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Two-weight norm inequalities for some fractional type operators related to Schrödinger operator on weighted Morrey spaces

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Abstract: In this paper, we establish the two-weight norm inequalities for fractional maximal functions and fractional integral operators related to Schrödinger differential operator on weighted Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class.

Key words: Fractional maximal functions, fractional integral, weighted Morrey spaces, Schrödinger operator

1. Introduction

In this section, we consider Schrödinger operator

$$L = -\Delta + V(x) \quad on \quad \mathbb{R}^n, n \geqslant 3,$$

where $V(x)$ is a nonnegative potential satisfying certain reverse Hölder class. A nonnegative locally L^q integral function $V(x)$ on $\mathbb{R}^n (n \geqslant 3)$ is said to belong $B_q (1 < q \leqslant \infty)$ if there exist a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|} \int_{B(x,r)} V^q(y)dy \right)^{\frac{1}{q}} \leqslant C \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} V(y)dy \right)$$

holds for every $x \in \mathbb{R}^n$ and $0 < r < \infty$, where $B(x,r)$ denotes the ball centered at x with radius r; see[1]. In particular, if V is a nonnegative polynomial, then $V \in B_\infty$. For $x \in \mathbb{R}^n$, the function $\rho(x)$ is defined by

$$\rho(x) := \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leqslant 1 \right\} = \frac{1}{m_V(x)}.$$

Obviously, $0 < m_V(x) < \infty$ if $V \neq 0$. In particular, $m_V(x) = 1$ with $V = 1$ and $m_V(x) \sim (1 + |x|)$ with $V = |x|^2$.

The weighted Morrey spaces related to certain nonnegative potentials V was introduced by Guixia Pan [5], which can be considered as an extension of weighted Lebesgue spaces. Let $1 \leqslant p < \infty, 0 < \lambda < 1, -\infty < \beta < \infty$.

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For $f \in L^p_{loc}(\mathbb{R}^n)$ and $V \in B_q(q > 1)$, we say $f \in L^{p,\lambda}_{\beta,V,(\sigma,u)}$ (weighted Morrey spaces related to the potential V) provided that

$$\|f\|^p_{L^{p,\lambda}_{\beta,V,(\sigma,u)}} := \sup_{B(x_0,r)\subset\mathbb{R}^n} \left(1+\frac{r}{\rho(x_0)}\right)^{\beta} \frac{1}{u(B)^{\lambda}} \int_B |f(x)|^p \sigma(x)dx < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$, $B := B(x_0, r)$ denotes the ball centered at x_0 and with radius r.

And the weak weighted Morrey spaces related to the potential V is defined by

$$WL^{p,\lambda}_{\beta,V,(\sigma,u)} = \{f : \|f\|_{WL^{p,\lambda}_{\beta,V,(\sigma,u)}} < \infty\},$$

where

$$\|f\|_{WL^{p,\lambda}_{\beta,V,(\sigma,u)}} := \sup_{B(x_0,r)\subset\mathbb{R}^n} \sup_{t>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\beta} \frac{1}{u(B)^{\lambda/p}} \sigma(\{x \in Q : |f(x) > t\})^{1/p}.$$

In particular, when $\beta = 0$ or $V = 0$, $u = \sigma = 1$, and $0 < \lambda < 1$, the space $L^{p,\lambda}_{\beta,V,(\sigma,u)}(\mathbb{R}^n)$ is the classic Morrey space $L^{p,\lambda}(\mathbb{R}^n)$; (see[6]). When $\beta = 0$ or $V = 0$, $u = \sigma$, and $0 < \lambda < 1$, $L^{p,\lambda}_u(\mathbb{R}^n)$ was first introduced in [7], where $u \in A_p(\mathbb{R}^n)$ (Muckenhoupt weights class).

The study of Schrödinger operator $L = -\Delta + V$ attracted much attention; see [1–3, 8–11]. From [1], we know some Schrödinger type operators, such as $\nabla(-\Delta + V)^{-1}\nabla$ with $V \in B_n$, $\nabla(-\Delta + V)^{-1/2}$ with $V \in B_n$, $(-\Delta + V)^{-1/2}$ with $V \in B_n$, $(-\Delta + V)^{i\gamma}$ with $\gamma \in \mathbb{R}$ and $V \in B_{n/2}$, and $\nabla^2(-\Delta + V)^{-1}$ with V is a nonnegative polynomial, are standard Calderón–Zygmund operators.

Recently, Bongioanni, Harboure and Salina [3] proved $L^p(\mathbb{R}^n)(1 < p < \infty)$ boundedness for commutators of Riesz transforms associated with Schrödinger operator with $BMO_\theta(\rho)$ functions which includes the class of BMO functions, and they [4] also obtained the weighted boundedness for Riesz transforms and fractional integrals associated with Schrödinger operator with weight $A^{\rho,\theta}_p$ class which includes the Muckenhoupt weight class. Tang [5, 26] established the boundedness of some Schrödinger type operators on (weighted) Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class.

Two-weight norm inequalities for the linear and multilinear fractional maximal operators and fractional integral operators on Lebesgue spaces were widely study; see [14, 16–19]. The weighted estimates with Muckenhoupt A_p weights on Morrey spaces were studies in [12, 20].

Very recently, M. Amelia Vignatti, Oscar Salinas and Silvia Hartzstein[15] gets two-weighted boundedness results for the Schrödinger fractional integral and its commutators, they applied the boundedness results in the setting of finite measure spaces of homogeneous type and Fefferman–Stein type inequalities that connect maximal operators naturally associated with Schrödinger operator. Sun [13] proved the two-weight norm inequalities for fractional maximal functions and fractional integral operators on weighted Morrey spaces with suitable weights. Naturally, it will be a very interesting problem to ask whether we can establish the two-weight norm inequalities for fractional maximal operators and fractional integrals associated with Schrödinger operators on weighted Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class.

In this paper, we give a positive answer. To obtain the conclusion, we will use the property of sparse sets. We study two-weight norm inequalities on weighted Morrey spaces related to the potential V for fractional maximal operators and fractional integral operators associated with Schrödinger operators when a pair of weights (u,σ) is in A_p^ρ (see next section). For the fractional maximal operators associated with Schrödinger operators, we have following result.

Theorem 1.1 Suppose $V \in B_{n/2}$, let p, α, β, λ and η be constants such that $0<\alpha<n, 1<p<n/\alpha, -\infty<\beta<\infty, 0<\lambda<1$ and $0<\eta<\infty$. Define the number q and $s(p)$ by (2.1) and (2.2), respectively. Then for any $(u,\sigma) \in A^\rho_{s(p)}$ with $u,\sigma \in A^\rho_\infty$, we have

$$\|M_{\alpha,\eta}(f\sigma)\|_{L^{q,\lambda}_{\beta,V,u}} \lesssim \|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}.$$

For the fractional integrals associated with Schrödinger operators, we have following estimate.

Theorem 1.2 Under the same hypotheses as in Theorem 1.1, we have

$$\|\mathcal{I}_\alpha(f\cdot\sigma)\|_{L^{q,\lambda}_{\beta,V,u}} \lesssim \|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}.$$

For the weak estimate, we have following result.

Theorem1.3 Let the constants p,q,α,β,λ and $s(p)$ be defined as in Theorem 1.1. Suppose that $(u,\sigma)\in A^\rho_{s(p)}$ and $u \in A_\infty$. Then we have

$$\|\mathcal{I}_\alpha(f\cdot\sigma)\|_{WL^{q,\lambda}_{\beta,V,u}} \lesssim \|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}.$$

The paper is organized as follows. In Section 2, we give some notation and basic results, these basic results play a crucial role in this paper. In Section 3, we give proofs for the above theorems.

In proving inequalities, if we write $A \lesssim B$, we mean that $A \leqslant CB$, where the constant C can depend on p, α, β, λ and η, but does not depend on the weights u or σ, nor on the function. If we write $A \simeq B$, then $A \lesssim B$ and $B \lesssim A$.

2. Some notation and basic results

We first recall some notations. Given $B=B(x,r)$ and $\lambda>0$, we will write λB for the $\lambda-$dilate ball, which is the ball with the same center x and with radius λr. Similarly, $Q(x,r)$ denotes the cube centered at x with the sidelength r (here and below only cubes with sides parallel to the coordinate axes are considered), and $\lambda Q(x,r) = Q(x,\lambda r)$. Given a Lebesgue measurable set E and a weight w, $|E|$ will denote the Lebesgue measure of E and

$$w(E) = \int_Q w dx.$$

For $0<p<\infty$,

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(y)|^p w(y) dy\right)^{\frac{1}{p}}.$$

In this paper, we write $\Psi_\theta(B) = (1+r/\rho(x_0))^\theta$, where $\theta>0$, x_0 and r denote the center and radius of B, respectively.

A weight will always mean a nonnegative function which is locally integrable. As in [2], we say that a weight w belongs to the class $A_p^{\rho,\theta}$ for $1 < p < \infty$ if there is a constant C such that for all ball $B = B(x,r)$

$$\left(\frac{1}{\Psi_\theta(B)|B|}\int_B w(y)dy\right)\left(\frac{1}{\Psi_\theta(B)|B|}\int_B w^{-1/(p-1)}(y)dy\right)^{p-1} \leqslant C.$$

We also say that a nonnegative function w satisfies the $A_1^{\rho,\theta}$ condition if there exists a constant C such that for all balls B

$$M_V^\theta(w)(x) \leqslant Cw(x) \quad for \quad a.e. \quad x \in \mathbb{R}^n,$$

where

$$M_V^\theta f(x) = \sup_{x\in B} \frac{1}{\Psi_\theta(B)|B|}\int_B |f(y)|dy.$$

Since $\Psi_\theta(B) \geqslant 1$, obviously, $A_p \subset A_p^{\rho,\theta}$ for $1 \leqslant p < \infty$, where A_p denote the classic Muckenhoupt weights; see [27]. In fact, let $\theta > 0$ and $0 \leqslant \gamma \leqslant \theta$; it is easy to check that

$$w(x) = (1+|x|)^{-(n+\gamma)} \notin A_\infty = \bigcup_{p\geqslant 1} A_p$$

and $w(x)dx$ is not a doubling measure, but

$$w(x) = (1+|x|)^{-(n+\gamma)} \in A_1^{\rho,\theta}$$

provided that $V = 1$ and

$$\Psi_\theta(B(x_0,r)) = (1+r)^\theta.$$

We remark that balls can be replaced by cubes in the definitions of $A_p^{\rho,\theta}$ for $p \geqslant 1$ and M_V^θ, since

$$\Psi_\theta(B) \leqslant \Psi_\theta(2B) \leqslant 2^\theta \Psi_\theta(B).$$

When $V = 0$ and $\theta = 0$, we denote $M_0^0 f(x)$ by $Mf(x)$ (the standard Hardy–Littlewood maximal function). It easy to see that

$$|f(x)| \leqslant M_V^\theta f(x) \leqslant Mf(x) \quad for \quad a.e. \quad x \in \mathbb{R}^n$$

and $\theta \geqslant 0$. For convenience, in the rest of this paper, let $\theta \geqslant 0$ be fixed, and we always assume that $\Psi(B)$ denotes $\Psi_\vartheta(B)$ and A_p^ρ denotes $A_p^{\rho,\theta}$, respectively.

Given positive numbers α and p such that $0 < \alpha < n$ and $1 < p < n/\alpha$, we define the number q by

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n} \tag{2.1}$$

and set

$$s(p) := 1 + \frac{q}{p'} \tag{2.2}$$

Cruz–Uribe and Moen [21] introduced the two-weight A_p condition, which is a natural generalization of A_p condition. Recall that we say a pair of weights (u, σ) is in A_p if

$$[u,\sigma]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q u(x)dx \right) \left(\frac{1}{|Q|} \int_Q \sigma(x)dx \right)^{p-1} < \infty.$$

Inspired by Tang [28], we adapted the two-weight condition, we say a pair of weights (u, σ) is in A_p^ρ if

$$[u,\sigma]_{A_p^\rho} := \sup_Q \left(\frac{1}{\Psi(Q)|Q|} \int_Q u(x)dx \right) \left(\frac{1}{\Psi(Q)|Q|} \int_Q \sigma(x)dx \right)^{p-1} < \infty.$$

Recall that the fractional maximal operator M_α and fractional maximal operator I_α is defined by

$$M_\alpha f(x) := \sup_{x\in Q} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|dy, \quad 0<\alpha<n,$$

and

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0<\alpha<n,$$

respectively, where f is locally integrable function defined on $\mathbb{R}^n$ and Q takes over all cubes in $\mathbb{R}^n$ which contain x.

With function $\rho(x)$, we define the fractional maximal operator associated with Schrödinger operator $M_{\alpha,\eta}$ (introduced by Tang in [28]) as

$$M_{\alpha,\eta}(f)(x) := \sup_{x\in Q} \frac{1}{\Psi(Q)^\eta} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|dy;$$

where the supremum is taken over all the cubes $Q = Q(x_0, r)$ including x and $\Psi(Q) = (1 + r/\rho(x_0))$.

The fractional integral operator associated with Schrödinger operator defined by

$$\begin{aligned}\mathcal{I}_\alpha f(x) &:= L^{-\alpha/2} f(x) \\ &= \int_0^\infty e^{-tL} f(x) t^{\alpha/2-1} dt = \int_{\mathbb{R}^n} k_\alpha(x,y) f(y) dy \quad 0<\alpha<n.\end{aligned}$$

For each $t > 0$, e^{-tL} is an integral operator with kernel $k_t(x, y)$ having a better behaviour far from the diagonal than the kernel of the classic heat semigroup, associated with the Laplacian differential operator. More precisely,

Lemma 2.1[29] Given $N > 0$ there exists C_N such that for all $x, y \in \mathbb{R}^n$

$$k_t(x,y) \leqslant C_N t^{-n/2} e^{-\frac{|x-y|^2}{5t}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

A consequence of the above lemma is that $k_\alpha \leqslant \frac{C}{|x-y|^{n-\alpha}}$ for all $x, y \in \mathbb{R}^n$, where $k_\alpha(x, y) = \int_0^\infty k_t(x, y) t^{\alpha/2-1} dt$ is the kernel $\mathcal{I}_\alpha (0 < \alpha < n)$. It is then clear that

$$\mathcal{I}_\alpha(f)(x) \lesssim I_\alpha(f)(x),$$

where the right hand side is the classic fractional integral.

Now, we introduce some preliminary results.

Let $\mathcal{D}$ be a set of cubes in $\mathbb{R}^n$. Recall that $\mathcal{D}$ is said to be a general dyadic grid if it satisfies the following:

(i) for any $Q \in \mathcal{D}$, its side length $l(Q)$ is of the form 2^k for some $k \in \mathbb{Z}$;

(ii) $Q_1 \cap Q_2 \in \{Q_1, Q_2, \emptyset\}$ for any $Q_1, Q_2 \in \mathcal{D}$;

(iii) the cubes of a fixed side length 2^k form a partition of $\mathbb{R}^n$.

Given a gengral dyadic grid $\mathcal{D}$, if a subset $\mathcal{S} \subset \mathcal{D}$ satisfies

$$\left| \bigcup_{Q' \in \mathcal{S}, Q' \subsetneq Q} Q' \right| \leqslant \frac{1}{2}|Q|, \qquad for \quad all \quad Q \in \mathcal{S},$$

we say that $\mathcal{S}$ is a sparse family in $\mathcal{D}$, For $Q \in \mathcal{S}$, denote

$$E(Q) := Q \backslash \left(\bigcup_{Q' \in \mathcal{S}, Q' \subsetneq Q} Q' \right).$$

We see from the definition of sparse family that $|E(Q)| \geqslant \frac{1}{2}|Q|$ for any $Q \in \mathcal{S}$.

Given a constant $0 < \alpha < n$ and a gengral dyadic grid $\mathbb{D}$ in $\mathbb{R}^n$, we define the dyadic fractional maximal operator associated with Schrödinger operator and dyadic fractional integral operator associated with Schrödinger operator $M^{\mathcal{D}}_{\alpha,\eta}$ and $I^{\mathcal{D}}_{\alpha}$ by

$$M^{\mathcal{D}}_{\alpha,\eta}(f)(x) := \sup_{x \in Q, Q \in \mathcal{D}} \frac{1}{\Psi(Q)^{\eta}} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| dy$$

and

$$\begin{aligned} \mathcal{I}^{\mathcal{D}}_{\alpha} f(x) &:= L^{-\frac{\alpha}{2}} f(x) \\ &= \int_0^{\infty} e^{-tL} f(x) t^{\alpha/2-1} dt = \sum_{Q \in \mathcal{D}} \int_Q k_{\alpha}(x,y) f(y) dy \quad 0 < \alpha < n, \end{aligned}$$

respectively. For a sparse family $\mathcal{S}$, the sparse dyadic fractional integral operator associated with Schrödinger operator $\mathcal{I}^{\mathcal{S}}_{\alpha}$ are defined similarly.

For $t \in \{0, 1/3\}^n := \{(t_1, \ldots, t_n) : t_i = 0 \quad or \quad 1/3, 1 \leqslant i \leqslant n\}$, define

$$\mathcal{D}^t := \{2^{-k}([0,1)^n + m + (-1)^k t) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}. \tag{2.3}$$

The importance of this grids is shown by the following proposition.

Proposition 2.1[22] There are 2^n dyadic grids $\mathcal{D}^t, t \in \{0, 1/3\}^n$, such that for any cube $Q \subset \mathbb{R}^n$ there exist some $t \in \{0, 1/3\}^n$ and a cube $Q_t \in \mathcal{D}^t$ satisfying $Q \subset Q_t$ and $l(Q_t) \leqslant 6l(Q)$.

Proposition 2.2[16] There exists a constants $C(n, \alpha)$ such that for every function $f \in L^1_{loc}$ and $1 \leqslant t \leqslant 3^n$,

$$M^{\mathcal{D}^t}_{\alpha} f(x) \leqslant M_{\alpha} f(x) \leqslant C(n, \alpha) \sup_t M^{\mathcal{D}^t}_{\alpha} f(x),$$

where the grids $\mathcal{D}^t$ are defined by (2.3).

We see from the above proposition that

$$M_\alpha f(x) \simeq \sum_{t\in\{0,1/3\}^n} M_\alpha^{\mathcal{D}^t}.$$

Applying proposition 2.1 and proposition 2.2, we can obtain the following result.

Proposition 2.3 There exists a constants $C(n,\alpha)$ such that for every function $f \in L^1_{loc}$ and $1 \leqslant t \leqslant 3^n$,

$$M_{\alpha,\eta}^{\mathcal{D}^t} f(x) \leqslant M_{\alpha,\eta} f(x) \leqslant C(n,\alpha) \sup_t M_{\alpha,\eta}^{\mathcal{D}^t} f(x), \tag{2.4}$$

where the grids $\mathcal{D}^t$ are defined by (2.1).

Proof. The first inequality is immediate. To prove the second, fix x and a cube Q containing x. Then by proposition 2.1 there exists t and $P \in \mathcal{D}^t$ such that $Q \subset P$ and $|P| \leqslant 3^n|Q|$. Therefore,

$$\frac{1}{\Psi(Q)^\eta}\frac{1}{|Q|^{1-\alpha/n}}\int_Q |f(y)|dy \leqslant \frac{1}{\Psi(P)^\eta}\frac{1}{|P|^{1-\alpha/n}}\int_P |f(y)|dy \leqslant C(n,\alpha) M_{\alpha,\eta}^{\mathcal{D}^t} f(x) \leqslant C(\alpha,\eta)\sup_t M_{\alpha,\eta}^{\mathcal{D}^t} f(x).$$

If we take the supremum over all cubes Q containing x, we get the desired inequality.

Cruz–Uribe and Moen [21] proved that the dyadic fractional integral operator and the fractional integral operator are equivalent in some sense. See also works by Sawyer and Wheeden [25] and Pérez [24].

Proposition 2.4[21] Given $0 < \alpha < n$ and a nonnegative function f, for any general dyadic grid $\mathcal{D}$, we have

$$I_\alpha^{\mathcal{D}} f(x) \lesssim I_\alpha f(x).$$

Conversely,we have that

$$I_\alpha f(x) \lesssim \sum_{t\in\{0,1/3\}^n} I_\alpha^{\mathcal{D}^t} f(x).$$

As a result, the fractional integral operator $I_\alpha f$ is pointwise equivalent to a linear combination of dyadic fractional integral operator, that is

$$I_\alpha f(x) \simeq \sum_{t\in\{0.1/3\}^n} I_\alpha{}^{\mathcal{D}} t f(x).$$

Cruz–Uribe and Moen [21] also proved the equivalence between the dyadic fractional integral operator and its sparse counterpart. See also works by Sawyer and Wheeden [25] and Pérez [24].

Proposition 2.5[21] Given a bounded, nonnegative function f with compact support and a general dyadic grid $\mathcal{D}$, there exists a sparse family $\mathcal{S}$ such that for all α with $0 < \alpha < n$,

$$I_\alpha^{\mathcal{D}} \lesssim I_\alpha^{\mathcal{S}} f(x).$$

3. Proof of the main results

In this section, we give proofs for the main results. First, we consider Theorem 1.1.

proof of Theorem 1.1. By (2.4), it suffices to show that

$$\|M_{\alpha,\eta}^{\mathcal{D}}(f\cdot\sigma)\|_{L^{q,\lambda}_{\beta,V,u}} \lesssim \|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}.$$

Fix some constant $a > 1$. For each integer k, let

$$\Omega_k := \{x \in \mathbb{R}^n : M^{\mathcal{D}}_{\alpha,\eta} > a^k\}.$$

We can decompose Ω_k into a sequence of maximal disjoint dyadic cubes $\{Q_j^k\}$ in $\mathcal{D}$ such that

$$a^k < \frac{1}{\Psi(Q_j^k)^\eta} \frac{1}{|Q_j^k|^{1-\alpha/n}} \int_{Q_j^k} |f \cdot \sigma| \leqslant 2^{n-\alpha} a^k.$$

It was shown in [16] that for a sufficiently large, the collection $\{Q_j^k\}_{j,k}$ is a sparse family.

Fix some dyadic cube $Q \in \mathcal{D}$. If there exists an integer k such that $Q \subset \Omega_k$, then there is some gengral dyadic cube $Q' \supset Q$ such that

$$a^k < \frac{1}{\Psi(Q')^\eta} \frac{1}{|Q'|^{1-\alpha/n}} \int_{Q'} |f \cdot \sigma|.$$

Since $(u, \sigma) \in A^\rho_{s(p)}$, we have

$$\frac{1}{|Q'|^{1-\alpha/n}} \lesssim \left(\frac{1}{\Psi(Q')} \int_{Q'} u \right)^{-1/q} \left(\frac{1}{\Psi(Q')} \int_{Q'} \sigma \right)^{-1/p'}.$$

Now, we see from Hölder inequality that

$$\begin{aligned} a^k &\lesssim \frac{1}{\Psi(Q')^\eta} \left(\frac{1}{\Psi(Q')} \int_{Q'} u \right)^{-1/q} \left(\frac{1}{\Psi(Q')} \int_{Q'} \sigma \right)^{-1/p'} \int_{Q'} |f \cdot \sigma| \\ &\leqslant \frac{1}{\Psi(Q')^{\eta+1/p'}} \left(\frac{1}{\Psi(Q')} \int_{Q'} u \right)^{-1/q} \left(\int_Q |f|^p \sigma \right)^{1/p} \\ &\leqslant \frac{1}{\Psi(Q')^{\eta+\beta/p+1/q+1/p'}} \frac{1}{u(Q')^{1/q}} u(Q')^{\lambda/q} \|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}. \end{aligned}$$

Since $\lambda < 1$ and $Q \subset Q'$, we conclude that k satisfies

$$a^k \lesssim \frac{1}{\Psi(Q)^{\eta+\beta/p+1/q+1/p'}} \frac{1}{u(Q)^{(\lambda-1)/q}} \|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}.$$

Thus there is some integer k_0 such that $Q \subset \Omega_{k_0}$ and $Q \not\subseteq \Omega_k$ for $k > k_0$. The above estimates show that

$$a^{k_0} \lesssim \frac{1}{\Psi(Q)^{\eta+\beta/p+1/q+1/p'}} \frac{1}{u(Q)^{(\lambda-1)/q}} \|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}.$$

Therefore,

$$\begin{aligned}\Psi(Q)^{\beta}\frac{1}{u(Q)^{\lambda}}\int_Q\Big(M^{\mathcal{D}}_{\alpha,\eta}(f\cdot\sigma)\Big)^q du &\leqslant \Psi(Q)^{\beta}\frac{1}{u(Q)^{\lambda}}a^{qk_0}u(Q)\\ &\quad+\Psi(Q)^{\beta}\frac{1}{u(Q)^{\lambda}}\sum_{Q_j^k\subset Q}\bigg(\frac{1}{\Psi(Q_j^k)^{\eta}}\frac{1}{|Q_j^k|^{1-\alpha/n}}\int_{Q_j^k}f\sigma\bigg)^q u(Q_j^k)\\ &\lesssim \frac{\Psi(Q)^{\beta(1-q/p)}}{\Psi(Q)^{q\eta+1+q/p'}}\frac{1}{u(Q)^{\lambda-1}}\|f\|^q_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}\\ &\quad+\Psi(Q)^{\beta}\frac{1}{u(Q)^{\lambda}}\sum_{Q_j^k\subset Q}\bigg(\frac{1}{\Psi(Q_j^k)^{\eta}}\frac{1}{|Q_j^k|^{1-\alpha/n}}\int_{Q_j^k}f\sigma\bigg)^q u(Q_j^k)\\ &\lesssim \|f\|^q_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}+\Psi(Q)^{\beta}\frac{1}{u(Q)^{\lambda}}\sum_{Q_j^k\subset Q}\bigg(\frac{1}{\Psi(Q_j^k)^{\eta}}\frac{1}{|Q_j^k|^{1-\alpha/n}}\int_{Q_j^k}f\sigma\bigg)^q u(Q_j^k).\end{aligned}$$

It remains to estimate the last term in the above inequalities.

Since $u\in A_{\infty}^{\rho}$, for which we have the reverse Hölder inequality

$$\bigg(\frac{1}{\Psi(Q)}\frac{1}{|Q|}\int_Q\sigma^r\bigg)^{1/r}\lesssim\frac{1}{\Psi(Q)}\frac{1}{|Q|}\int_Q\sigma. \tag{3.1}$$

Let $s=(p'r)'$, keeping in mind that $p>s$, we have

$$\begin{aligned}&\Psi(Q)^{\beta}\frac{1}{u(Q)^{\lambda}}\sum_{Q_j^k\subset Q}\bigg(\frac{1}{\Psi(Q_j^k)^{\eta}|Q_j^k|^{1-\alpha/n}}\int_{Q_j^k}f\sigma\bigg)^q u(Q_j^k)\\ &=\Psi(Q)^{\beta}\frac{1}{u(Q)^{\lambda}}\sum_{Q_j^k\subset Q}\frac{1}{|Q_j^k|^{-\alpha q/n}}\bigg(\frac{1}{\Psi(Q_j^k)^{\eta}|Q_j^k|}\int_{Q_j^k}f\sigma^{1/p}\sigma^{1/p'}\bigg)^q\\ &\leqslant\Psi(Q)^{\beta}\frac{1}{u(Q)^{\lambda}}\sum_{Q_j^k\subset Q}\frac{1}{|Q_j^k|^{-\alpha q/n}}\bigg(\frac{1}{\Psi(Q_j^k)^{\eta}|Q_j^k|}\int_{Q_j^k}f^s\sigma^{s/p}\bigg)^{q/s}\\ &\quad\cdot\bigg(\frac{1}{\Psi(Q_j^k)|Q_j^k|}\int_{Q_j^k}\sigma^r\bigg)^{q/p'r}u(Q_j^k)\\ &=\Psi(Q)^{\beta}\Psi(Q_j^k)\frac{1}{u(Q)^{\lambda}}\sum_{Q_j^k\subset Q}\frac{|Q_j^k|}{|Q_j^k|^{-\alpha p/n}}\bigg(\frac{1}{\Psi(Q_j^k)^{\eta}|Q_j^k|}\int_{Q_j^k}f^s\sigma^{s/p}\bigg)^{q/s}\\ &\quad\cdot\bigg(\frac{1}{\Psi(Q_j^k)|Q_j^k|}\int_{Q_j^k}\sigma^r\bigg)^{q/p'r}\bigg(\frac{1}{\Psi(Q_j^k)|Q_j^k|}\int_{Q_j^k}u\bigg)\\ &\lesssim\Psi(Q)^{\beta}\Psi(Q_j^k)\frac{1}{u(Q)^{\lambda}}\sum_{Q_j^k\subset Q}\frac{|Q_j^k|}{|Q_j^k|^{-\alpha p/n}}\bigg(\frac{1}{\Psi(Q_j^k)^{\eta}|Q_j^k|}\int_{Q_j^k}f^s\sigma^{s/p}\bigg)^{q/s}\\ &\quad\cdot\bigg(\frac{1}{\Psi(Q_j^k)|Q_j^k|}\int_{Q_j^k}\sigma^r\bigg)^{q/p'}\bigg(\frac{1}{\Psi(Q_j^k)|Q_j^k|}\int_{Q_j^k}u\bigg),\end{aligned}$$

where we use (3.1) in the last step.

Since $(u,\sigma)\in A^{\rho}_{s(p)}$, we have

$$\left(\frac{1}{\Psi(Q_j^k)|Q_j^k|}\int_{Q_j^k}u(x)\right)\left(\frac{1}{\Psi(Q_j^k)|Q_j^k|}\int_{Q_j^k}\sigma(x)\right)^{q/p'}\leqslant[u,\sigma]_{A^{\rho}_{s(p)}}.$$

It follows that

$$\begin{aligned}&\Psi(Q)^{\beta}\frac{1}{u(Q)^{\lambda}}\sum_{Q_j^k\subset Q}\left(\frac{1}{\Psi(Q_j^k)^{\eta}|Q_j^k|^{1-\alpha/n}}\int_{Q_j^k}f\sigma\right)^{q}u(Q_j^k)\\&\lesssim\Psi(Q)^{\beta}\Psi(Q_j^k)\frac{1}{u(Q)^{\lambda}}\sum_{Q_j^k\subset Q}|Q_j^k|^{1+\alpha q/n}\left(\frac{1}{\Psi(Q_j^k)^{\eta}|Q_j^k|}\int_{Q_j^k}f^s\sigma^{s/p}\right)^{q/s}\\&=\Psi(Q)^{\beta}\Psi(Q_j^k)\frac{1}{u(Q)^{\lambda}}\sum_{Q_j^k\subset Q}|Q_j^k|\left(\frac{1}{\Psi(Q_j^k)|Q_j^k|^{1-\alpha s/n}}\int_{Q_j^k}f^s\sigma^{s/p}\right)^{q/s}\\&\leqslant\Psi(Q)^{\beta}\Psi(Q_j^k)\frac{1}{u^{\lambda}(Q)}\sum_{Q_j^k\subset Q}|E(Q_j^k)|\left(\frac{1}{\Psi(Q_j^k)|Q_j^k|^{1-\alpha s/n}}\int_{Q_j^k}f^s\sigma^{s/p}\right)^{q/s},\end{aligned}$$

where we used the sparsity of $\{Q_j^k\}_{j,k}$ in the last step.

Recall that $1/p-1/q=\alpha/n$. We have

$$\frac{s}{p}-\frac{s}{q}=\frac{\alpha s}{n}\tag{3.2}$$

Since $Q_j^k\subset Q$, for any $x\in E(Q_j^k)$, we have

$$\frac{1}{\Psi(Q_j^k)^{\eta}}\frac{1}{|Q_j^k|^{1-\alpha s/n}}\int_{Q_j^k}f^s\sigma^{s/p}\leqslant M_{\alpha s/n,\eta}(f^s\sigma^{s/p}\cdot\chi_Q)(x),$$

where $M_{\alpha s/n,\eta}$ is the standard fractional maximal operator associated with Schrödinger operator. Consequently,

$$\begin{aligned}&\Psi(Q)^{\beta}\frac{1}{u(Q)^{\lambda}}\sum_{Q_j^k\subset Q}\left(\frac{1}{\Psi(Q_j^k)^{\eta}|Q_j^k|^{1-\alpha/n}}\int_{Q_j^k}f\sigma\right)^{q}u(Q_j^k)\\&\lesssim\Psi(Q)^{\beta}\Psi(Q_j^k)\frac{1}{u(Q)^{\lambda}}\sum_{Q_j^k\subset Q}|E(Q_j^k)|\left(M_{\alpha s/n,\eta}(f^s\sigma^{s/p}\cdot\chi_Q)(x)\chi_{E(Q_j^k)}x)\right)^{q/s}\\&\lesssim\Psi(Q)^{\beta}\Psi(Q_j^k)\frac{1}{u(Q)^{\lambda}}\int\left(M_{\alpha s/n,\eta}(f^s\sigma^{q/s}\cdot\chi_Q)\right)^{q/s}.\end{aligned}$$

By (3.2), we see from the property of maximal functions that

$$\begin{aligned}&\Psi(Q)^{\beta}\frac{1}{u(Q)^{\lambda}}\sum_{Q_j^k\subset Q}\left(\frac{1}{\Psi(Q_j^k)^{\eta}|Q_j^k|^{1-\alpha/n}}\int_{Q_j^k}f\sigma\right)^{q}u(Q_j^k)\\&\lesssim\Psi(Q)^{\beta}\Psi(Q_j^k)\frac{1}{u(Q)^{\lambda}}\left(\int_Q f^p\sigma\right)^{q/p}.\end{aligned}$$

Hence

$$\Psi(Q)^{\beta}\frac{1}{u(Q)^{\lambda}}\sum_{Q_j^k\subset Q}\left(\frac{1}{\Psi(Q_j^k)^{\eta}|Q_j^k|^{1-\alpha/n}}\int_{Q_j^k} f\sigma\right)^q u(Q_j^k)\lesssim \|f\|^q_{L^{p.\lambda p/q}_{\beta,V,(\sigma,u)}}.$$

This completes the proof.

Next we give a proof of Theorem 1.2.

Proof of Theorem 1.2. First, we show that for any general dyadic grid and sparse family $\mathcal{S}$,

$$\sup_{R\in\mathcal{D}}\Psi(R)^{\beta/q}\frac{1}{u(R)^{\lambda/q}}\|I_\alpha^{\mathcal{S}}(f\sigma)\cdot\chi_R\|_{L^q(u)}\lesssim\|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}} \tag{3.3}$$

Fix some $\mathcal{R}\in\mathcal{D}$. we have

$$\begin{aligned}
&\Psi(R)^{\beta/q}\frac{1}{u(R)^{\lambda/q}}\|I_\alpha^{\mathcal{S}}(f\sigma)\cdot\chi_R\|_{L^q(u)}\\
&=\Psi(R)^{\beta/q}\frac{1}{u(R)^{\lambda/q}}\sup_{\|g\|_{L^{q'}(u)}=1}\int\sum_{Q\in\mathcal{S}}|Q|^{\alpha/n-1}\int_Q f\sigma dy\cdot\chi_Q\cdot\chi_R\cdot gudx\\
&\leqslant\Psi(R)^{\beta/q}\frac{1}{u(R)^{\lambda/q}}\sup_{\|g\|_{L^{q'}(u)}=1}\int\sum_{Q\in\mathcal{S},Q\subset R}|Q|^{\alpha/n-1}\int_Q f\sigma dy\cdot\chi_Q\cdot\chi_R\cdot gudx\\
&+\Psi(R)^{\beta/q}\frac{1}{u(R)^{\lambda/q}}\sup_{\|g\|_{L^{q'}(u)}=1}\int\sum_{Q\in\mathcal{S},R\subset Q}|Q|^{\alpha/n-1}\int_Q f\sigma dy\cdot\chi_Q\cdot\chi_R\cdot gudx\\
&=: J_1+J_2.
\end{aligned}$$

First, we estimate the term J_1. Since $s(p)=1+q/p'$ and $(u,\sigma)\in A^{\rho}_{s(p)}$, we have

$$\left(\frac{1}{\Psi(Q)|Q|}\int_Q u(x)dx\right)\left(\frac{1}{\Psi(Q)|Q|}\int_Q\sigma(x)dx\right)^{q/p'}\leqslant[u,\sigma]_{A^{\rho}_{s(p)}}.$$

Now we see from $1/p-1/q=\alpha/n$ that

$$|Q|^{\alpha/n-1}\leqslant[u,\sigma]^{1/q}_{A^{\rho}_{s(p)}}\Psi(Q)^{1/q+1/p'}u(Q)^{-1/q}\sigma(Q)^{-1/p'}.$$

Hence,

$$\begin{aligned}
J_1&=\Psi(R)^{\beta/q}\cdot\frac{1}{u(R)^{\lambda/q}}\sup_{\|g\|_{L^{q'}(u)}=1}\int\sum_{Q\in\mathcal{S},Q\subset R}|Q|^{\alpha/n-1}\int_Q f\sigma dy\cdot\chi_Q\cdot\chi_R\cdot gudx\\
&\lesssim\Psi(R)^{\beta/q}\Psi(Q)^{1/q+1/p'}\cdot\frac{1}{u(R)^{\lambda/q}}\sup_{\|g\|_{L^{q'}(u)}=1}\sum_{Q\in\mathcal{S},Q\subset R}u(Q)^{-1/q}\sigma(Q)^{-1/p'}\int_Q|f|\sigma dy\cdot\int_Q|g|udx\\
&=\Psi(R)^{\beta/q}\Psi(Q)^{1/q+1/p'}\cdot\frac{1}{u(R)^{\lambda/q}}\sup_{\|g\|_{L^{q'}(u)}=1}\sum_{Q\in\mathcal{S},Q\subset R}u(Q)^{1/q'}\sigma(Q)^{1/p}\cdot\frac{1}{\sigma(Q)}\int d_Q|f|\sigma dy\frac{1}{u(Q)}\int_Q|g|udx.
\end{aligned}$$

Note that $|E(Q)|\geqslant\frac{1}{2}|Q|$ and $u,\sigma\in A^{\rho}_{\infty}$, we have

$$u(Q)\lesssim u(E(Q)),\quad\sigma(Q)\lesssim\sigma(E(Q)).$$

Hence,

$$J_1 \lesssim \Psi(R)^{\beta/q}\Psi(Q)^{1/q+1/p'} \cdot \frac{1}{u(R)^{\lambda/q}} \sup_{\|g\|_{L^{q'}(u)}=1} \sum_{Q\in\mathcal{S},Q\subset R} u(Q)^{1/q'}\sigma(Q)^{1/p} \cdot \frac{1}{\sigma(Q)}\int d_Q|f|\sigma dy \frac{1}{u(Q)}\int_Q |g|udx.$$

$$\lesssim \Psi(R)^{\beta/q}\Psi(Q)^{1/q+1/p'} \cdot \frac{1}{u(Q)^{\lambda}} \sup_{\|g\|_{L^{q'}(u)}=1} \left(\sum_{Q\in\mathcal{S},Q\subset R} \sigma(E(Q))\left(\frac{1}{\sigma(Q)}\int_Q |f|\sigma dy\right)^p\right)^{1/p}$$

$$\cdot \left(\sum_{Q\in\mathcal{S},Q\subset R} u(E(Q))^{p'/q'}\left(\frac{1}{u(Q)}\int_Q |g|udx\right)^{p'}\right)^{1/p'}.$$

Observe that $1/p-1/q>0,$ we have $p'>q'.$ Thus

$$\left(\sum_{Q\in\mathcal{S},Q\subset R} u(E(Q))^{p'/q'}\left(\frac{1}{u(Q)}\int_Q |g|udx\right)^{p'}\right)^{1/p'} \leqslant \left(\sum_{Q\in\mathcal{S},Q\subset R} u(E(Q))\left(\frac{1}{u(Q)}\int_Q |g|udx\right)^{q'}\right)^{1/q'}.$$

Since $Q\subset R$ and $x\in E(Q),$ we have

$$\frac{1}{\sigma(Q)}\int_Q |f|\sigma dy \leqslant M_\sigma(f\cdot\chi_R)(x).$$

Similarly, we see from $x\in E(Q)$ that

$$\frac{1}{u(Q)}\int_Q |g|udy \leqslant M_u g(x).$$

So

$$J_1 \lesssim \Psi(R)^{\beta/q}\Psi(Q)^{1/q+1/p'} \cdot \frac{1}{u(Q)^{\lambda}} \sup_{\|g\|_{L^{q'}(u)}=1} \left(\sum_{Q\in\mathcal{S},Q\subset R} \sigma(E(Q))\left(\frac{1}{\sigma(Q)}\int_Q |f|\sigma dy\right)^p\right)^{1/p}$$

$$\cdot \left(\sum_{Q\in\mathcal{S},Q\subset R} u(E(Q))^{p'/q'}\left(\frac{1}{u(Q)}\int_Q |g|udx\right)^{p'}\right)^{1/p'}$$

$$\lesssim \Psi(R)^{\beta/q}\Psi(Q)^{1/q+1/p'} \cdot \frac{1}{u(R)^{\lambda/q}} \sup_{\|g\|_{L^{q'}(u)}=1} \left(\sum_{Q\in\mathcal{S},Q\subset R} \int_{E(Q)} (M_\sigma(f\cdot\chi_R)(x))^p\sigma dy\right)^{1/p}$$

$$\cdot \left(\sum_{Q\in\mathcal{S},Q\subset R} \int_{E(Q)} (M_u g(x))^{q'}udy\right)^{1/q'}$$

$$\lesssim \Psi(R)^{\beta/q}\Psi(Q)^{1/q+1/p'} \cdot \frac{1}{u(R)^{\lambda/q}} \sup_{\|g\|_{L^{q'}(u)}=1} \|M_\sigma(f\cdot\chi_R)\|_{L^p(\sigma)}\cdot\|M_u g\|_{L^{q'}(u)}$$

$$\lesssim \Psi(R)^{\beta/q}\Psi(Q)^{1/q+1/p'} \cdot \frac{1}{u(Q)^{\lambda}}\|f\|_{L^p(\sigma)}$$

$$\lesssim \|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}. \tag{3.4}$$

Next, we estimate the term J_2. We have

$$\begin{aligned}
J_2 &= \Psi(R)^{\beta/q}\frac{1}{u(R)^{\lambda/q}}\sup_{\|g\|_{L^{q'}(u)}=1}\int\sum_{Q\in\mathcal{S},R\subset Q}|Q|^{\alpha/n-1}\int_Q f\sigma dy\cdot\chi_Q\cdot\chi_R\cdot gudx\\
&= \Psi(R)^{\beta/q}\frac{1}{u(R)^{\lambda/q}}\sum_{Q\in\mathcal{S},R\subset Q}|Q|^{\alpha/n-1}\int_Q f\sigma dy\cdot\int_R gudx\\
&\leqslant \Psi(R)^{\beta/q}\frac{1}{u(R)^{\lambda/q}}\sum_{Q\in\mathcal{S},R\subset Q}|Q|^{\alpha/n-1}\left(\int_Q|f|^p\sigma dy\right)^{1/p}\cdot\sigma(Q)^{1/p'}\cdot\left(\int_R|g|^{q'}udx\right)^{1/q'}\cdot u(R)^{1/q}\\
&\leqslant \Psi(R)^{\beta/q}\Psi(Q)^{1/q+1/p'}\sum_{Q\in\mathcal{S},R\subset Q}\frac{1}{u(R)^{\lambda/q}}\cdot\frac{1}{u(Q)^{1/q}}\cdot u(Q)^{\lambda/q}\left(\frac{1}{u(Q)^{\lambda/q}}\left(\int_Q|f|^p\sigma dy\right)^{1/p}\right)\cdot u(R)^{1/q}\\
&\leqslant \|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}\sum_{Q\in\mathcal{S},R\subset Q}\frac{u(Q)^{\lambda/q}\cdot u(R)^{1/q}}{u(R)^{\lambda/q}\cdot u(Q)^{1/q}}\\
&= \|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}\sum_{Q\in\mathcal{S},R\subset Q}\frac{u(R)^{1/q-\lambda/q}}{u(Q)^{1/q-\lambda/q}}. \qquad (3.5)
\end{aligned}$$

Given a general dyadic cube $Q\in\mathcal{D}$, let $Q^{(1)}$ denote the parent cube of Q. Since $u\in A^{\rho}_{\infty}$, we see from the $u\in A^{\rho}_{\infty}$, condition that there exists some constant $c_0<1$ such that

$$u(Q)\leqslant c_0\cdot u(Q^{(1)}).$$

Consequently,

$$\begin{aligned}
\|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}\sum_{Q\in\mathcal{S},R\subset Q}\frac{u(R)^{1/q-\lambda/q}}{u(Q)^{1/q-\lambda/q}} &\leqslant \|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}\sum_{k=0}^{\infty}c_0^{k(1/q-\lambda/q)}\\
&\lesssim \|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}.
\end{aligned}$$

Now we see from (3.4) and (3.5) that (3.3) is true.

Take some cube $Q\in\mathbb{R}^n$. Then there exist 2^n cubes $Q_i\in\mathcal{D}, 1\leqslant i\leqslant 2^n$, such that $Q\subset\bigcup_{i=1}^{2^n}$ and $l(Q)\leqslant l(Q_i)=l(Q_j)<2l(Q)$ for all $1\leqslant i,j\leqslant 2^n$ Hence $|Q|\simeq\left|\bigcup_{i=1}^{2^n}Q_i\right|\simeq|Q_i|$. Note that $u\in A^{\rho}_{\infty}$. We have

$$u(Q)\simeq u(Q_i),\quad i\leqslant i\leqslant 2^n.$$

Therefore,

$$\Psi(Q)^{\beta/q}\frac{1}{u(Q)^{\lambda/q}}\|I^{\mathcal{S}}_{\alpha}(fsigma)\cdot\chi_Q\|_{L^q(u)}\leqslant\sum_{i=1}^{2^n}\Psi(Q_i)^{\beta/q}\frac{1}{u(Q_i)^{\lambda/q}}\|I^{\mathcal{S}}_{\alpha}(f\sigma)\cdot\chi_{Q_i}\|_{L^q(u)}.$$

Since $Q_i\in\mathcal{D},\quad 1\leqslant i\leqslant 2^n$, we see from (3.3) that for any cube Q,

$$\Psi(Q)^{\beta/q}\frac{1}{u(Q)^{\lambda/q}}\|I^{\mathcal{S}}_{\alpha}(f\sigma)\cdot\chi_Q\|_{L^q(u)}\lesssim\|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}.$$

Taking the supremum over all cubes in $\mathbb{R}^n$, we get

$$\|I_\alpha^{\mathcal{S}}(f\cdot\sigma)\|_{L^{q,\lambda}_{\beta,V,u}} \lesssim \|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}.$$

By lemma 2.1, we know that $\mathcal{I}_\alpha(f)(x) \lesssim I_\alpha(f)(x)$, hence,

$$\|\mathcal{I}_\alpha^{\mathcal{S}}(f\cdot\sigma)\|_{L^{q,\lambda}_{\beta,V,u}} \lesssim \|I_\alpha^{\mathcal{S}}(f\cdot\sigma)\|_{L^{q,\lambda}_{\beta,V,u}}.$$

Using proposition 2.5

$$\|\mathcal{I}_\alpha(f\cdot\sigma)\|_{L^{q,\lambda}_{\beta,V,u}} \lesssim \|f\|_{L^{p,\lambda p/q}_{\beta,V,(\sigma,u)}}.$$

This comletes the proof.

Given a general dyadic grid $\mathcal{D}$, let $\mathcal{D}$, and $Q\in\mathcal{D}$. Denote

$$I_\alpha^{\mathcal{S}(Q)}f(x) := \sum_{Q'\in\mathcal{S},Q'\subset Q} \frac{1}{|Q'|^{1-\alpha/n}}\int_{Q'} f dx\cdot\chi_{Q'}(x). \tag{3.6}$$

For $1<p\leqslant q<\infty$ and a pair of weights (u,σ), define the testing condition of (u,σ) by

$$[u,\sigma]^{\mathcal{D}}_{(I_\alpha^{\mathcal{S}})^{q',p'}} := \sup_{Q\in\mathcal{D}}\left(\frac{1}{\Psi(Q)}\right)^{-1/q'}\left(\frac{1}{\Psi(Q)}\int_Q I_\alpha^{\mathcal{S}(Q)}(\chi_Q u)^{p'}\sigma dx\right)^{1/p'}.$$

To prove Theorem 1.3, we need following result.

Lemma 3.1. Given $0<\alpha<n$ and $1<p<n/\alpha$, define q by (2.1). Suppose that (u,σ) is a pair of weights with $[u,\sigma]_{A^\rho_{s(p)}}<\infty$ and $\mathcal{D}$ is a general dyadic grid with sparse subset $\mathcal{S}$. If $u\in A^\rho_\infty$, then

$$[u,\sigma]^{\mathcal{D}}_{(I_\alpha^{\mathcal{S}})^{q',p'}} \lesssim [u,\sigma]^{1/q}_{A^\rho_{s(p)}}[u]^{1/p'}_{A^\rho_\infty},$$

where the constant in the above inequality is independent of $\mathcal{D}$ or $\mathcal{S}$.

This is a improved version by Cruz–Uribe and Moen[21].

Proof of Theorem 1.3. For any $t>0$, denote $\Omega_t := \{x : I_\alpha^{\mathcal{S}} > t\}$. Let $\mathcal{Q}_t$ be the collection of all maximal general dyadic cubes in the sparse family $\mathcal{S}$ which is contained in Ω_t. For $R\in\mathcal{D}$, denote

$$\mathcal{Q}_{R,t} := \{Q\cap R : Q\in\mathcal{Q}_t\}.$$

There are two cases.

(i). $\mathcal{Q}_{R,t} = R$.

In this case, there exists some $Q_0\in\mathcal{Q}_t$ such that $R\subset Q_0$. Since $\mathcal{S}$ is a sparse family, there is some $x_0\in Q_0\backslash\cup_{Q'\in\mathcal{S},Q'\subsetneq Q_0}$. Note that $x_0\in\Omega_t$. We have

$$t < I_\alpha^{\mathcal{S}}(f\sigma)(x_0) = \sum_{Q'\in\mathcal{S},Q_0\subset Q'} |Q'|^{\alpha/n-1}\int_{Q'} f\sigma dy.$$

Since $(u,\sigma)\in A^{\rho}_{s(p)}$, we have $|Q'|^{\alpha/n-1}\lesssim \Psi(Q')^{1/q+1/p'}u(Q')^{-1/q}\sigma(Q')^{-1/p'}$. Hence,

$$
\begin{aligned}
t &\lesssim \Psi(Q')^{1/q+1/p'}\sum_{Q'\in\mathcal{S},Q_0\subset Q'} u(Q')^{-1/q}\sigma(Q')^{-1/p'}\int_{Q'} f\sigma dy\\
&\leqslant \Psi(Q')^{1/q+1/p'}\sum_{Q'\in\mathcal{S},Q_0\subset Q'} u(Q')^{-1/q}\sigma(Q')^{-1/p'}\sigma(Q)^{1/p'}\left(\int_{Q'}|f|^p\sigma dy\right)^{1/p}\\
&\leqslant \Psi(Q')^{1/q+1/p'-\beta/q}\sum_{Q'\in\mathcal{S},Q_0\subset Q'} u(Q')^{-1/q}u(Q')^{\lambda/q}\|f\|_{L^{p,\lambda p/q}_{\beta,V,(u,\sigma)}}.
\end{aligned}
$$

On the other hand, let $Q^{(1)}$ denote the parent cube of Q, since $u\in A^{\rho}_{\infty}$, there exists some constant $0<c_0<1$, such that

$$u(Q)\leqslant c_0\cdot u(Q^{(1)}).$$

This give us

$$
\begin{aligned}
t &\lesssim \Psi(R)^{1/q+1/p'-\beta/q}u(Q_0)^{-1/q+\lambda/q}\|f\|_{L^{p,\lambda p/q}_{\beta,V,(u,\sigma)}}\sum_{k=0}^{\infty}c_0^{k(1/q-\lambda/q)}\\
&\lesssim \Psi(R)^{1/q+1/p'-\beta/q}u(R)^{-1/q+\lambda/q}\|f\|_{L^{p,\lambda p/q}_{\beta,V,(u,\sigma)}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Psi(R)^{\beta/q}u(R)^{-\lambda/q}\cdot t\cdot u(\{x\in R: I^{\mathcal{S}}_{\alpha}(f\sigma)(x)>t\})^{1/q}\\
&=\Psi(R)^{\beta/q}\cdot u(R)^{-\lambda/q}\cdot t\cdot u(R)^{1/q}\\
&\lesssim \Psi(R)^{(\beta+1)(1/q-1/p)+1}u(R)^{-\lambda/q}\cdot u(R)^{-1/q+\lambda/q}\|f\|_{L^{p,\lambda p/q}_{\beta,V,(u,\sigma)}}u(R)^{1/q}\\
&\lesssim \|f\|_{L^{p,\lambda p/q}_{\beta,V,(u,\sigma)}}.
\end{aligned}
$$

(ii). $\mathcal{Q}_{R,t}\neq\{R\}$

In this case, for any $Q\in\mathcal{Q}_{R,t}$, we have $Q\subsetneq R$, and $Q\in\mathcal{Q}_t$. Set $\mathcal{Q}^{*}_{R,t}:=\{Q: Q\in\mathcal{Q}_{R,t}, Q\cup\Omega_{2t}\neq\emptyset\}$. Recall that for any general dyadic cube Q, $I^{\mathcal{S}(Q)}_{\alpha}$ is defined by (3.6).

Fix some $Q\in\mathcal{Q}^{*}_{R,t}$, since Q is the maximal general dyadic cube in Ω_t, we have

$$\sum_{Q'\in\mathcal{S},Q\subsetneq Q'}\frac{1}{|Q'|^{1-\alpha/n}}\int_{Q'} f\sigma dy\leqslant t \tag{3.7}$$

On the other hand, since $Q\cap\Omega_{2t}\neq\emptyset$, for any $x\in Q\cap\Omega_{2t}$, we have

$$2t\leqslant I^{\mathcal{S}}_{\alpha}(f\sigma)(x).$$

It follows from that

$$t\leqslant I^{\mathcal{S}(Q)}_{\alpha}(f\sigma)(x),\quad x\in Q\cap\Omega_{2t}.$$

Let $\eta = 2^{-q-1}$. We split $\mathcal{Q}^*_{R,t}$ into two subsets ε and $\mathcal{F}$, where

$$\begin{aligned} \varepsilon &:= \{Q \in \mathcal{Q}^*_{R,t} : u\big(Q \cap \Omega_{2t} < \eta u(Q)\big)\}, \\ \mathcal{F} &:= \mathcal{Q}^*_{R,t} \setminus \varepsilon. \end{aligned}$$

We have

$$\begin{aligned} &\Psi(R)^{\beta} u(R)^{-\lambda} (2t)^q u(\{x \in R : x \in \Omega\}) \\ &= \Psi(R)^{\beta} u(R)^{-\lambda} (2t)^q \big(u(\cup_{Q \in \varepsilon} Q \cap \Omega_{2t}) + u(\cup_{Q \in \mathcal{F}} Q \cap \Omega_{2t})\big) \\ &\leqslant \frac{1}{2} \cdot \Psi(R)^{\beta} \cdot u(R)^{-\lambda} \cdot t^q \cdot u\big(\{x \in R, x \in \Omega_t\}\big) \\ &+ \Psi(R)^{\beta} \cdot u(R)^{-\lambda} \cdot 2^q \cdot \eta^{-q} \sum_{Q \in \mathcal{Q}^*_{R,t}} \left(\frac{1}{u(Q)} \int_Q I_{\alpha}^{\mathcal{S}(Q)}(f\sigma) u dx \right)^q u(Q) \\ &\leqslant \frac{1}{2} \cdot \Psi(R)^{\beta} \cdot u(R)^{-\lambda} \cdot t^q \cdot u\big(\{x \in R, x \in \Omega_t\}\big) \\ &+ \Psi(R)^{\beta} \cdot u(R)^{-\lambda} \cdot 2^q \cdot \eta^{-q} \sum_{Q \in \mathcal{Q}^*_{R,t}} \left(\frac{1}{u(Q)} \int_Q f\sigma I_{\alpha}^{\mathcal{S}(Q)}(\chi_Q u) dx \right)^q u(Q)^{1-q}. \end{aligned}$$

Since $(u, \sigma) \in A^{\rho}_{s(p)}$ and $u \in A^{\rho}_{\infty}$, we see from lemma 3.1 that

$$[u, \sigma]^{\mathcal{S})q',p'}_{(I_{\alpha}} \mathcal{D} = \sup_{Q \in \mathcal{D}} \left(\frac{1}{\Psi(Q)} \int_Q u dx \right)^{-1/q'} \left(\frac{1}{\Psi(Q)} \int_Q I_{\alpha}^{\mathcal{S}(D)}(\chi_Q u)^{p'} \sigma dx \right)^{1/p'} < \infty.$$

Hence

$$\Psi(Q)^{q-1-q/p'} \left(\int_Q I_{\alpha}^{\mathcal{S}(Q)}(\chi_Q u)^{p'} \sigma dx \right)^{q/p'} u(Q)^{1-q} \lesssim u(Q)^{q/q'+1-q}.$$

It follows that

$$\begin{aligned} \left(\frac{1}{u(Q)} \int_Q f\sigma I_{\alpha}^{\mathcal{S}(Q)}(\chi_Q u) dx \right)^q u(Q)^{1-q} &= \left(\frac{1}{u(Q)} \int_Q f\sigma^{1/p} \sigma^{1/p'} I_{\alpha}^{\mathcal{S}(Q)}(\chi_Q u) dx \right)^q u(Q)^{1-q} \\ &\leqslant \left(\int_Q |f|^p \sigma \right)^{q/p} \left(\int_Q I_{\alpha}^{\mathcal{S}(Q)}(\chi_Q u)^{p'} \sigma dx \right)^{q/p'} u(Q)^{1-q} \\ &\lesssim \left(\int_Q |f|^p \sigma \right)^{q/p} \Psi(Q)^{1+q/p'-q} u(Q)^{q/q'+1-q}. \qquad (3.8) \end{aligned}$$

By (3.8), we have

$$
\begin{aligned}
&\Psi(R)^{\beta}u(R)^{-\lambda}(2t)^{q}u(\{x\in R: x\in\Omega_t\})\\
&\leqslant \frac{1}{2}\Psi(R)^{\beta}\cdot u(R)^{-\lambda}\cdot t^{q}u(\{x\in R: x\in\Omega_t\})+C\frac{\Psi(R)^{\beta}}{\Psi(Q)^{q-1-q/p'}}\cdot u(R)^{-\lambda}\sum_{Q\in\mathcal{Q}^{*}_{R,t}}\left(\int_Q |f|^{p}\sigma dy\right)^{q/p}u(Q)^{q/q'+1-q}\\
&\leqslant \frac{1}{2}\Psi(R)^{\beta}\cdot u(R)^{-\lambda}\cdot t^{q}u(\{x\in R: x\in\Omega_t\})+C\frac{\Psi(R)^{\beta}}{\Psi(Q)^{q-1-q/p'}}\cdot u(R)^{-\lambda}\left(\sum_{Q\in\mathcal{Q}^{*}_{R,t}}\int_Q |f|^{p}\sigma dy\right)^{q/p}\\
&\leqslant \frac{1}{2}\Psi(R)^{\beta}\cdot u(R)^{-\lambda}\cdot t^{q}u(\{x\in R: x\in\Omega_t\})+C\frac{\Psi(R)^{\beta}}{\Psi(Q)^{q-1-q/p'}}\cdot u(R)^{-\lambda}\left(\int_R |f|^{p}\sigma dy\right)^{q/p}\\
&\leqslant \frac{1}{2}\Psi(R)^{\beta}\cdot u(R)^{-\lambda}\cdot t^{q}u(\{x\in R: x\in\Omega_t\})+C\|f\|^{q}_{L^{p,\lambda p/q}_{\beta,V,(u,\sigma)}}.
\end{aligned}
$$

Taking the supremum over all positive numbers t and all cubes R in $\mathcal{D}$, we get

$$
\sup_{t>0}\sup_{R\in\mathcal{D}}\Psi(R)^{\beta/q}\cdot u(R)^{-\lambda/q}\cdot t^{1/q}\cdot u(\{x\in R: I^{\mathcal{S}}_{\alpha}(f\sigma)(x)>t\})\lesssim\|f\|_{L^{p.\lambda p/q}_{\beta,V,(u,\sigma)}}.
$$

By lemma 2.1, we know that $\mathcal{I}_{\alpha}(f)(x)\lesssim I_{\alpha}(f)(x)$, hence,

$$
\|\mathcal{I}^{\mathcal{S}}_{\alpha}(f\cdot\sigma)\|_{WL^{q,\lambda}_{\beta,V,u}}\lesssim\|I^{\mathcal{S}}_{\alpha}(f\cdot\sigma)\|_{WL^{q,\lambda}_{\beta,V,u}}.
$$

For any cube $Q\in\mathbb{R}^n$, similarly to Theorem 1.2 we obtain

$$
\begin{aligned}
\|\mathcal{I}^{\mathcal{S}}_{\alpha}(f\cdot\sigma)\|_{WL^{q,\lambda}_{\beta,V,u}}&=\sup_{t>0}\sup_{Q\in\mathcal{D}}\Psi(Q)^{\beta/q}\cdot u(Q)^{-\lambda/q}\cdot t^{1/q}\cdot u(\{x\in Q:\mathcal{I}^{\mathcal{S}}_{\alpha}(f\cdot\sigma)(x)>t\})\\
&\lesssim\|f\|_{L^{p.\lambda p/q}_{\beta,V,(u,\sigma)}}.
\end{aligned}
$$

This completes the proof.

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