Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 2614 – 2622
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doi:10.3906/mat-2006-24

On Lyapunov-type inequalities for $(n+1)$st order nonlinear differential equations with the antiperiodic boundary conditions

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Received: 08.06.2020 • **Accepted/Published Online:** 07.10.2021 • **Final Version:** 29.11.2021

Abstract: In this paper, we establish new Lyapunov-type inequalities for $(n+1)$st order nonlinear differential equation including p-relativistic operator and q-prescribed curvature operator under the antiperiodic boundary conditions.

Key words: Lyapunov-type inequality, p-relativistic operator, q-prescribed curvature operator, the antiperiodic boundary condition

1. Introduction

In [12], Lyapunov obtained the following remarkable result: If $q \in C([0,\infty),\mathbb{R})$ and $x(t)$ is a solution of second order linear problem

$$x'' + q(t)\,x = 0 \tag{1.1}$$

with Dirichlet boundary condition

$$x(a) = x(b) = 0, \tag{1.2}$$

where $a, b \in \mathbb{R}$ with $b > a$ and $x(t) \not\equiv 0$ for $t \in (a,b)$, then the Lyapunov inequality

$$\int_a^b |q(s)|\,\mathrm{d}s \geq \frac{4}{b-a} \tag{1.3}$$

holds. Thus, the Lyapunov inequality (1.3) provides a lower bound for the distance between two consecutive zeros of x. The inequality (1.3) is best possible in the sense that the constant 4 in the left hand side of (1.3) cannot be replaced by a larger number ([11, p. 267]). This result has found many applications in areas like eigenvalue problems, stability, oscillation theory, disconjugacy, etc. Since then, there have been several results to improve and generalize the linear equation (1.1) in many directions [1–4, 13–16].

More recently, by replacing x'' and a relativistic acceleration $\left(\frac{x'}{\sqrt{1-x'^2}}\right)'$ in Eq. (1.1), Yang et al. [16] have proved that if we assume that

$$q \in \mathcal{A} = \left\{ q \in L^1_{loc}((a,b),[0,\infty)) : \infty > \int_a^b (s-a)(b-s)\,q(s)\,\mathrm{d}s \right\} \tag{1.4}$$

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2010 *AMS Mathematics Subject Classification:* 34B15, 34C10, 34L15

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and second order nonlinear problem

$$\left(\frac{x'}{\sqrt{1-x'^2}}\right)' + q(t)x = 0, \quad 1 > |x'|, \tag{1.5}$$

with Dirichlet boundary condition (1.2) has a positive solution, then the Lyapunov-type inequality

$$\int_a^b (s-a)(b-s)\, q(s)\, \mathrm{d}s > b-a \tag{1.6}$$

holds.

In 2012, Wang [14] considered the following $(n+1)$st order nonlinear differential equation

$$\left(\left|x^{(n)}\right|^{p-2} x^{(n)}\right)' + q(t)\left|x\right|^{p-2} x = 0, \tag{1.7}$$

where $p > 1$, $q(t) \in C([0,\infty), \mathbb{R})$, and $x(t)$ is a solution of (1.7) satisfying the antiperiodic boundary conditions

$$x^{(k)}(a) + x^{(k)}(b) = 0, \quad k = 0, 1, ..., n, \tag{1.8}$$

where $a, b \in \mathbb{R}$ with $b > a$ and $x(t) \not\equiv 0$ for $t \in (a,b)$ and so it is obtained the following inequality

$$\int_a^b |q(s)|\, \mathrm{d}s > 2\left(\frac{2}{b-a}\right)^{n(p-1)}. \tag{1.9}$$

In 2015, Wang et al. [13] considered the following $(n+1)$st order nonlinear differential equation

$$\left(\left|x^{(n)}\right|^{p-2} x^{(n)}\right)' + \sum_{i=0}^{n} q_i(t)\left|x^{(i)}\right|^{p-2} x^{(i)} = 0, \tag{1.10}$$

where $p > 1$, $q_k(t) \in C([0,\infty), \mathbb{R})$, $k = 0, 1, ..., n$, $n \in \mathbb{N}$, and $x(t)$ is a solution of (1.10) satisfying the antiperiodic boundary conditions (1.8) and so they obtained the following inequality

$$\int_a^b |q_n(s)|\, \mathrm{d}s + \sum_{i=0}^{n-1}\Big((b-a)\, S_{n-i}\Big)^{(p-1)/2} \int_a^b |q_i(s)|\, \mathrm{d}s > 2, \tag{1.11}$$

where

$$S_n = \frac{\left(2^{2n}-1\right)(b-a)^{2n-1}}{2^{2n-1}\pi^{2n}}\, \zeta(2n), \quad n = 1, 2, ..., \tag{1.12}$$

and $\zeta(z) = \sum\limits_{i=1}^{+\infty} \frac{1}{i^z}$, $Re(z) > 1$ is the Riemann zeta function, and the constants $\{S_n\}$ are sharp.

Remark 1.1 *If we take $q_0(t) = q(t)$ and $q_k(t) = 0$, $k = 1, 2, ..., n$ in (1.10), then, from*

$$1 > \frac{2^2\left(2^{2n}-1\right)}{\pi^{2n}} > \frac{2\left(2^{2n}-1\right)\zeta(2n)}{\pi^{2n}}, \quad n > 1, \tag{1.13}$$

the inequality (1.11) is sharper than (1.9) when $n > 1$ [13].

We now consider the $(n+1)$st order nonlinear differential equation of the form

$$\left(\frac{\left|x^{(n)}\right|^{p-2}x^{(n)}}{\left(1-\left|x^{(n)}\right|^{p}\right)^{(p-1)/p}}\right)'+\sum_{i=0}^{n-1}q_i(t)\frac{\left|x^{(i)}\right|^{p-2}x^{(i)}}{\left(1+\left|x^{(i)}\right|^{p}\right)^{(p-1)/p}}=0, \tag{1.14}$$

where $p>1$, $q_k(t)\in C([0,\infty),\mathbb{R})$, $k=0,1,...,n-1$, $n\in\mathbb{N}$, and $x(t)$ is a solution of (1.14) satisfying the antiperiodic boundary conditions (1.8).

In this paper, we prove new Lyapunov-type inequalities for the problem (1.14) with (1.8) including p-relativistic operator

$$\varphi_p(u)=\frac{|u|^{p-2}u}{\left(1-|u|^p\right)^{(p-1)/p}},\quad 1>|u|, \tag{1.15}$$

and q-prescribed curvature operator

$$\varphi_q(u)=\frac{|u|^{q-2}u}{\left(1+|u|^q\right)^{(q-1)/q}},\quad u\in\mathbb{R}, \tag{1.16}$$

[5, 8–10]. Note that if we take $p=q=2$, then they turn out to be the relativistic and the prescribed curvature operator φ_2, which have been attracting much attention in differential geometry and partial differential equation. It has many important applications in physics, biology and other inter disciplines [6, 7]. To the best of our knowledge, this is the first paper using the p-relativistic operator φ_p and q-prescribed curvature operator φ_q to study the above problem.

2. Main results

Firstly, we state some lemmas which we will use in the proof of our main results.

Lemma 2.1 *[13, Lemma 3.2] If $x(t)$ is a nontrivial solution of the problem (1.14) with (1.8), then we have*

$$\sqrt{(b-a)S_{n-k}}M_n\geq M_k,\quad k=0,1,...,n-1, \tag{2.1}$$

where S_n is given in (1.12) and

$$M_n=\sup_{a\leq t\leq b}\left|x^{(n)}(t)\right|. \tag{2.2}$$

Lemma 2.2 *[14, Lemma 3.2] If $x(t)$ is a nontrivial solution of the problem (1.14) with (1.8), then we have*

$$\frac{1}{2}\left(\frac{b-a}{2}\right)^{(n-k)(p-1)}\int_a^b\left|x^{(n)}(s)\right|^p\mathrm{d}s\geq\left|x^{(n-1)}(t)\right|\left|x^{(k)}(t)\right|^{p-1} \tag{2.3}$$

for $k=0,1,...,n-1$.

The proof of Lemma 2.2 is similar to that of Lemma 3.2 given by Wang [14], and hence is omitted.

Now, we give main results obtained which are inspired by the results obtained by Wang et al. [13] and Wang [14].

Theorem 2.3 *If $x(t)$ is a nontrivial solution of the problem (1.14) with (1.8), then the following inequality*

$$\sum_{i=0}^{n-1}\Big((b-a)\,S_{n-i}\Big)^{(p-1)/2}\int_a^b |q_i(s)|\,\mathrm{d}s > 2 \tag{2.4}$$

holds, where S_n is given in (1.12).

Proof Assume that $x(t)$ is a solution of Eq. (1.14) satisfying the antiperiodic boundary conditions $x^{(k)}(a)+x^{(k)}(b)=0,\ k=0,1,...,n$, where $a,b\in\mathbb{R}$ with $b>a$ and $x(t)\not\equiv 0$ for $t\in(a,b)$. Now, we define

$$H(t,s)=\begin{cases} 1/2 & , a\le s\le t,\\ -1/2 & , t\le s\le b.\end{cases} \tag{2.5}$$

Multiplying both sides of (1.14) by the function H, integrating from a to b, and using the antiperiodic boundary conditions (1.8) with $i=n$, we have

$$\begin{aligned}\frac{\left|x^{(n)}(t)\right|^{p-2}x^{(n)}(t)}{\left(1-\left|x^{(n)}(t)\right|^{p}\right)^{(p-1)/p}} &= \int_a^b H(t,s)\left(\frac{\left|x^{(n)}(s)\right|^{p-2}x^{(n)}(s)}{\left(1-\left|x^{(n)}(s)\right|^{p}\right)^{(p-1)/p}}\right)'\mathrm{d}s\\ &= -\sum_{i=0}^{n-1}\int_a^b H(t,s)\,q_i(s)\frac{\left|x^{(i)}(s)\right|^{p-2}x^{(i)}(s)}{\left(1+\left|x^{(i)}(s)\right|^{p}\right)^{(p-1)/p}}\mathrm{d}s.\end{aligned} \tag{2.6}$$

Taking absolute value of (2.6) and using Lemma 2.1, we get

$$\begin{aligned}\left|x^{(n)}(t)\right|^{p-1} &\le \frac{\left|x^{(n)}(t)\right|^{p-1}}{\left(1-\left|x^{(n)}(t)\right|^{p}\right)^{(p-1)/p}}\\ &\le \sum_{i=0}^{n-1}\int_a^b |H(t,s)|\,|q_i(s)|\,\frac{\left|x^{(i)}(s)\right|^{p-1}}{\left(1+\left|x^{(i)}(s)\right|^{p}\right)^{(p-1)/p}}\mathrm{d}s\\ &= \frac{1}{2}\sum_{i=0}^{n-1}\int_a^b |q_i(s)|\,\frac{\left|x^{(i)}(s)\right|^{p-1}}{\left(1+\left|x^{(i)}(s)\right|^{p}\right)^{(p-1)/p}}\mathrm{d}s\\ &\le \frac{1}{2}\sum_{i=0}^{n-1}\int_a^b |q_i(s)|\left|x^{(i)}(s)\right|^{p-1}\mathrm{d}s\\ &< \frac{1}{2}\sum_{i=0}^{n-1}M_i^{p-1}\int_a^b |q_i(s)|\,\mathrm{d}s\\ &\le \frac{M_n^{p-1}}{2}\sum_{i=0}^{n-1}\Big((b-a)\,S_{n-i}\Big)^{(p-1)/2}\int_a^b |q_i(s)|\,\mathrm{d}s\end{aligned} \tag{2.7}$$

and so

$$\frac{M_n^{p-1}}{2}\sum_{i=0}^{n-1}\Big((b-a)\,S_{n-i}\Big)^{(p-1)/2}\int_a^b |q_i(s)|\,\mathrm{d}s > M_n^{p-1}, \tag{2.8}$$

where S_n and M_n are given in (1.12) and (2.2), respectively. Now, we claim that $M_n > 0$. In fact, if it is not true, then we have $M_n = 0$ or $x^{(n)}(t) = 0$ for $t \in [a,b]$. By the antiperiodic boundary conditions (1.8), we obtain $x(t) = 0$ for $t \in [a,b]$, which contradicts to the assumption that $x(t)$ is a nontrivial solution of (1.14) with (1.8). Therefore, we obtain $M_n > 0$. Dividing both sides of the inequality (2.8) by M_n^{p-1}, we obtain

$$\sum_{i=0}^{n-1}\Big((b-a)\,S_{n-i}\Big)^{(p-1)/2}\int_a^b |q_i(s)|\,\mathrm{d}s > 2. \tag{2.9}$$

□

Theorem 2.4 *If $x(t)$ is a nontrivial solution of the problem (1.14) with (1.8), then the following inequality*

$$\sum_{i=0}^{n-1}\left(\frac{b-a}{2}\right)^{(n-i)(p-1)}\int_a^b |q_i(s)|\,\mathrm{d}s > 2 \tag{2.10}$$

holds.

Proof Assume that $x(t)$ is a solution of Eq. (1.14) satisfying the antiperiodic boundary conditions $x^{(k)}(a) + x^{(k)}(b) = 0$, $k = 0, 1, ..., n$, where $a, b \in \mathbb{R}$ with $b > a$ and $x(t) \not\equiv 0$ for $t \in (a,b)$. Multiplying both sides of (1.14) by $x^{(n-1)}(t)$ and integrating from a to b, we have

$$\int_a^b x^{(n-1)}(s)\left(\frac{\left|x^{(n)}(s)\right|^{p-2}x^{(n)}(s)}{\left(1-\left|x^{(n)}(s)\right|^p\right)^{(p-1)/p}}\right)'\mathrm{d}s + \sum_{i=0}^{n-1}\int_a^b q_i(s)\,x^{(n-1)}(s)\,\frac{\left|x^{(i)}(s)\right|^{p-2}x^{(i)}(s)}{\left(1+\left|x^{(i)}(s)\right|^p\right)^{(p-1)/p}}\mathrm{d}s = 0. \tag{2.11}$$

Integrating by parts the first integral of equation (2.11) and using the antiperiodic boundary conditions (1.8), we get

$$\int_a^b \frac{\left|x^{(n)}(s)\right|^p}{\left(1-\left|x^{(n)}(s)\right|^p\right)^{(p-1)/p}}\mathrm{d}s = \sum_{i=0}^{n-1}\int_a^b q_i(s)\,x^{(n-1)}(s)\,\frac{\left|x^{(i)}(s)\right|^{p-2}x^{(i)}(s)}{\left(1+\left|x^{(i)}(s)\right|^p\right)^{(p-1)/p}}\mathrm{d}s. \tag{2.12}$$

By Lemma 2.2, we have

$$\begin{aligned}
\int_a^b \left|x^{(n)}(s)\right|^p \mathrm{d}s &\le \int_a^b \frac{\left|x^{(n)}(s)\right|^p}{\left(1-\left|x^{(n)}(s)\right|^p\right)^{(p-1)/p}}\mathrm{d}s \\
&= \sum_{i=0}^{n-1}\int_a^b q_i(s)\, x^{(n-1)}(s)\, \frac{\left|x^{(i)}(s)\right|^{p-2} x^{(i)}(s)}{\left(1+\left|x^{(i)}(s)\right|^p\right)^{(p-1)/p}}\mathrm{d}s \\
&\le \sum_{i=0}^{n-1}\int_a^b |q_i(s)|\, \frac{\left|x^{(n-1)}(s)\right|\left|x^{(i)}(s)\right|^{p-1}}{\left(1+\left|x^{(i)}(s)\right|^p\right)^{(p-1)/p}}\mathrm{d}s \\
&\le \sum_{i=0}^{n-1}\int_a^b |q_i(s)|\left|x^{(n-1)}(s)\right|\left|x^{(i)}(s)\right|^{p-1}\mathrm{d}s \\
&< \sum_{i=0}^{n-1}\max_{a\le t\le b}\left|x^{(n-1)}(t)\right|\left|x^{(i)}(t)\right|^{p-1}\int_a^b |q_i(s)|\,\mathrm{d}s \\
&\le \frac{1}{2}\int_a^b \left|x^{(n)}(s)\right|^p \mathrm{d}s \sum_{i=0}^{n-1}\left(\frac{b-a}{2}\right)^{(n-i)(p-1)}\int_a^b |q_i(s)|\,\mathrm{d}s. \qquad (2.13)
\end{aligned}$$

Now, we claim that $\int_a^b \left|x^{(n)}(s)\right|^p \mathrm{d}s > 0$. In fact, if it is not true, then we have $\int_a^b \left|x^{(n)}(s)\right|^p \mathrm{d}s = 0$ or $x^{(n)}(t) = 0$ for $t \in [a,b]$. By the antiperiodic boundary conditions (1.8), we obtain $x(t) = 0$ for $t \in [a,b]$, which contradicts to the assumption that $x(t)$ is a nontrivial solution of (1.14) with (1.8). Therefore, we have $\int_a^b \left|x^{(n)}(s)\right|^p \mathrm{d}s > 0$. Dividing both sides of the inequality (2.13) by $\int_a^b \left|x^{(n)}(s)\right|^p \mathrm{d}s$, we obtain

$$\sum_{i=0}^{n-1}\left(\frac{b-a}{2}\right)^{(n-i)(p-1)}\int_a^b |q_i(s)|\,\mathrm{d}s > 2. \qquad (2.14)$$

□

Remark 2.5 *If we take $n = 1$ in Eq. (1.14), then the inequality (2.4) is equal to (2.10). Note that as m increases, the ratio $R = \left(\frac{2\left(2^{2m}-1\right)\zeta(2m)}{\pi^{2m}}\right)^{1/2}$ decreases rapidly (especially when $m \ge 2$, R is smaller than 1) [15]. Therefore, the inequality (2.4) is sharper than (2.10) when $n > 1$.*

As immediate consequences of Theorems 2.3 and 2.4, the following result gives sufficient conditions for the nonexistence of nontrivial solutions of the boundary value problem (1.14) with (1.8).

Corollary 2.6 (a) *Assume*

$$2 \ge \sum_{i=0}^{n-1}\Big((b-a)\,S_{n-i}\Big)^{(p-1)/2}\int_a^b |q_i(s)|\,\mathrm{d}s, \qquad (2.15)$$

where S_n is given in (1.12). The problem (1.14) with (1.8) has no nontrivial solution.
(b) Assume

$$2 \geq \sum_{i=0}^{n-1} \left(\frac{b-a}{2} \right)^{(n-i)(p-1)} \int_a^b |q_i(s)| \, \mathrm{d}s. \tag{2.16}$$

The problem (1.14) with (1.8) has no nontrivial solution.

Now we give a sufficient condition for the uniqueness of the solution of the following nonhomogenous boundary value problem

$$\begin{cases} \left(\frac{\left|x^{(n)}\right|^{p-2} x^{(n)}}{\left(1-\left|x^{(n)}\right|^{p}\right)^{(p-1)/p}} \right)' + \sum\limits_{i=0}^{n-1} q_i(t) \frac{\left|x^{(i)}\right|^{p-2} x^{(i)}}{\left(1+\left|x^{(i)}\right|^{p}\right)^{(p-1)/p}} = f(t), \quad t \in (a,b), \\ x^{(k)}(a) + x^{(k)}(b) = 0, \quad k = 0,1,...,n, \end{cases} \tag{2.17}$$

where $p > 1$ and $f, q_k \in C([a,b], \mathbb{R})$, $k = 0,1,...,n-1$, $n \in \mathbb{N}$, as an application of Lyapunov-type inequalities *(2.4) and (2.10).*

Theorem 2.7 *If (2.15) or (2.16) holds, then nonhomogenous boundary problem (2.17) has a unique solution.*

Proof To prove the uniqueness, it is sufficient to show that the homogeneous boundary value problem (1.14) with (1.8) has only trivial solution. Assume on the contrary that $x(t) \not\equiv 0$ is a solution of the homogeneous boundary value problem (1.14) with (1.8). Then by using Lyapunov-type inequality *(2.4)* or *(2.10)*, we have

$$\sum_{i=0}^{n-1} \left((b-a) S_{n-i}\right)^{(p-1)/2} \int_a^b |q_i(s)| \, \mathrm{d}s > 2 \tag{2.18}$$

or

$$\sum_{i=0}^{n-1} \left(\frac{b-a}{2} \right)^{(n-i)(p-1)} \int_a^b |q_i(s)| \, \mathrm{d}s > 2 \tag{2.19}$$

which gives contradiction to *(2.15)* or *(2.16)*, respectively. Therefore the homogeneous boundary value problem (1.14) with (1.8) has only trivial solution. Because of the theory of boundary value problems, the nonhomogenous boundary problem *(2.17)* has a unique solution. □

As an example of Theorem 2.7, we consider the following nonhomogenous boundary value problem

$$\begin{cases} \left(\frac{x'}{\sqrt{1-|x'|^2}} \right)' + \frac{x}{4\sqrt{1+|x|^2}} = -\frac{1}{\sin^2 t} + \frac{\sin t}{4\sqrt{1+\sin^2 t}}, \quad t \in (0,\pi), \\ x(0) + x(\pi) = 0, \\ x'(0) + x'(\pi) = 0. \end{cases} \tag{2.20}$$

Since the condition (2.16) is satisfied, the problem (2.20) has a unique solution. One such solution of the problem (2.20) is $x(t) = \sin t$. Note that $|x'|$ is bounded with 1 and the homogeneous boundary value problem corresponding to the problem (2.20) has only trivial solution.

In the following part, we apply the obtained Lyapunov-type inequalities (2.4) and (2.10) to the eigenvalue problems associated with the nonlinear boundary value problem

$$\begin{cases} \left(\frac{\left|x^{(n)}\right|^{p-2} x^{(n)}}{\left(1-\left|x^{(n)}\right|^{p}\right)^{(p-1)/p}} \right)' + \lambda \sum_{i=0}^{n-1} q_i(t) \frac{\left|x^{(i)}\right|^{p-2} x^{(i)}}{\left(1-\left|x^{(i)}\right|^{p}\right)^{(p-1)/p}} = 0, \\ x^{(k)}(a) + x^{(k)}(b) = 0, \quad k = 0, 1, ..., n, \end{cases} \tag{2.21}$$

where $p > 1$, $q_k \in C([a,b], \mathbb{R})$, $k = 0, 1, ..., n-1$, $n \in \mathbb{N}$, and $\lambda \in \mathbb{R}$ is an eigenvalue parameter. As direct consequences of Theorems 2.3 and 2.4, we obtain the following result.

Theorem 2.8 (a) *Assume λ is an eigenvalue of the boundary value problem (2.21). Then*

$$|\lambda| > \frac{2}{\sum_{i=0}^{n-1} \left((b-a) S_{n-i}\right)^{(p-1)/2} \int_a^b |q_i(s)| \, \mathrm{d}s}, \tag{2.22}$$

where S_n is given in (1.12).
(b) *Assume λ is an eigenvalue of the boundary value problem (2.21). Then*

$$|\lambda| > \frac{2}{\sum_{i=0}^{n-1} \left(\frac{b-a}{2}\right)^{(n-i)(p-1)} \int_a^b |q_i(s)| \, \mathrm{d}s}. \tag{2.23}$$

Open Problem. Note that we have only considered the case $p = q$ in the following $(n+1)$st order nonlinear differential equation

$$\left(\varphi_p\left(x^{(n)}(t)\right)\right)' + \sum_{i=0}^{n-1} q_i(t) \varphi_q\left(x^{(i)}(t)\right) = 0, \tag{2.24}$$

where $p, q > 1$, $q_k \in C([0, \infty), \mathbb{R})$, and the operators φ_p and φ_q are given in (1.15) and (1.16), respectively. It is an open problem to obtain similar results for Eq. (2.24) without any restrictions on p and q.

Acknowledgement The author thanks the anonymous referee for his/her valuable suggestions and comments that helped to improve the presentation of this paper.

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