right) - b + \frac{\log(f_k(\alpha)/(3^a \pm 1))}{\log 3}\right| < 1.5 \cdot 3^{-b}. \tag{3.17}$$

To apply Lemma 2.2 to (3.17), this time for $1 \le a \le 249$ we take

$$\gamma := \frac{\log\alpha}{\log 3}, \quad \mu_a := \frac{\log(f_k(\alpha)/(3^a \pm 1))}{\log 3}, \ (1 \le a \le 249), \quad A := 1.5 \quad \text{and} \quad B := 3.$$

As seen before $\gamma \notin \mathbb{Q}$. Again, for each $(k, a) \in [2, 400] \times [1, 249]$, we find a good approximation of γ and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M_k$ and $\varepsilon = \varepsilon(k) = ||\mu q|| - M_k||\gamma q|| > 0$, where $M_k := \lfloor 1.36 \times 10^{29} k^8 \log^5 k \rfloor$, which is is an upper bound of $n-1$ from Lemma 3.1. After doing this, we use Lemma 2.2 on inequality (3.17). A computer search with Mathematica revealed that the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$ over all $(k, a) \in [1, 400] \times [1, 249]$ is $250.158\ldots$, which according to Lemma 2.2, is an upper bound of b.

Hence, we deduce that the possible solutions (a, b, k, n) of equation (1.1) for which $k \in [2, 400]$ satisfy $1 \le a \le b \le 250$. Therefore, we use inequalities (3.1) to obtain $n \le 1380$.

Finally, we used Mathematica to compare $F_n^{(k)}$ and $(3^a \pm 1) \cdot (3^b \pm 1)$, for $4 \le n \le 1380$ and $1 \le a \le b \le 250$, with $n < 2.71(a + b) + 3.71$ and checked that equation (1.1) has also the solutions:

$$F_{12}^{(6)} = (3^1 + 1)(3^5 + 1), \quad F_{10}^{(4)} = (3^2 - 1)(3^3 - 1), \quad \text{and} \quad F_{13}^{(6)} = (3^2 - 1)(3^5 - 1).$$

3.4. The case $k > 400$

In this subsection, we will show that Equation (1.1) has no solutions when $k > 400$.

3.4.1. An absolute upper bound on k

Lemma 3.2 *If (a, b, k, n) is a solution of Diophantine equation (1.1) with $k > 400$ and $n \ge k + 2$, then k and n are bounded by*

$$k < 1.19 \times 10^{25} \quad \textit{and} \quad n < 3.51 \times 10^{238}. \tag{3.18}$$

Proof For $k > 400$, it is easy to check that

$$n < 1.36 \times 10^{29} k^8 \log^5 k < 2^{k/2}.$$

Thus, from [3], $F_n^{(k)}$ can be rewritten into the form

$$F_n^{(k)} = 2^{n-2}(1 + \zeta), \quad \text{where } |\zeta| < \frac{1}{2^{k/2}}. \tag{3.19}$$

Substituting (3.19) in (3.3), we obtain

$$\left|3^{a+b} - 2^{n-2}\right| = \left|2^{n-2}\zeta \mp 3^a \mp 3^b - 1\right| \le \frac{2^{n-2}}{2^{k/2}} + 3^{b+1}.$$

This and the fact that $3^{a+b+1} < \alpha^{n-2} < 2^{n-2}$ yields

$$|\Gamma_3| < \frac{1}{2^{k/2}} + \frac{2}{3^a} < \frac{3}{2^{\min\{k/2, a\}}}, \quad \text{where} \quad \Gamma_3 := 3^{a+b} \cdot 2^{-n+2} - 1. \tag{3.20}$$

In order to apply Theorem 2.1 to Γ_3, we take

$$t := 2, \quad (\eta_1, b_1) := (3, a+b), \quad (\eta_2, b_2) := (2, -n+2).$$

If $\Gamma_3 = 0$, then $3^{a+b} = 2^{n-2}$, which is impossible since $b \geq a \geq 1$. Therefore, $\Gamma_3 \neq 0$. Since $\eta_1, \eta_2, \eta_3 \in \mathbb{K} := \mathbb{Q}$, then the degree is $d_{\mathbb{K}} = 1$. We can choose $B := n$ because $a + b \leq n$. On the other hand, as

$$h(\eta_1) = \log 3 \quad \text{and} \quad h(\eta_2) = \log 2,$$

then we take

$$A_1 := \log 3 \quad \text{and} \quad A_2 := \log 2.$$

Therefore, by Theorem 2.1, we have

$$|\Gamma_3| > \exp\left(-3.91 \times 10^7 \log n\right), \tag{3.21}$$

where we have use the fact that $1 + \log n < 2 \log n$ for all $n \geq 2$. The comparison of (3.20) and (3.21) gives

$$\min\{k/2, a\} < 5.65 \times 10^7 \log n. \tag{3.22}$$

Now, we distinguish two cases according to the value of $\min\{k/2, a\}$.

Case 1: $\min\{k/2, a\} = k/2$**.** In this case it follows from (3.22) and Lemma 3.1 that

$$k < 1.13 \times 10^8 \log\left(1.36 \times 10^{29} k^8 \log^5 k\right).$$

Solving the above inequality gives

$$k < 3.13 \times 10^{10} \quad \text{and} \quad n < 1.04 \times 10^{120}. \tag{3.23}$$

Case 2: $\min\{k/2, a\} = a$**.** In this case, it follows from (3.22) that

$$a < 5.65 \times 10^7 \log n. \tag{3.24}$$

We go back to Equation (3.8) and we use (3.19) to obtain

$$\frac{2^{n-2}(\zeta+1)}{3^a \pm 1} \mp 1 = 3^b$$

giving

$$\left|3^b - \frac{2^{n-2}}{3^a \pm 1}\right| = \left|\mp 1 + \frac{2^{n-2}\zeta}{3^a \pm 1}\right| \leq 1 + \frac{2^{n-2}}{(3^a \pm 1)2^{k/2}}.$$

Multiplying both sides by $(3^a \pm 1) \cdot 2^{-(n-2)}$ and using the facts that $n \geq k+2$ and $a < k/2$, we get

$$\left|\Gamma_4^{(\pm)}\right| \leq \frac{3^{a+1}}{2^{n-2}} + \frac{1}{2^{k/2}} < \frac{4}{2^{k/2}}, \quad \text{where} \quad \Gamma_4^{(\pm)} := (3^a \pm 1) \cdot 3^b \cdot 2^{-n+2} - 1. \tag{3.25}$$

So, the conditions are met to apply Theorem 2.1 with

$$t := 3, \quad (\eta_1, b_1) := (3^a \pm 1, 1), \quad (\eta_2, b_2) := (2, -n+2), \quad (\eta_3, b_3) := (3, b).$$

If $\Gamma_4^{(\pm)} = 0$, then $(3^a \pm 1) \cdot 3^b = 2^{n-2}$. Thus, 3 divides 2^{n-2} which is false. Therefore, $\Gamma_4^{(\pm)} \neq 0$. As calculated before, we take

$$d_{\mathbb{K}} := 1, \quad A_2 := \log 2, \quad A_3 := \log 3 \quad \text{and} \quad B := n.$$

Moreover, by (3.24), we have

$$\begin{array}{rcl} h(\eta_1) & \leq & ah(3) + h(1) + \log 2 \\ & \leq & 5.65 \times 10^7 \log n \times \log 3 + \log 2 \\ & \leq & 6.21 \times 10^7 \log n. \end{array}$$

Hence, we take $A_1 := 6.21 \times 10^7 \log n$. Using Theorem 2.1 and inequality (3.25), we get that

$$\exp(-1.36 \times 10^{19} (\log n)^2) < \frac{4}{2^{k/2}}$$

and so

$$k < 3.93 \times 10^{19} (\log n)^2.$$

From this and Lemma 3.1, we conclude that

$$k < 1.19 \times 10^{25} \quad \text{and} \quad n < 3.51 \times 10^{238}. \tag{3.26}$$

So, in both cases, inequalities (3.26) hold. This completes the proof of Lemma 3.2. □

3.4.2. Reducing the bound on k

To reduce the bound on k, we will use Lemmas 2.2 and 2.3 several times. First, consider

$$\Lambda_3 := (a + b) \log 3 - (n - 2) \log 2 = \log(\Gamma_3 + 1).$$

Assume that $a \geq 2$. Thus from (3.20) we get $|\Lambda_3| < 0.75$. Hence, by Lemma 2.5 with $d = 0.75$, one has

$$|\Lambda_3| < -\frac{\log(0.25)}{0.75} \cdot |\Gamma_3| < 5.55 \cdot 2^{-\min\{k/2, a\}}. \tag{3.27}$$

So, we get

$$\left| (n-2) \left(\frac{\log 2}{\log 3} \right) - (a + b) \right| < 5.1 \cdot 2^{-\min\{k/2, a\}}. \tag{3.28}$$

To obtain a lower bound for the left-hand side of (3.28), we will apply Lemma 2.3 with

$$\gamma := \frac{\log 2}{\log 3} \notin \mathbb{Q}, \quad x := n - 2 \quad \text{and} \quad y := a + b.$$

Since $n < 3.51 \times 10^{238}$ by Lemma 3.2, then we can take $M := 3.51 \times 10^{238}$. Let

$$[a_0, a_1, a_2, a_3, \ldots] = [0, 1, 1, 1, 2, 2, 3, 1, 5, 2, 23, 2 \ldots]$$

be the continued fraction of γ. Using Maple, it seen that $q_{465} < M < q_{466}$ and since $\max_{1 \leq i \leq 466} a_i = a_{331} = 2436$, then by Lemma 2.3, we obtain

$$\left| (n-2) \left(\frac{\log 2}{\log 3} \right) - (a + b) \right| > \frac{1}{2438(n-2)}. \tag{3.29}$$

Comparing estimates (3.28), (3.33), and by Lemma 3.2 we get

$$\min\{k/2, a\} < \frac{\log\left(5.1 \times 2438 \times 3.51 \times 10^{238}\right)}{\log 2} < 807.$$

Case 1: $\min\{k/2, a\} = k/2$**.** In this case, we get

$$k \leq 1614. \tag{3.30}$$

Case 2: $\min\{k/2, a\} = a$**.** In this case, we obtain that $a \leq 806$. Now let us consider

$$\Lambda_4^{(\pm)} := b \log 3 - (n-2)\log 2 + \log(3^a \pm 1) = \log(\Gamma_4^{(\pm)} + 1).$$

Since $k > 400$, then from (3.25) we have $\left|\Gamma_4^{(\pm)}\right| < 0.01$. Hence, by Lemma 2.5, one gets

$$\left|\Lambda_4^{(\pm)}\right| < -\frac{\log(0.99)}{0.01} \cdot \left|\Gamma_4^{(\pm)}\right| < 4.03 \cdot 2^{-k/2}. \tag{3.31}$$

Thus, we obtain

$$\left|(n-2)\left(\frac{\log 2}{\log 3}\right) - b - \frac{\log(3^a \pm 1)}{\log 3}\right| < 3.7 \cdot 2^{-k/2}.$$

For $a \neq 1$ and $a \neq 2$ in the case $(-)$, we will apply Lemma 2.2 with the parameters

$$\gamma := \frac{\log 2}{\log 3} \notin \mathbb{Q}, \quad \mu_a := -\frac{\log(3^a \pm 1)}{\log 3}, \ (1 \leq a \leq 806), \quad A := 3.7 \quad \text{and} \quad B := 2.$$

Moreover, Lemma 3.2 implies that we can take $M := 3.51 \times 10^{238}$. Using Mathematica, we find that $q_{610} \approx 1.52 \times 10^{318}$ satisfies the hypotheses of Lemma 2.2. Furthermore, by Lemma 2.2 we obtain $k < 1280$. For the other cases, $\Lambda_4^{(\pm)}$ turns into

$$\Lambda_4^{(\pm)} = \begin{cases} b\log 3 - (n-3)\log 2, & \text{if } a = 1 \text{ in the case } (-), \\ b\log 3 - (n-4)\log 2, & \text{if } a = 1 \text{ in the case } (+), \\ b\log 3 - (n-5)\log 2, & \text{if } a = 2 \text{ in the case } (-). \end{cases}$$

Hence, we get

$$\left|(n-\ell)\left(\frac{\log 2}{\log 3}\right) - b\right| < 3.7 \cdot 2^{-k/2}, \quad \text{where } \ell \in \{3, 4, 5\}. \tag{3.32}$$

To obtain a lower bound for the left-hand side of (3.32), we will apply Lemma 2.3 with

$$\gamma := \frac{\log 2}{\log 3} \notin \mathbb{Q}, \quad x := n - \ell \quad \text{and} \quad y := b.$$

Since $n < 3.51 \times 10^{238}$ by Lemma 3.2, then we can take $M := 3.51 \times 10^{238}$. Let

$$[a_0, a_1, a_2, a_3, \ldots] = [0, 1, 1, 1, 2, 2, 3, 1, 5, 2, 23, 2 \ldots]$$

be the continued fraction of γ. Using Maple, it seen that $q_{465} < M < q_{466}$ and since $\max_{1\le i\le 466} a_i = a_{331} = 2436$, then by Lemma 2.3, we obtain that

$$\left|(n-\ell)\left(\frac{\log 2}{\log 3}\right) - b\right| > \frac{1}{2438(n-\ell)}. \tag{3.33}$$

We compare estimates (3.28) and (3.33), and then we use Lemma 3.2 to obtain

$$k < \frac{2\log\left(3.7\times 2438\times 3.51\times 10^{238}\right)}{\log 2} < 1612.$$

So, the inequality $k < 1614$ holds always. With this new upper bound of k, we get

$$n < 1.38\times 10^{56}.$$

We now proceed as we did before but with $M = 1.38\times 10^{56}$ and we obtain $k < 390$, which contradicts our assumption $k > 400$. This completes the proof of Theorem 1.1.

4. Conclusion

For any integer $k \ge 2$, the sequence of the k-Fibonacci numbers $(F_n^{(k)})_{n\ge 2-k}$ is defined by the k initial values $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$, $F_1^{(k)} = 1$ and such that each term afterwards is the sum of the k preceding ones. In this paper, we investigate solutions of the Diophantine equation $F_n^{(k)} = (3^a \pm 1)(3^b \pm 1)$, where a and b are nonnegative integers. One can study equation used other generalized sequences of such form.

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