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On the geometry of tangent bundle of a hypersurface in $\mathbb{R}^{n+1}$

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Abstract: In this paper, tangent bundle TM of the hypersurface M in $\mathbb{R}^{n+1}$ has been studied. For hypersurface M given by immersion $f : M \to \mathbb{R}^{n+1}$, considering the fact that $F = df : TM \to \mathbb{R}^{2n+2}$ is also immersion, TM is treated as a submanifold of $\mathbb{R}^{2n+2}$. Firstly, an induced metric which is called rescaled induced metric has been defined on TM, and the Levi-Civita connection has been calculated for this metric. Next, curvature tensors of tangent bundle TM have been obtained. Finally, the orthonormal frame at the point $(p, u) \in TM$ has been defined and some curvature properties of such a tangent bundle by means of orthonormal frame for a given point have been investigated.

Key words: Tangent bundle, hypersurface, rescaled induced metric, curvature tensor, orthonormal frame

1. Introduction

Theory of tangent bundle has attracted the attention of scientists working in math and physics and has received great importance in these areas. In recent years, there have been many publications examining the differential geometric properties of the tangent bundle using different approaches, methods, and notations. In 1958, Sasaki defined a new metric on the tangent bundle [12] and this metric became the focus of interest for many mathematicians working on the theory of tangent bundle. Moreover, after Dombrowski has established the relationship between the geometry of the tangent bundle with Sasaki metric and the base manifold [6], interest in tangent bundle theory has grown even more. Due to the relationship between almost complex structure J and the tangent bundle TM of a Riemannian manifold M, one naturally expects very good features associated with this almost complex structure vis-a-vis the complex geometry. However, the Sasaki metric on TM causes hindrance on the almost complex structure and does not allow it even to be a complex structure unless the base manifold is flat. This problem in the Sasaki metric has led researchers to search for other metrics on the tangent bundle for example Cheeger-Gromoll metric, Oproiu metric (cf. [1, 3, 7–11]). Since the projection $\pi : TM \to M$ is a Riemannian submersion, all metrics defined on TM are appropriate with this smooth projection. This is why these metrics are called natural metrics. However, in [4] Deshmukh et al. demonstrated that induced metric on TM is not generally a natural metric. In addition, they proved that the smooth map is an immersion and therefore considered tangent bundle TM of a hypersurface M as an immersed submanifold. In [5], Deshmukh and Al-Shaikh defined induced metric on TM to search its geometry using the fact that TM is a submanifold of the Euclidean space $\mathbb{R}^{2n+2}$. Moreover, the authors found that there is a reduction in the codimension of TM, so they examined the differential geometric properties of TM in terms of being hypersurface of $\mathbb{R}^{2n+1}$.

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In 2018, Al-Shaikh constituted an orthonormal frame on M^3 which is submanifold of $\mathbb{R}^4$ considering a special case of [5] and examined the properties of tangent bundle through this frame [2]. Because tangent bundle of a manifold is not compact, it cannot admit a Riemannian metric of strictly positive sectional curvature, it is important to find a metric on the tangent bundle TM which has nonnegative sectional curvature.

The aim of this study is to introduce an induced metric on tangent bundle, which is called a rescaled induced metric, and to investigate some curvature properties of such a tangent bundle.

For this purpose, firstly, the covariant derivatives of the horizontal and vertical vector fields in the direction of each other are calculated, then the curvature tensors (0,4) and (1,3) types are obtained. Later on, the orthonormal frame at the point $(p,u) \in TM$ is defined and the mean curvature and sectional curvatures of tangent bundle TM were obtained by means of the orthonormal frame for a given point.

2. Tangent bundle of a hypersurface

Consider the orientable hypersurface M given by immersion $f : M \to \mathbb{R}^{n+1}$ and tangent bundle TM of the hypersurface M with immersion $F : TM \to \mathbb{R}^{2n+2}$ (See. Theorem 3.2 in [4]). We denote the induced metrics on M, TM by $g, \widetilde{g}$ and the Euclidean metric on $\mathbb{R}^{n+1}$ as well as that on $\mathbb{R}^{2n+2}$ by $\langle\ ,\ \rangle$ and the Riemannian connections on M, TM, $\mathbb{R}^{n+1}$, $\mathbb{R}^{2n+2}$ by $\nabla, \widetilde{\nabla}, D, \widetilde{D}$, respectively. Let N and S be the unit normal vector field and the shape operator of the hypersurface M. Then we have the following:

Lemma 2.1 *If an oriantable hypersurface of $\mathbb{R}^{n+1}$ is M and its tangent bundle as submanifold of $\mathbb{R}^{2n+2}$ is TM, then the metric $\widetilde{g}$ on TM, which we call rescaled induced metric, for $P = (p,u) \in TM$ satisfies:*

(i) $\widetilde{g}_P\left(X_P^h, Y_P^h\right) = a\left\{g_p\left(X_p, Y_p\right) + g_p\left(S_p\left(X_p\right), u\right) g_p\left(S_p\left(Y_p\right), u\right)\right\}$

(ii) $\widetilde{g}_P\left(X_P^h, Y_P^v\right) = 0$

(iii) $\widetilde{g}_P\left(X_P^v, Y_P^v\right) = g_p\left(X_p, Y_p\right)$

where $a : M \to \mathbb{R}$ is a positive function.

Theorem 2.2 *Let M be an oriantable hypersurface of the Euclidean space $\mathbb{R}^{n+1}$ and TM be its tangent bundle as submanifold of $\mathbb{R}^{2n+2}$. Then*

(i) $\widetilde{\nabla}_{X^h} Y^h = \left(\nabla_X Y\right)^h + \left(A_f\left(X,Y\right)\right)^h - \frac{1}{2}\left(R\left(X,Y\right)u\right)^v$

(ii) $\widetilde{\nabla}_{X^h} Y^v = \left(\nabla_X Y\right)^v + g\left(S\left(X\right), Y\right) N^v$

(iii) $\widetilde{\nabla}_{X^v} Y^h = g\left(S\left(Y\right), X\right) N^v$

(iv) $\widetilde{\nabla}_{X^v} Y^v = 0$

Proof If Kozul's formula is used

$$\begin{aligned}2\widetilde{g}\left(\widetilde{\nabla}_{X^h}Y^h,Z^h\right) &= X^h\widetilde{g}\left(Y^h,Z^h\right)+Y^h\widetilde{g}\left(Z^h,X^h\right)\\ &\quad -Z^h\widetilde{g}\left(X^h,Y^h\right)-\widetilde{g}\left(X^h,\left[Y^h,Z^h\right]\right)\\ &\quad +\widetilde{g}\left(Y^h,\left[Z^h,X^h\right]\right)+\widetilde{g}\left(Z^h,\left[X^h,Y^h\right]\right)\end{aligned}$$

from Lemma 2.1 and the fact $\left[X^h,Y^h\right]=[X,Y]^h-(R(X,Y)u)^v$, we get

$$\begin{aligned}2\widetilde{g}\left(\widetilde{\nabla}_{X^h}Y^h,Z^h\right) &= \left(X^h a\right)\left[g(Y,Z)+g(S(Y),u)\,g(S(Z),u)\right]\\ &\quad +a\left[\begin{array}{c}X^h g(Y,Z)+g(S(Z),u)\,X^h g(S(Y),u)\\ +g(S(Y),u)\,X^h g(S(Z),u)\end{array}\right]\\ &\quad +\left(Y^h a\right)\left[g(Z,X)+g(S(Z),u)\,g(S(X),u)\right]\\ &\quad +a\left[\begin{array}{c}Y^h g(Z,X)+g(S(X),u)\,Y^h g(S(Z),u)\\ +g(S(Z),u)\,Y^h g(S(X),u)\end{array}\right]\\ &\quad -\left(Z^h a\right)\left[g(X,Y)+g(S(X),u)\,g(S(Y),u)\right]\\ &\quad -a\left[\begin{array}{c}Z^h g(X,Y)+g(S(Y),u)\,Z^h g(S(X),u)\\ +g(S(X),u)\,Z^h g(S(Y),u)\end{array}\right]\\ &\quad -\widetilde{g}\left(X^h,[Y,Z]^h\right)+\widetilde{g}\left(Y^h,[Z,X]^h\right)+\widetilde{g}\left(Z^h,[X,Y]^h\right).\end{aligned}$$

From the fact that $X^h g(Y,u)=g(\nabla_X Y,u)$, $X^h g(Y,e_i)=g(\nabla_X Y,e_i)$ and the shape operator gives $S_p : T_pM\to T_pM$ a linear map for each $p\in M$, choosing a basis $\{e_1,e_2,...,e_n\}$ of T_pM that diagonalizes S_p with $S_p(e_i)=\lambda_i e_i$, we get

$$\begin{aligned}X^h g(Y,S(u)) &= X^h g\left(Y,S\left(\sum g_p(u,e_i)\,e_i\right)\right)\\ &= \sum\lambda_i g_p(u,e_i)\,X^h g(Y,e_i)\\ &= \sum\lambda_i g_p(u,e_i)\,g(\nabla_X Y,e_i)\\ &= g(\nabla_X Y,S(u)).\end{aligned}$$

Considering these results and

$$X^h\widetilde{g}(Y^v,Z^v)=\widetilde{g}\left((\nabla_X Y)^v,Z^v\right)+\widetilde{g}\left(Y^v,(\nabla_X Z)^v\right)$$

we get

$$
\begin{aligned}
2\widetilde{g}\left(\widetilde{\nabla}_{X^h}Y^h, Z^h\right) &= (Xa)\left[g(Y,Z)+g(S(Y),u)\,g(S(Z),u)\right] \\
&+(Ya)\left[g(Z,X)+g(S(Z),u)\,g(S(X),u)\right] \\
&-(Za)\left[g(X,Y)+g(S(X),u)\,g(S(Y),u)\right] \\
&+a\left\{\begin{array}{c}
\widetilde{g}\left((\nabla_X Y)^v+(\nabla_Y X)^v, Z^v\right) \\
+\widetilde{g}\left([X,Z]^v, Y^v\right)+\widetilde{g}\left([Y,Z]^v, X^v\right) \\
+g(S(Z),u)\left[\begin{array}{c} g(S(\nabla_X Y),u) \\ +g(S(\nabla_Y X),u)\end{array}\right] \\
+g(S(Y),u)\left[\begin{array}{c} g(S(\nabla_X Z),u) \\ -g(S(\nabla_Z X),u)\end{array}\right] \\
+g(S(X),u)\left[\begin{array}{c} g(S(\nabla_Y Z),u) \\ -g(S(\nabla_Z Y),u)\end{array}\right] \\
-g(X,[Y,Z])-g(S(X),u)\,g(S([Y,Z]),u) \\
+g(Y,[Z,X])+g(S(Y),u)\,g(S([Z,X]),u) \\
+g(Z,[X,Y])+g(S(Z),u)\,g(S([X,Y]),u)
\end{array}\right\}
\end{aligned}
$$

and thus we have

$$
\begin{aligned}
2\widetilde{g}\left(\widetilde{\nabla}_{X^h}Y^h, Z^h\right) &= \frac{1}{a}\left\{\begin{array}{c}
\widetilde{g}\left((X(a)Y)^h, Z^h\right) \\
+\widetilde{g}\left((Y(a)X)^h, Z^h\right) \\
-\widetilde{g}\left((g(X,Y)\circ df^*)^h, Z^h\right)
\end{array}\right\} \\
&\quad +2\widetilde{g}\left((\nabla_X Y)^h, Z^h\right)
\end{aligned}
$$

where $(g(X,Y)\circ df^*)$ is locally expressed as

$$g(X,Y)\circ df^* = g_{ij}f^k = g_{ij}\partial^k f.$$

In this way, we obtain

$$
\begin{aligned}
&2\widetilde{g}\left(\widetilde{\nabla}_{X^h}Y^h, Z^h\right) \\
=\ &2\widetilde{g}\left((\nabla_X Y)^h+\left(\frac{1}{2a}\left(\begin{array}{c} X(a)Y+Y(a)X \\ -g(X,Y)\circ df^*\end{array}\right)\right)^h, Z^h\right) \\
=\ &2\widetilde{g}\left((\nabla_X Y+A_f(X,Y))^h, Z^h\right)
\end{aligned}
$$

and herefrom we get the horizontal part of vector field $\widetilde{\nabla}_{X^h}Y^h$ as $(\nabla_X Y)^h+(A_f(X,Y))^h$. Now we will search the vertical part of vector field $\widetilde{\nabla}_{X^h}Y^h$. From the equation

$$X^h\left(\widetilde{g}(Y^v,Z^v)\right)=\widetilde{g}\left((\nabla_X Y)^v, Z^v\right)+\widetilde{g}\left(Y^v,(\nabla_X Z)^v\right)$$

and Kozul's formula, we get

$$\begin{aligned} 2\widetilde{g}\left(\widetilde{\nabla}_{X^h}Y^h, Z^v\right) &= \widetilde{g}\left(Z^v, [X,Y]^h - (R(X,Y)u)^v\right) \\ &\quad -Z^v\{a\{g(X,Y)+g(S(X),u)\,g(S(Y),u)\}\} \\ &\quad -\widetilde{g}\left(X^h, (\nabla_Y Z)^v\right) - \widetilde{g}\left(Y^h, (\nabla_X Z)^v\right) \\ &= -\widetilde{g}\left((R(X,Y)u)^v, Z^v\right). \end{aligned}$$

Hence, vertical part of vector field $\widetilde{\nabla}_{X^h}Y^h$ is found as $-\frac{1}{2}(R(X,Y)u)^v$. Finally, combining expressions $\widetilde{g}\left(\widetilde{\nabla}_{X^h}Y^h, Z^h\right)$ and $\widetilde{g}\left(\widetilde{\nabla}_{X^h}Y^h, Z^v\right)$, we obtain (i) as follows

$$\widetilde{\nabla}_{X^h}Y^h = (\nabla_X Y)^h + (A_f(X,Y))^h - \frac{1}{2}(R(X,Y)u)^v.$$

To prove (ii), we use the immersion $F: TM \to \mathbb{R}^{2n+2}$ to write the Gauss equation $\widetilde{D}_X Y = \widetilde{\nabla}_X Y + \widetilde{h}(X,Y)$ for TM (eq. 3 in [4]), in the form

$$\widetilde{D}_{dF(X^h)}dF(Y^v) = dF\left(\widetilde{\nabla}_{X^h}Y^v\right) + \widetilde{h}\left(X^h, Y^v\right).$$

We get from Lemma 3.2 in [4]

$$\widetilde{D}_{[df(X)]^h+V}[df(Y)]^v = dF\left(\widetilde{\nabla}_{X^h}Y^v\right) + \widetilde{h}\left(X^h, Y^v\right).$$

Note that the metric on $\mathbb{R}^{2n+2}$ being Sasaki metric, using the corresponding equation in [8], we obtain

$$[df(\nabla_X Y) + g(S(X),Y)N]^v = dF\left(\widetilde{\nabla}_{X^h}Y^v\right) + \widetilde{h}\left(X^h, Y^v\right)$$

Since N^v is tangent to TM, equating the tangential and normal components, we get

$$\begin{aligned} dF\left(\widetilde{\nabla}_{X^h}Y^v\right) &= dF\left((\nabla_X Y)^v\right) + g(S(X),Y)N^v, \\ \widetilde{h}\left(X^h, Y^v\right) &= 0 \end{aligned} \tag{2.1}$$

that gives

$$\widetilde{\nabla}_{X^h}Y^v = (\nabla_X Y)^v + g(S(X),Y)N^v$$

which proves (ii).

For (iii), we use the fact that $\left[Y^h, X^v\right] = (\nabla_Y X)^v$ (cf. [8], proposition 5.1) and (ii) to get

$$(\nabla_Y X)^v + g(S(Y),X)N^v - \widetilde{\nabla}_{X^v}Y^h = (\nabla_Y X)^v,$$

so we obtain

$$\widetilde{\nabla}_{X^v} Y^h = g\left(S\left(Y\right), X\right) N^v$$

which proves (iii).

To proof (iv), we use the Kozul's formula and Proposition 5.1 in [8] together with Lemma 2.1, so we obtain

$$\begin{aligned} 2\widetilde{g}\left(\widetilde{\nabla}_{X^v} Y^v, Z^h\right) &= -Z^h \widetilde{g}\left(X^v, Y^v\right) \\ &\quad -\widetilde{g}\left(X^v, \left[Y^v, Z^h\right]\right) + \widetilde{g}\left(Y^v, \left[Z^h, X^v\right]\right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} 2\widetilde{g}\left(\widetilde{\nabla}_{X^v} Y^v, Z^v\right) &= X^v g(Y,Z) + Y^v g(Z,X) - Z^v g(X,Y) \\ &= 0 \end{aligned}$$

which proves (iv). □

Lemma 2.3 *Let $X, Y \in \chi(M)$ be two vector fields on the hypersurface M in $\mathbb{R}^{n+1}$. Then the second fundamental form $\widetilde{h}$ of the submanifold TM satisfies*

(i) $\widetilde{h}\left(X^v, Y^v\right) = 0,$

(ii) $\widetilde{h}\left(X^h, Y^v\right) = 0,$

(iii) $\widetilde{h}\left(X^h, Y^h\right) = g\left(S\left(X\right), Y\right) N^h.$

Proof The truth of equation (ii) is clear from Eq. 2.1. For (i), we have from equations $\widetilde{D}_X Y = \widetilde{\nabla}_X Y + \widetilde{h}\left(X, Y\right)$, $\widetilde{D}_X \widetilde{N} = -\widetilde{S}_{\widetilde{N}}\left(X\right) + \nabla^{\perp}_X \widetilde{N}$ for the submanifold TM that $\widetilde{D}_{X^v} Y^v = \widetilde{\nabla}_{X^v} Y^v + \widetilde{h}\left(X^v, Y^v\right)$ which together with the fact that metric on the tangent bundle $T\mathbb{R}^{n+1}$ is a Sasaki metric, that is, $\widetilde{D}_{X^v} Y^v = 0$, we get $\widetilde{h}\left(X^v, Y^v\right) = 0$. For (iii), we have local orthonormal unit normal vector fields $\widetilde{N} = \frac{1}{\sqrt{a}} N^h$, N^* for the submanifold TM as described in Lemmas 3.3 and 3.4 of [4], where N^* is vertical on the tangent bundle $T\mathbb{R}^{n+1}$. Therefore, we get

$$\begin{aligned} \widetilde{h}\left(X^h, Y^h\right) &= \widetilde{g}\left(\widetilde{h}\left(X^h, Y^h\right), \widetilde{N}\right) + \widetilde{g}\left(\widetilde{h}\left(X^h, Y^h\right), N^*\right) \\ &= \widetilde{g}\left(\widetilde{S}_{\widetilde{N}}\left(X^h\right), Y^h\right) \widetilde{N} + \widetilde{g}\left(\widetilde{S}_{N^*}\left(X^h\right), Y^h\right) N^* \end{aligned}$$

Considering equations $\widetilde{D}_X Y = \widetilde{\nabla}_X Y + \widetilde{h}\left(X, Y\right)$, $\widetilde{D}_X \widetilde{N} = -\widetilde{S}_{\widetilde{N}}\left(X\right) + \nabla^{\perp}_X \widetilde{N}$ together with Lemma 3.2 of [4] and Proposition 7.2 in [8] for Sasaki metric on $T\mathbb{R}^{n+1}$ we find

$$\begin{aligned}
\widetilde{g}\left(\widetilde{S}_{\widetilde{N}}\left(X^h\right), Y^h\right)\widetilde{N} &= \frac{1}{\sqrt{a}}\widetilde{g}\left(\widetilde{D}_{X^h}\frac{1}{\sqrt{a}}N^h, Y^h\right)N^h \\
&= \frac{1}{\sqrt{a}}\widetilde{g}\left(\left[D_X\left(\frac{1}{\sqrt{a}}\right)\right]N^h + \frac{1}{\sqrt{a}}\left(D_X N\right)^h, Y^h\right)N^h \\
&= \frac{1}{\sqrt{a}}D_X\left(\frac{1}{\sqrt{a}}\right)\widetilde{g}\left(N^h, Y^h\right)N^h \\
&\quad +\frac{1}{\sqrt{a}}\frac{1}{\sqrt{a}}\widetilde{g}\left(\left(D_X N\right)^h, Y^h\right)N^h \\
&= \frac{1}{\sqrt{a}}D_X\left(\frac{1}{\sqrt{a}}\right)a.g\left(N, Y\right) \\
&\quad +\frac{1}{a}.a.g\left(D_X N, Y\right)N^h \\
&= g\left(S\left(X\right), Y\right)N^h. \qquad (2.2)
\end{aligned}$$

Since N^v is tangent to TM and the unit normal vector field N^* is vertical on the tangent bundle $T\mathbb{R}^{n+1}$ and from Lemma 3.2 of [4] and Proposition 7.2 in [8], it is taken

$$\begin{aligned}
\widetilde{g}\left(\widetilde{h}\left(X^h, Y^h\right), N^*\right) &= \widetilde{g}\left(\widetilde{D}_{dF(X^h)}dF\left(Y^h\right), N^*\right) \\
&= \widetilde{g}\left(\left(D_{dF(X)}dF\left(Y\right)\right)^h + \left(Xg\left(S\left(Y\right), u\right)\right)N^v, N^*\right) \\
&\quad +\widetilde{g}\left(g\left(S\left(Y\right), u\right)\left(D_{dF(X)}N\right)^v, N^*\right) \\
&= -g\left(S\left(Y\right), u\right)\widetilde{g}\left(\left(S\left(X\right)\right)^v, N^*\right) = 0
\end{aligned}$$

where we used $\left(S\left(X\right)\right)^v \in \chi\left(TM\right)$ and that N^* is normal vector field to the submanifold TM. Combining the above equation with equation 2.2, we get

$$\widetilde{h}\left(X^h, Y^h\right) = g\left(S\left(X\right), Y\right)N^h.$$

□

Lemma 2.4 *[5] The covariant derivatives in the direction of X^h and X^v for $X \in \chi\left(M\right)$ of the vertical and horizontal lifts of the unit normal vector field N for an orientable hypersurface M in $\mathbb{R}^{n+1}$ are given by*

(i) $\widetilde{D}_{X^v}N^v = 0,$

(ii) $\widetilde{D}_{X^v}N^h = 0,$

(iii) $\widetilde{D}_{X^h}N^v = -\left(S\left(X\right)\right)^v,$

(iv) $\widetilde{D}_{X^h}N^h = -\left(S\left(X\right)\right)^h.$

Lemma 2.5 *[5] For the tangent bundle TM of an orientable hypersurface M of the Euclidean space $\mathbb{R}^{n+1}$, we have*

(i) $\widetilde{h}(X^v, N^v) = 0,$

(ii) $\widetilde{\nabla}_{X^v} N^v = 0,$

(iii) $\widetilde{h}\left(X^h, N^v\right) = 0,$

(iv) $\widetilde{\nabla}_{X^h} N^v = -(S(X))^v, \quad X \in \chi(M).$

3. Curvature tensors of tangent bundle of hypersurfaces in a euclidean Space

In this section, we obtain expressions for the curvature tensors with type (1,3) and (0,4) of the tangent bundle TM of the hypersurface M in a Euclidean space $\mathbb{R}^{n+1}$ as a submanifold with unit normal N, as well as study the properties of the vector field N^v. The tangent bundle TM is now a submanifold of the Euclidean space $\mathbb{R}^{2n+2}$. Then using the Gauss equation expressing curvature tensor field for submanifold TM in the Euclidean space $\mathbb{R}^{2n+2}$ together with Lemma 2.1, we have the following theorems:

Theorem 3.1 *Let (M, g) be a hypersurface in $\mathbb{R}^{n+1}$ and $\left(TM, \widetilde{g}\right)$ be its tangent bundle. Then $\widetilde{R}$ the Riemannian curvature tensor field with type (1,3) of the tangent bundle equipped with the rescaled induced metric $\widetilde{g}$ is as follows:*

(i) $$\widetilde{R}(X^h, Y^h)Z^h = \left\{ \begin{array}{c} R(X,Y)Z + (\nabla_X A_f)(Y,Z) - (\nabla_Y A_f)(X,Z) \\ +A_f(X, A_f(Y,Z)) - A_f(Y, A_f(X,Z)) \end{array} \right\}^h$$

$$+ \left\{ \begin{array}{c} \frac{1}{2}(\nabla_Y R)(X,Z)u - \frac{1}{2}(\nabla_X R)(Y,Z)u \\ -\frac{1}{2}R(X, A_f(Y,Z)) + \frac{1}{2}R(Y, A_f(X,Z)) \\ -\frac{1}{2}g(S(X), R(Y,Z)u)N \\ +\frac{1}{2}g(S(Y), R(X,Z)u)N \\ +g(S(Z), R(X,Y)u)N \end{array} \right\}^v,$$

(ii) $$\widetilde{R}(X^h, Y^h)Z^v = \left\{ \begin{array}{c} R(X,Y)Z - g((\nabla_Y S)X, Z)N \\ +g((\nabla_X S)Y, Z)N \\ -g(S(Y), Z)S(X) \\ +g(S(X), Z)S(Y) \end{array} \right\}^v,$$

(iii) $$\widetilde{R}(X^h, Y^v)Z^h = \left\{ \begin{array}{c} g((\nabla_X S)Y, Z)N \\ -g(S(Y), Z)S(X) \\ -g(S(Y), A_f(X,Z))N \end{array} \right\}^v,$$

(iv) $\widetilde{R}(X^h, Y^v)Z^v = 0,$

(v) $\widetilde{R}(X^v, Y^v)Z^h = 0,$

(vi) $\widetilde{R}(X^v, Y^v)Z^v = 0, \qquad X, Y, Z \in \chi(M)$.

Proof If we use formula of the curvature tensor together with Theorem 2.2, we get for (i)

$$\begin{aligned}
\widetilde{R}(X^h, Y^h)Z^h &= \widetilde{\nabla}_{X^h}\left\{(\nabla_Y Z)^h + (A_f(Y,Z))^h - \frac{1}{2}(R(Y,Z)u)^v\right\} \\
&\quad -\widetilde{\nabla}_{Y^h}\left\{(\nabla_X Z)^h + (A_f(X,Z))^h - \frac{1}{2}(R(X,Z)u)^v\right\} \\
&\quad -\widetilde{\nabla}_{[X,Y]^h-(R(X,Y)u)^v} Z^h \\
&= \left\{\begin{array}{c} R(X,Y)Z + (\nabla_X A_f)(Y,Z) - (\nabla_Y A_f)(X,Z) \\ +A_f(X, A_f(Y,Z)) - A_f(Y, A_f(X,Z)) \end{array}\right\}^h \\
&\quad + \left\{\begin{array}{c} \frac{1}{2}(\nabla_Y R)(X,Z)u - \frac{1}{2}(\nabla_X R)(Y,Z)u \\ -\frac{1}{2}R(X, A_f(Y,Z)) + \frac{1}{2}R(Y, A_f(X,Z)) \\ -\frac{1}{2}g(S(X), R(Y,Z)u)N \\ +\frac{1}{2}g(S(Y), R(X,Z)u)N \\ +g(S(Z), R(X,Y)u)N \end{array}\right\}^v.
\end{aligned}$$

For (ii), we get

$$\begin{aligned}
\widetilde{R}(X^h, Y^h)Z^v &= \widetilde{\nabla}_{X^h}\{(\nabla_Y Z)^v + g(S(Y), Z)N^v\} \\
&\quad -\widetilde{\nabla}_{Y^h}\{(\nabla_X Z)^v + g(S(X), Z)N^v\} \\
&\quad -\widetilde{\nabla}_{[X,Y]^h-(R(X,Y)u)^v} Z^v \\
&= (\nabla_X \nabla_Y Z)^v + g(S(X), \nabla_Y Z)N^v \\
&\quad + (\nabla_X g(S(Y), Z)N^v)^v \\
&\quad + g(S(X), g(S(Y), Z)N^v)N^v \\
&\quad - (\nabla_Y \nabla_X Z)^v - g(S(Y), \nabla_X Z)N^v \\
&\quad - (\nabla_Y g(S(X), Z)N^v)^v \\
&\quad - g(S(Y), g(S(X), Z)N^v)N^v \\
&\quad - \left(\nabla_{[X,Y]} Z\right)^v - g(S([X,Y]), Z)N^v
\end{aligned}$$

where if we consider that

$$\begin{aligned}
(\nabla_Y g)(S(X), Z) &= \nabla_Y g(S(X), Z) - g(\nabla_Y S(X), Z) - g(S(X), \nabla_Y Z) \\
&= \nabla_Y g(S(X), Z) - g((\nabla_Y S)X, Z) \\
&\quad - g(S(\nabla_Y X), Z) - g(S(X), \nabla_Y Z)
\end{aligned}$$

and from lemma 2.5, we obtain

$$\widetilde{R}(X^h, Y^h)Z^v = \left\{\begin{array}{c} R(X,Y)Z - g((\nabla_Y S)X, Z)N + g((\nabla_X S)Y, Z)N \\ -g(S(Y), Z)S(X) + g(S(X), Z)S(Y) \end{array}\right\}^v.$$

For (iii), we can write

$$\begin{aligned}\widetilde{R}(X^h,Y^v)Z^h &= \widetilde{\nabla}_{X^h}\left\{g\left(S\left(Y\right),Z\right)N^v\right\}-\widetilde{\nabla}_{(\nabla_X Y)^v}Z^h \\ &\quad -\widetilde{\nabla}_{Y^v}\left\{\left(\nabla_X Z\right)^h+\left(A_f\left(X,Z\right)\right)^h-\frac{1}{2}\left(R\left(X,Z\right)u\right)^v\right\} \\ &= \left(\nabla_X g\left(S\left(Y\right),Z\right)\right)N^v+g\left(S\left(Y\right),Z\right)\widetilde{\nabla}_{X^h}N^v \\ &\quad -g\left(S\left(Y\right),\nabla_X Z\right)N^v-g\left(S\left(Y\right),A_f\left(X,Z\right)\right)N^v \\ &\quad -g\left(S\left(\nabla_X Y\right),Z\right)N^v\end{aligned}$$

where since

$$\nabla_X S\left(Y\right)=\left(\nabla_X S\right)Y+S\left(\nabla_X Y\right)$$

and from Lemma 2.5 we get

$$\begin{aligned}\widetilde{R}(X^h,Y^v)Z^h &= \left(\nabla_X g\left(S\left(Y\right),Z\right)\right)N^v-g\left(S\left(Y\right),\nabla_X Z\right)N^v \\ &\quad -g\left(\nabla_X S\left(Y\right),Z\right)N^v+g\left(\left(\nabla_X S\right)Y,Z\right)N^v \\ &\quad -g\left(S\left(Y\right),Z\right)\left(S\left(X\right)\right)^v-g\left(S\left(Y\right),A_f\left(X,Z\right)\right)N^v\end{aligned}$$

also because

$$\nabla_X g\left(S\left(Y\right),Z\right)=\left(\nabla_X g\right)\left(S\left(Y\right),Z\right)+g\left(\nabla_X S\left(Y\right),Z\right)+g\left(S\left(Y\right),\nabla_X Z\right)$$

and $\nabla_X g=0$, it is found that

$$\nabla_X g\left(S\left(Y\right),Z\right)-g\left(\nabla_X S\left(Y\right),Z\right)-g\left(S\left(Y\right),\nabla_X Z\right)=0.$$

Thus, we find (iii) as

$$\widetilde{R}(X^h,Y^v)Z^h=\left\{\begin{array}{c} g\left(\left(\nabla_X S\right)Y,Z\right)N-g\left(S\left(Y\right),Z\right)S\left(X\right) \\ -g\left(S\left(Y\right),A_f\left(X,Z\right)\right)N\end{array}\right\}^v.$$

For (iv), we can obtain easily that

$$\begin{aligned}\widetilde{R}(X^h,Y^v)Z^v &= -\widetilde{\nabla}_{Y^v}\left\{\left(\nabla_X Z\right)^v+g\left(S\left(X\right),Z\right)N^v\right\} \\ &= 0.\end{aligned}$$

For (v) and (vi), we can similarly see that

$$\begin{aligned}\widetilde{R}(X^v,Y^v)Z^h &= \widetilde{\nabla}_{X^v}\left\{g\left(S\left(Y\right),Z\right)N^v\right\}-\widetilde{\nabla}_{Y^v}\left\{g\left(S\left(X\right),Z\right)N^v\right\} \\ &= 0\end{aligned}$$

and

$$\widetilde{R}(X^v,Y^v)Z^v=0.$$

□

Theorem 3.2 *Let* (M, g) *be a hypersurface in* $\mathbb{R}^{n+1}$ *and* $\left(TM, \widetilde{g}\right)$ *be its tangent bundle. Then* $\widetilde{R}$ *the Riemannian curvature tensor field with type (0,4) of the tangent bundle equipped with the rescaled induced metric* $\widetilde{g}$ *is as follows:*

$$\text{(i)}\quad \widetilde{R}\left(X^h, Y^h; Z^h, W^h\right) = a.g\left(\left\{\begin{array}{c} R(X,Y)Z + (\nabla_X A_f)(Y,Z) \\ -(\nabla_Y A_f)(X,Z) \\ +A_f(X, A_f(Y,Z)) \\ -A_f(Y, A_f(X,Z)) \end{array}\right\}, W\right)$$

$$+a.g(S(W), u).g\left(\left\{\begin{array}{c} R(X,Y)Z \\ +(\nabla_X A_f)(Y,Z) \\ -(\nabla_Y A_f)(X,Z) \\ +A_f(X, A_f(Y,Z)) \\ -A_f(Y, A_f(X,Z)) \end{array}\right\}, S(u)\right),$$

$$\text{(ii)}\quad \widetilde{R}\left(X^h, Y^h; Z^h, W^v\right) = \tfrac{1}{2}g\left(\left\{\begin{array}{c} (\nabla_Y R)(X,Z)u \\ -(\nabla_X R)(Y,Z)u \\ -R(X, A_f(Y,Z)) \\ +R(Y, A_f(X,Z)) \end{array}\right\}, W\right),$$

$$\text{(iii)}\quad \widetilde{R}\left(X^h, Y^h; Z^v, W^h\right) = 0,$$

$$\text{(iv)}\quad \widetilde{R}\left(X^h, Y^h; Z^v, W^v\right) = g\left(\left\{\begin{array}{c} R(X,Y)Z \\ -g(S(Y), Z)S(X) \\ +g(S(X), Z)S(Y) \end{array}\right\}, W\right),$$

$$\text{(v)}\quad \widetilde{R}\left(X^v, Y^v; Z^v, W^h\right) = 0,$$

$$\text{(vi)}\quad \widetilde{R}(X^v, Y^v; Z^v, W^v) = 0,$$

$$\text{(vii)}\quad \widetilde{R}\left(X^h, Y^h; Z^h, N^v\right) = 0,$$

$$\text{(viii)}\quad \widetilde{R}\left(X^h, Y^h; Z^v, N^v\right) = g\left((\nabla_X S)Y - (\nabla_Y S)X, Z\right), \qquad X, Y, Z, W \in \chi(M).$$

Proof Now, to prove the above equations, we will use the fact $\widetilde{R}\left(X^h, Y^h; Z^h, W^h\right) = \widetilde{g}\left(\widetilde{R}(X^h, Y^h)Z^h, W^h\right)$

and together with Theorem 3.1. Thus, for (i), we get

$$\begin{aligned}
\widetilde{R}(X^h, Y^h, Z^h, W^h) &= \widetilde{g}\left(\left\{\begin{array}{c} R(X,Y)Z + (\nabla_X A_f)(Y,Z) \\ -(\nabla_Y A_f)(X,Z) \\ +A_f(X, A_f(Y,Z)) \\ -A_f(Y, A_f(X,Z)) \end{array}\right\}^h, W^h\right) \\
&= a.g\left(\left\{\begin{array}{c} R(X,Y)Z + (\nabla_X A_f)(Y,Z) \\ -(\nabla_Y A_f)(X,Z) \\ +A_f(X, A_f(Y,Z)) \\ -A_f(Y, A_f(X,Z)) \end{array}\right\}, W\right) \\
&\quad + a.g(S(W), u).g\left(\left\{\begin{array}{c} R(X,Y)Z \\ +(\nabla_X A_f)(Y,Z) \\ -(\nabla_Y A_f)(X,Z) \\ +A_f(X, A_f(Y,Z)) \\ -A_f(Y, A_f(X,Z)) \end{array}\right\}, S(u)\right).
\end{aligned}$$

Similarly, for (ii), it is obtained that

$$\begin{aligned}
\widetilde{R}(X^h, Y^h, Z^h, W^v) &= \widetilde{g}\left(\left\{\begin{array}{c} \frac{1}{2}(\nabla_Y R)(X,Z)u \\ -\frac{1}{2}(\nabla_X R)(Y,Z)u \\ -\frac{1}{2}R(X, A_f(Y,Z)) \\ +\frac{1}{2}R(Y, A_f(X,Z)) \\ -\frac{1}{2}g(S(X), R(Y,Z)u)N \\ +\frac{1}{2}g(S(Y), R(X,Z)u)N \\ +g(S(Z), R(X,Y)u)N \end{array}\right\}^v, W^v\right) \\
&= \frac{1}{2}g\left(\left\{\begin{array}{c} (\nabla_Y R)(X,Z)u - (\nabla_X R)(Y,Z)u \\ -R(X, A_f(Y,Z)) + R(Y, A_f(X,Z)) \end{array}\right\}, W\right).
\end{aligned}$$

For (iii),(iv),(v), and (vi), it is easy to see respectively the following equations

$$\begin{aligned}
\widetilde{R}\left(X^h, Y^h; Z^v, W^h\right) &= \widetilde{g}\left(\left\{\begin{array}{c} R(X,Y)Z - g((\nabla_Y S)X, Z)N \\ +g((\nabla_X S)Y, Z)N \\ -g(S(Y), Z)S(X) \\ +g(S(X), Z)S(Y) \end{array}\right\}^v, W^h\right) \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\widetilde{R}\left(X^h, Y^h; Z^v, W^v\right) &= \widetilde{g}\left(\left\{\begin{array}{c} R(X,Y)Z \\ -g((\nabla_Y S)X, Z)N \\ +g((\nabla_X S)Y, Z)N \\ -g(S(Y), Z)S(X) \\ +g(S(X), Z)S(Y) \end{array}\right\}^v, W^v\right) \\
&= g\left(\left\{\begin{array}{c} R(X,Y)Z - g(S(Y), Z)S(X) \\ +g(S(X), Z)S(Y) \end{array}\right\}, W\right),
\end{aligned}$$

$$\widetilde{R}\left(X^v, Y^v; Z^v, W^h\right) = 0,$$

$$\widetilde{R}\left(X^v, Y^v; Z^v, W^v\right) = 0.$$

Finally, for equations (vii) and (viii), we can obtain that

$$\begin{aligned}\widetilde{R}\left(X^h,Y^h;Z^h,N^v\right) &= g\left(\left\{\begin{array}{c}\frac{1}{2}\left(\nabla_Y R\right)(X,Z)\,u\\ -\frac{1}{2}\left(\nabla_X R\right)(Y,Z)\,u\\ -\frac{1}{2}R\left(X,A_f\left(Y,Z\right)\right)\\ +\frac{1}{2}R\left(Y,A_f\left(X,Z\right)\right)\\ -\frac{1}{2}g\left(S\left(X\right),R\left(Y,Z\right)u\right)N\\ +\frac{1}{2}g\left(S\left(Y\right),R\left(X,Z\right)u\right)N\\ +g\left(S\left(Z\right),R\left(X,Y\right)u\right)N\end{array}\right\},N\right)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\widetilde{R}\left(X^h,Y^h;Z^v,N^v\right) &= \widetilde{g}\left(\left\{\begin{array}{c}R\left(X,Y\right)Z\\ -g\left(\left(\nabla_Y S\right)X,Z\right)N\\ +g\left(\left(\nabla_X S\right)Y,Z\right)N\\ -g\left(S\left(Y\right),Z\right)S\left(X\right)\\ +g\left(S\left(X\right),Z\right)S\left(Y\right)\end{array}\right\}^v,N^v\right)\\ &= g\left(\left(\nabla_X S\right)Y-\left(\nabla_Y S\right)X,Z\right).\end{aligned}$$

□

4. Orthonormal frame on the tangent bundle

In this section, considering the orthonormal basis $\{e_1,e_2,...,e_n\}$ on the hypersurface M at p, orthonormal frame on the tangent bundle TM at (p,u) will be introduced. Here, it is assumed that parameter curves on M are principal curvature lines.

Lemma 4.1 *Let (M,g) be a hypersurface in $\mathbb{R}^{n+1}$ and $\left(TM,\widetilde{g}\right)$ be its tangent bundle as submanifold of $\mathbb{R}^{2n+2}$ and $\{e_1,e_2,...,e_n\}$ be an orthonormal basis for M at p, such that $e_1=\frac{u}{|u|}$. Thus, for a given point $(p,u)\in TM$, with $u\neq 0$, the set $\{f_1,f_2,...,f_{2n}\}$ formed from horizontal and vertical lifts of $\{e_1,e_2,...,e_n\}$ is an orthonormal basis for TM. Here, $f_1=\frac{1}{\sqrt{\lambda}}e_1^h$ with $\lambda=a\left(1+\kappa_1^2.\left|u\right|^2\right)$, $f_i=\frac{1}{\sqrt{a}}e_i^h$ for $i=2,...,n$ and $f_{j+n}=e_j^v$ for $j=1,...,n$.*

Proof We have

$$\begin{aligned}\widetilde{g}\left(f_1,f_1\right) &= \widetilde{g}\left(\frac{1}{\sqrt{\lambda}}e_1^h,\frac{1}{\sqrt{\lambda}}e_1^h\right)=\frac{1}{\lambda}\widetilde{g}\left(e_1^h,e_1^h\right)\\ &= \frac{1}{\lambda}a\left\{g\left(e_1,e_1\right)+g\left(S\left(e_1\right),u\right)g\left(S\left(e_1\right),u\right)\right\}\\ &= \frac{a}{\lambda}\left(1+\kappa_1^2.\left|u\right|^2\right)\\ &= 1\end{aligned}$$

$$\begin{aligned}\widetilde{g}\left(f_2,f_2\right) &= \widetilde{g}\left(\frac{1}{\sqrt{a}}e_2^h,\frac{1}{\sqrt{a}}e_2^h\right)=\frac{1}{a}\widetilde{g}\left(e_2^h,e_2^h\right)\\ &= \frac{1}{a}a\left\{g\left(e_2,e_2\right)+g\left(S\left(e_2\right),u\right)g\left(S\left(e_2\right),u\right)\right\}\\ &= 1\end{aligned}$$

so we get

$$\begin{aligned}\widetilde{g}\left(f_n,f_n\right) &= \widetilde{g}\left(\frac{1}{\sqrt{a}}e_n^h,\frac{1}{\sqrt{a}}e_n^h\right)=\frac{1}{a}\widetilde{g}\left(e_n^h,e_n^h\right)\\ &= \frac{1}{a}a\left\{g\left(e_n,e_n\right)+g\left(S\left(e_n\right),u\right)g\left(S\left(e_n\right),u\right)\right\}\\ &= 1\end{aligned}$$

$$\begin{aligned}\widetilde{g}\left(f_{n+1},f_{n+1}\right) &= \widetilde{g}\left(e_1^v,e_1^v\right)=1\\ &\vdots\\ \widetilde{g}\left(f_{2n},f_{2n}\right) &= \widetilde{g}\left(e_n^v,e_n^v\right)=1.\end{aligned}$$

Moreover, we can easily show that the inner product of any pair of the set $\{f_1,f_2,...,f_{2n}\}$ is zero. □

Theorem 4.2 *Let (M,g) be a hypersurface in $\mathbb{R}^{n+1}$ and $\left(TM,\widetilde{g}\right)$ be its tangent bundle as submanifold of $\mathbb{R}^{2n+2}$. Then, the mean curvature denoted by $\widetilde{H}$ of TM is*

$$\widetilde{H}=\frac{1}{2n}\left\{\frac{1}{\lambda}h\left(e_1,e_1\right)+\frac{1}{a}\left[h\left(e_2,e_2\right)+\cdots+h\left(e_n,e_n\right)\right]\right\}N^h.$$

Proof For the orthonormal basis $\{e_1,e_2,...,e_n\}$ of the hypersurface M, we have the orthonormal basis of TM at the point (p,u) as

$$\{f_1,f_2,...,f_{2n}\}=\left\{\frac{1}{\sqrt{\lambda}}e_1^h,\frac{1}{\sqrt{a}}e_2^h,...,\frac{1}{\sqrt{a}}e_n^h,e_1^v,e_2^v,...,e_n^v\right\}.$$

Hence, the mean curvature of TM is calculated as

$$\begin{aligned}\widetilde{H} &= \frac{1}{2n}\left\{\sum_{i=1}^{2n}\widetilde{h}\left(f_i,f_i\right)\right\}\\ &= \frac{1}{2n}\left\{\begin{array}{c}\frac{1}{\lambda}\widetilde{h}\left(e_1^h,e_1^h\right)\\ +\frac{1}{a}\widetilde{h}\left(e_2^h,e_2^h\right)+\cdots+\frac{1}{a}\widetilde{h}\left(e_n^h,e_n^h\right)\\ +\widetilde{h}\left(e_1^v,e_1^v\right)+\cdots+\widetilde{h}\left(e_n^v,e_n^v\right)\end{array}\right\}\\ &= \frac{1}{2n}\left\{\begin{array}{c}\frac{1}{\lambda}g\left(S\left(e_1\right),e_1\right)\\ +\frac{1}{a}g\left(S\left(e_2\right),e_2\right)+\cdots+\frac{1}{a}g\left(S\left(e_n\right),e_n\right)\end{array}\right\}N^h\\ &= \frac{1}{2n}\left\{\begin{array}{c}\frac{1}{\lambda}h\left(e_1,e_1\right)\\ +\frac{1}{a}\left[h\left(e_2,e_2\right)+\cdots+h\left(e_n,e_n\right)\right]\end{array}\right\}N^h.\end{aligned}$$

□

Theorem 4.3 *Let* (M, g) *be a hypersurface in* $\mathbb{R}^{n+1}$ *and* $\left(TM, \widetilde{g}\right)$ *be its tangent bundle as submanifold of* $\mathbb{R}^{2n+2}$. *Then, sectional curvatures denoted by* $\widetilde{K}$ *of* TM *are*

$$\begin{aligned}
\widetilde{K}_{1i} &= \frac{1}{a}\left\{\begin{array}{c} R_{1ii1} + g\left(\nabla_1 (A_f)_{ii} - \nabla_i (A_f)_{1i}, e_1\right) \\ +g\left(A_f\left(e_1, (A_f)_{ii}\right) - A_f\left(e_i, (A_f)_{1i}\right), e_1\right) \end{array}\right\}, \\
(i &= 2, \cdots, n) \\
\widetilde{K}_{ij} &= \frac{1}{a}\left\{\begin{array}{c} R_{ijji} + g\left(\nabla_i (A_f)_{jj} - \nabla_j (A_f)_{ij}, e_i\right) \\ +g\left(A_f\left(e_i, (A_f)_{jj}\right) - A_f\left(e_j, (A_f)_{ij}\right), e_i\right) \end{array}\right\}, \\
(i, j &= 2, \cdots, n, \quad i \neq j, \quad i < j) \\
\widetilde{K}_{1k} &= 0, \qquad (k = n+1, \cdots, 2n) \\
\widetilde{K}_{jk} &= 0, \qquad (j = 2, \cdots, n, \quad k = n+1, \cdots, 2n) \\
\widetilde{K}_{kl} &= 0, \qquad (k, l = n+1, \cdots, 2n, \quad k \neq l, \quad k < l).
\end{aligned}$$

Proof Using orthonormal frame $\{f_1, f_2, ..., f_{2n}\}$ of TM at the point (p, u), the sectional curvature $\widetilde{K}$ can be given by the following formulas

$$\begin{aligned}
\widetilde{K}_{1i} &= \widetilde{K}(f_1, f_i) = \widetilde{R}(f_1, f_i, f_i, f_1), \\
&\quad (i = 2, \cdots, n) \\
\widetilde{K}_{ij} &= \widetilde{K}(f_i, f_j) = \widetilde{R}(f_i, f_j, f_j, f_i), \\
&\quad (i, j = 2, \cdots, n, \quad i \neq j, \quad i < j) \\
\widetilde{K}_{1k} &= \widetilde{K}(f_1, f_k) = \widetilde{R}(f_1, f_k, f_k, f_1), \\
&\quad (k = n+1, \cdots, 2n) \\
\widetilde{K}_{jk} &= \widetilde{K}(f_j, f_k) = \widetilde{R}(f_j, f_k, f_k, f_j), \\
&\quad (j = 2, \cdots, n, \quad k = n+1, \cdots, 2n) \\
\widetilde{K}_{kl} &= \widetilde{K}(f_k, f_l) = \widetilde{R}(f_k, f_l, f_l, f_k), \\
&\quad (k, l = n+1, \cdots, 2n, \quad k \neq l, \quad k < l).
\end{aligned}$$

Firstly, from $\{f_1, f_2, ..., f_{2n}\} = \left\{\frac{1}{\sqrt{\lambda}} e_1^h, \frac{1}{\sqrt{a}} e_2^h, ..., \frac{1}{\sqrt{a}} e_n^h, e_1^v, e_2^v, ..., e_n^v\right\}$ the orthonormal basis of TM at the point

(p,u) and Theorem 3.2, we get

$$\begin{aligned}
\widetilde{K}_{1i} &= \widetilde{K}(f_1,f_i)=\widetilde{R}(f_1,f_i,f_i,f_1) \\
&= \widetilde{R}\left(\frac{1}{\sqrt{\lambda}}e_1^h,\frac{1}{\sqrt{a}}e_i^h,\frac{1}{\sqrt{a}}e_i^h,\frac{1}{\sqrt{\lambda}}e_1^h\right) \\
&= \frac{1}{a\lambda}\widetilde{R}\left(e_1^h,e_i^h,e_i^he_1^h\right) \\
&= \frac{1}{\lambda}g\left(\left\{\begin{array}{c} R(e_1,e_i)e_i+(\nabla_{e_1}A_f)(e_i,e_i) \\ -(\nabla_{e_i}A_f)(e_1,e_i) \\ +A_f(e_1,A_f(e_i,e_i)) \\ -A_f(e_i,A_f(e_1,e_i)) \end{array}\right\},e_1\right) \\
&\quad +\frac{1}{\lambda}g(S(e_1),u).g\left(\left\{\begin{array}{c} R(e_1,e_i)e_i \\ +(\nabla_{e_1}A_f)(e_i,e_i) \\ -(\nabla_{e_i}A_f)(e_1,e_i) \\ +A_f(e_1,A_f(e_i,e_i)) \\ -A_f(e_i,A_f(e_1,e_i)) \end{array}\right\},S(u)\right)
\end{aligned}$$

Herefrom, if we take expression with coordinates for brevity, we have

$$\begin{aligned}
\widetilde{K}_{1i} &= \frac{1}{\lambda}\left\{\begin{array}{c} R_{1ii1}+g\left(\nabla_1(A_f)_{ii}-\nabla_i(A_f)_{1i},e_1\right) \\ +g\left(A_f\left(e_1,(A_f)_{ii}\right)-A_f\left(e_i,(A_f)_{1i}\right),e_1\right) \end{array}\right\} \\
&\quad +\frac{\kappa_1.|u|}{\lambda}.g\left(\left\{\begin{array}{c} R(e_1,e_i)e_i \\ +\nabla_1(A_f)_{ii} \\ -\nabla_i(A_f)_{1i} \\ +A_f\left(e_1,(A_f)_{ii}\right) \\ -A_f\left(e_i,(A_f)_{1i}\right) \end{array}\right\},S(u)\right)
\end{aligned}$$

If we consider $S(u)=S(e_1.|u|)=|u|\,S(e_1)=|u|\,\kappa_1 e_1$ and $\lambda=a\left(1+\kappa_1^2.|u|^2\right)$, we get

$$\widetilde{K}_{1i}=\frac{1}{a}\left\{\begin{array}{c} R_{1ii1}+g\left(\nabla_1(A_f)_{ii}-\nabla_i(A_f)_{1i},e_1\right) \\ +g\left(A_f\left(e_1,(A_f)_{ii}\right)-A_f\left(e_i,(A_f)_{1i}\right),e_1\right) \end{array}\right\}.$$

Similarly, it is calculated as

$$\begin{aligned}
\widetilde{K}_{ij} &= \widetilde{K}(f_i,f_j)=\widetilde{R}(f_i,f_j,f_j,f_i) \\
&= \widetilde{R}\left(\frac{1}{\sqrt{a}}e_i^h,\frac{1}{\sqrt{a}}e_j^h,\frac{1}{\sqrt{a}}e_j^h,\frac{1}{\sqrt{a}}e_i^h\right) \\
&= \frac{1}{a^2}\widetilde{R}\left(e_i^h,e_j^h,e_j^h,e_i^h\right) \\
&= \frac{1}{a}\left\{\begin{array}{c} R_{ijji}+g\left(\nabla_i(A_f)_{jj}-\nabla_j(A_f)_{ij},e_i\right) \\ +g\left(\begin{array}{c} A_f\left(e_i,(A_f)_{jj}\right) \\ -A_f\left(e_j,(A_f)_{ij}\right),e_i \end{array}\right) \end{array}\right\}.
\end{aligned}$$

And other sectional curvatures are obtained as zero. □

Conclusion 4.4 *In this article, rescaled induced metric has been defined on TM which is considered a submanifold of $\mathbb{R}^{2n+2}$ and by means of this metric, orthonormal frame on TM at the point $P=(p,u)$ has been established. In this way, many differential geometric results in the theory of submanifolds can be obtained other than the results given in this paper.*

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