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Some qualitative results for functional delay dynamic equations on time scales

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Abstract: Let $\mathbb{T}$ be a nonempty, closed, and arbitrary set of real numbers, namely a time scale, and consider the following delay dynamical equation

$$x^{\Delta}(t) = a(t)x(t) + f(t, x(\vartheta(t))), \quad t \in \mathbb{T},$$

where ϑ stands for the abstract delay function. The main goal of this study is three-fold: obtaining the existence of an equi-bounded solution, proving the asymptotic stability of the zero solution, and showing the existence of a periodic solution based on new periodicity concept on time scales for the given delayed equation under certain conditions. In our analysis, we propose an alternative variation of parameters formulation by using an auxiliary function to invert a mapping for the utilization of fixed point theory.

Key words: Delay dynamic equation, time scale, equi-bounded, asymptotic stability, shift operator, periodicity

1. Introduction

Qualitative theory of differential and difference equations are extensively studied in the literature, and these studies have been unified and generalized on arbitrary, nonempty, and closed subset of real numbers, namely time scales. The theory of time scales has become a very popular research area in the last three decades since it avoids separate studies of continuous and discrete time mathematical models. For an excellent review on time scale calculus, we refer the readers to pioneering books [8, 9].

Fixed point theory is one of the fundamental tools in the qualitative analysis of dynamic equations. As it is well known, one has to invert the problem and obtain an appropriate mapping for the utilization of the fixed point theory to qualitatively analyze a dynamic equation. Contraction mapping principle plays a crucial role in this study, so boundedness, stability, and periodicity for solutions of the delayed dynamical equation

$$x^{\Delta}(t) = a(t)x(t) + f(t, x(\vartheta(t)))$$

are studied on time scales by proposing a mapping due to a new variation of parameters formula. Inspired by the papers [23, 24], we define a mapping with the help of an auxiliary function, which we use to construct the integrating factor, in the setup of our analysis. Thus, we obtain our results without using the conventional variation of constants formula (see [8, Theorem 2.77]) unlike the existing literature. It should be pointed out

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that this method does not only provide an alternative approach but also relaxes a restrictive condition, i.e. regressivity, on the delay dynamic equation.

We summarize the highlights of the manuscript as follows:

- Boundedness for the solutions of dynamic equations is considered by several researchers on continuous, discrete and hybrid time domains. We refer to the monograph [20], and the papers [18, 23] as the related studies focusing on boundedness of the solutions of differential and difference equations, respectively. As a result of the noticeable interest on the qualitative properties of dynamic equations on time scales, the existence of bounded solutions for the dynamic equations is also studied on hybrid time domains. We refer to readers [6, 13, 21, 26] as remarkable studies. Motivated by the above given literature, we focus on the equi-boundedness of the solutions for delay dynamic equations on time scales and provide sufficient conditions for the existence of an equi-bounded solution as the first objective of the manuscript.

- Stability theory of dynamic equations is a voguish research area in pure and applied mathematics because of its potential for application in numereous fields such as biology, physics, control theory, and economics, among others. Undoubtedly, there is a vast literature which focuses on the asymptotic properties and the stability of the solutions of dynamic equations. We refer to readers to [14, 27, 28] and [22, 23] as related papers which employs fixed point theory to analyze stability for the solutions of differential and difference equations, respectively. Additionally, we cite the inspiring papers [1, 7, 16, 25, 26] for the stability analysis of dynamic equations on time scales. As the second task of the paper, we aim to utilize the new variation of parameters formula for the stability analysis of delay dynamic equations on time scales. In our analysis, we get necessary and sufficient conditions for the asymptotic stability of the zero solution via fixed point theory. The use of the new variation of parameters formula enables us to obtain the stability result without assuming a restrictive condition, namely $e_a(t, t_0) \to 0$ as $t \to \infty$, which is used in the existing literature (see [1, 7, 23, 26]). Thus, the obtained stability result significantly improves the established literature.

- The shift operators denoted by $\delta_{\pm}(s, t)$ are introduced to construct delay dynamic equations and a new periodicity concept on time scales (see [2, 3]). The utilization of shift operators and the new periodicity concept to define periodic structures enables us to focus on a more general class of time scales which is not necessarily translation invariant, i.e. there exists a $P > 0$ such that $t \pm P \in \mathbb{T}$ for all $t \in \mathbb{T}$. We give a detailed information about the shift operators in further sections. We refer to [4, 11, 12, 17] as the examples of studies employing the new periodicity concept on time scales for the existence of periodic solutions of dynamic equations due to fixed point theory. Moreover, we refer to the brand new monograph [5], which excellently summarizes the recent advances in functional dynamic equations on time scales regarding periodicity, stability, and boundedness of solutions for further reading. As the final goal of the manuscript, we redefine the delay dynamic equation by using the backward shift operator $\delta_{-}(s, t)$ as the delay function and use the new variation of parameters formula to obtain the existence of the unique periodic solution in shifts $\delta_{\pm}$ in the light of new periodicity concept on time scales.

The paper is organized as follows. In the next section, we present an overview of time scale calculus which provides the fundamentals for the readership. Section 3 is devoted to presentation of our main results. In Subsection 3.1, we give the setup of our analysis, and present our boundedness and stability results. Besides, we briefly introduce the new periodicity concept on time scales based on shift operators and present the existence result for periodic solution of the delay dynamic equation with respect to new periodicity concept in Subsection 3.2. The last section includes some examples as an implementation of our results.

2. Time scale essentials

Throughout the manuscript, we assume a familiarity with the theory of time scales. In this section, we just give a short summary about the fundamentals of the time scale calculus. Given definitions, results, and examples can be found in [8] and [9].

A time scale, denoted by $\mathbb{T}$, is an arbitrary, nonempty, and closed subset of real numbers. The operator $\sigma : \mathbb{T} \to \mathbb{T}$, which is called forward jump operator, is defined by $\sigma(t) := \inf\{s \in \mathbb{T}, s > t\}$. The step size function $\mu : \mathbb{T} \to \mathbb{R}$ is given by $\mu(t) := \sigma(t) - t$. We say a point $t \in \mathbb{T}$ is right-dense if $\mu(t) = 0$, and right-scattered if $\mu(t) > 0$. Furthermore, a point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) := \sup\{s \in \mathbb{T}, s < t\} = t$ and left-scattered if $\rho(t) < t$. The notation $[s,t)_{\mathbb{T}}$ indicates the intersection $[s,t) \cap \mathbb{T}$ and the intervals $[s,t]_{\mathbb{T}}, (s,t)_{\mathbb{T}}$, and $(s,t]_{\mathbb{T}}$ can be defined similarly.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at right-dense points and its left-sided limits exist at left-dense points. Moreover, we use the notation C_{rd} in order to represent the set of all rd-continuous functions on $\mathbb{T}$. The set $\mathbb{T}^k$ is defined in the following way: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. Furthermore, the delta derivative of a function $f : \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}^k$ is defined by

$$f^{\Delta}(t) := \lim_{\substack{s \to t \\ s \neq \sigma(t)}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

We summarize the basic properties of delta derivative with the next result.

Theorem 2.1 *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then*

i. $(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$.

ii. $(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t)$ *for any constant* α.

iii. $(fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t) = f(t) g^{\Delta}(t) + f^{\Delta}(t) g(\sigma(t))$.

iv. If $f(t) f(\sigma(t)) \neq 0$, *then* $\frac{1}{f}$ *is differentiable at* t *with*

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t) f(\sigma(t))}.$$

v. If $g(t) g(\sigma(t)) \neq 0$, *then* $\frac{f}{g}$ *is differentiable at* t *with*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t) g(t) - f(t) g^{\Delta}(t)}{g(t) g(\sigma(t))}.$$

Table 1 shows the main characteristics of three important examples of time scales:

Table 1.

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$q^{\mathbb{Z}} \cup \{0\}$, $q > 1$
$\rho(t)$	t	$t-1$	$\frac{t}{q}$
$\sigma(t)$	t	$t+1$	qt
$\mu(t)$	0	1	$(q-1)t$
$f^{\Delta}(t)$	$f'(t)$	$\Delta f(t)$	$D_q f(t) = \frac{f(qt)-f(t)}{(q-1)t}$

Definition 2.2 *A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(t) p(t) \neq 0$ for all $t \in \mathbb{T}^k$. We denote the set of all regressive functions by $\mathcal{R}$. Moreover, $\mathcal{R}^+$ stands for the set of all positively regressive elements of $\mathcal{R}$ defined by*

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t) p(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Definition 2.3 (Exponential function) *For $h > 0$, set $\mathbb{C}_h := \{z \in \mathbb{C} : z \neq -1/h\}$, $\mathbb{J}_h := \{z \in \mathbb{C} : -\pi/h < Im(z) \leq \pi/h\}$ and $\mathbb{C}_0 := \mathbb{J}_0 := \mathbb{C}$. For $h \geq 0$ and $z \in \mathbb{C}_h$, the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{J}_h$ is given by*

$$\xi_h(z) := \begin{cases} z, & h = 0 \\ \dfrac{1}{h} \operatorname{Log}(1 + zh), & h > 0. \end{cases}$$

Then, the exponential function on $\mathbb{T}$ is presented in the form

$$e_p(t,s) := \exp\left\{ \int_s^t \xi_{\mu(\tau)}\big(p(\tau)\big) \Delta\tau \right\} \quad \text{for } s,t \in \mathbb{T}.$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Moreover, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^{\Delta} = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 2.4 *Let $p, q \in \mathcal{R}$. Then*

i. $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

ii. $e_p(\sigma(t), s) = (1 + \mu(t)p(t)) e_p(t,s)$;

iii. $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$ *where,* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;

iv. $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

v. $e_p(t,s) e_p(s,r) = e_p(t,r)$;

vi. $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

Let $\alpha \in \mathcal{R}$ be a constant. Table 2 demonstrates some exponential functions over particular time scales

Table 2.

$\mathbb{T}$	$e_\alpha(t,t_0)$
$\mathbb{R}$	$e^{\alpha(t-t_0)}$
$\mathbb{Z}$	$(1+\alpha)^{t-t_0}$
$h\mathbb{Z}$	$(1+h\alpha)^{(t-t_0)/h}$
$q^{\mathbb{N}_0}, q>1$	$\prod_{s\in[t_0,t)} [1+(q-1)\alpha s]\,, t>t_0$
$\frac{1}{n}\mathbb{Z}$	$\left(1+\frac{\alpha}{n}\right)^{n(t-t_0)}$

Theorem 2.5 (Variation of constants) *Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the regressive initial value problem*

$$x^{\Delta}(t) = a(t)\,x(t) + f(t)\,, \quad y(t_0) = y_0$$

is given by

$$x(t) = e_a(t,t_0)\,y_0 + \int_{t_0}^{t} e_a(t,\sigma(\tau))\,f(\tau)\,\Delta\tau.$$

3. Main results

3.1. Boundedness and stability

3.1.1. Setup

Let $\mathbb{T}$ be a time scale unbounded from above and $t_0 \in \mathbb{T}$ is fixed.

We focus on the following nonlinear delay dynamic equation

$$x^{\Delta}(t) = a(t)\,x(t) + f(t, x(\vartheta(t)))\,, \quad t \in \mathbb{T}, \tag{3.1}$$

where $a : [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}$ and $f : [t_0,\infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$. Together with the rd-continuity assumptions on the functions a and f, we also suppose that $f(t,0) = 0$. Moreover, $\vartheta : [t_0,\infty)_{\mathbb{T}} \to [\vartheta(t_0),\infty)_{\mathbb{T}}$ indicates the delay function with the properties

i $\vartheta(t) < t$ for all $t \in \mathbb{T}$

ii ϑ is strictly increasing with $\lim_{t\to\infty} \vartheta(t) = \infty$.

In Table 3, we give some examples of time scales with their delay functions

By letting $\xi : [\vartheta(t_0), t_0]_{\mathbb{T}} \to \mathbb{R}$, we denote the solution of (3.1) by $x^{\xi} := x(t,t_0,\xi)$ which satisfies (3.1) for all $t \in [t_0,\infty)_{\mathbb{T}}$ and $x^{\xi} = \xi(t)$ on $[\vartheta(t_0), t_0]_{\mathbb{T}}$. We assume $\xi \in \mathbb{B}$, where $\mathbb{B}$ is the Banach space of bounded functions with the supremum norm, i.e.

$$\|\xi\| = \sup_{t\in[\vartheta(t_0),t_0]_{\mathbb{T}}} |\xi(t)| < \infty.$$

We provide the following lemma which is fundamental for the proofs of main results.

Table 3.

Time scale	Delay function
$\mathbb{T}=\mathbb{R}$	$\vartheta(t)=t-\tau, \quad \tau\in\mathbb{R}_+$
$\mathbb{T}=\mathbb{Z}$	$\vartheta(t)=t-\tau, \quad \tau\in\mathbb{Z}$
$\mathbb{T}=\{t=k-q^m : k\in\mathbb{Z}, m\in\mathbb{N}_0, q\in(0,1)\}$	$\vartheta(t)=t-\tau, \quad \tau\in\mathbb{Z}_+$
$\mathbb{T}=h\mathbb{Z}$	$\vartheta(t)=t-h\tau, \quad \tau\in h\mathbb{Z}$
$\mathbb{T}=\mathbb{N}^{1/2}$	$\vartheta(t)=\sqrt{t^2-\tau^2}, \quad \tau\in\mathbb{N}^{1/2}$
$\mathbb{T}=q^{\mathbb{Z}}\cup\{0\}, q>1$	$\vartheta(t)=t/\tau, \quad \tau\in q^{\mathbb{N}}$

Lemma 3.1 *The nonlinear delay dynamic equation (3.1) has a solution x if and only if*

$$x(t)=\xi(t_0)e_p(t,t_0)+\int_{t_0}^{t}e_p(t,\sigma(\tau))\left([a(\tau)-p(\tau)]x(\tau)+f(\tau,x(\vartheta(\tau)))\right)\Delta\tau, \text{ for all } t\in[t_0,\infty)_{\mathbb{T}},$$

where $p:[t_0,\infty)_{\mathbb{T}}\to\mathbb{R}$ is regressive.

Proof We multiply both sides of (3.1) with $e_{\ominus p}(t,t_0)$ and obtain

$$x^{\Delta}(t)e_{\ominus p}(t,t_0)=a(t)x(t)e_{\ominus p}(t,t_0)+f(t,x(\vartheta(t)))e_{\ominus p}(t,t_0).$$

As a consequence of Lemma 2.4, we rewrite the above equation as follows

$$[x(t)e_{\ominus p}(t,t_0)]^{\Delta}=\frac{e_{\ominus p}(t,t_0)}{1+\mu(t)p(t)}\left([a(t)-p(t)]x(t)+f(t,x(\vartheta(t)))\right). \tag{3.2}$$

Integrating the equation (3.2) from t_0 to t yields

$$x(t)=\xi(t_0)e_p(t,t_0)+\int_{t_0}^{t}e_p(t,\sigma(\tau))\left([a(\tau)-p(\tau)]x(\tau)+f(\tau,x(\vartheta(\tau)))\right)\Delta\tau.$$

This proves the assertion. □

We construct the following sets as a preparation for the main results:

$$B_k\subset\mathbb{B} \text{ so that } B_k:=\{x\in\mathbb{B}:\|x\|\le k\}, \tag{3.3}$$

$$\mathbb{M}_1:=\{x:\mathbb{T}\to\mathbb{R}:x(t_0)=\xi(t_0) \text{ and } x\in B_k \text{ for } t\in[t_0,\infty)_{\mathbb{T}}\}, \tag{3.4}$$

and

$$\mathbb{M}_2:=\{x\in\mathbb{M}_1:x(t)\to 0 \text{ as } t\to\infty\}. \tag{3.5}$$

We should point it out that $\mathbb{M}_1$ and $\mathbb{M}_2$ are complete metric spaces with the metric

$$d(x_1,x_2)=\sup_{t\in[t_0,\infty)_{\mathbb{T}}}|x_1(t)-x_2(t)|.$$

We define the mapping H as follows:

$$(Hx)(t) := \begin{cases} \xi(t) \text{ if } t \in [\vartheta(t_0), t_0]_{\mathbb{T}} \\ \xi(t_0) e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(\tau)) ([a(\tau) - p(\tau)] x(\tau) + f(\tau, x(\vartheta(\tau)))) \Delta\tau \text{ if } t \in [t_0, \infty)_{\mathbb{T}} \end{cases}. \quad (3.6)$$

In the next part, we concentrate on the equi-boundedness of the solutions of (3.1).

3.1.2. Equi-boundedness

We shall start by presenting the following definition in the light of [20].

Definition 3.2 *The solution of (3.1) is said to be equi-bounded if for $t_0 \in \mathbb{T}$ and any $l > 0$, there exists a $k := k(t_0, l)$ so that $|\xi(t)| \leq l$ on $t \in [\vartheta(t_0), t_0]_{\mathbb{T}}$ implies $|x(t, t_0, \xi)| \leq k$ for $t \in [t_0, \infty)_{\mathbb{T}}$.*

For proving the existence of an equi-bounded solution of delay dynamic equation (3.1), we shall make the following assumptions: Let $g : [t_0, \infty)_{\mathbb{T}} \to (0, \infty)$ and $\alpha > 0$. We introduce the following conditions

C1

$$\int_{t_0}^{t} |e_p(t, \sigma(\tau))| (|a(\tau) - p(\tau)| + g(\tau)) \Delta\tau \leq \alpha < 1 \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}.$$

C2

$$|f(t, x) - f(t, y)| \leq g(t) \|x - y\| \text{ for any } x, y \in B_k,$$

where B_k is as in (3.3).

Theorem 3.3 *Assume **C1** and **C2**. If there exists a positive constant U such that $|e_p(t, t_0)| \leq U$ for all $t \in [t_0, \infty)_{\mathbb{T}}$, then (3.1) has an equi-bounded solution.*

Proof Suppose that **C1** and **C2** hold. Let $l > 0$ be given and the initial function ξ for the solution of (3.1) satisfies $|\xi(t)| \leq l$ for all $t \in [\vartheta(t_0), t_0]_{\mathbb{T}}$. Let the positive constant k be chosen so that the inequality

$$lU + \alpha k \leq k$$

holds. Now, we will show that H maps $\mathbb{M}_1$ into $\mathbb{M}_1$, where $\mathbb{M}_1$ and H are as in (3.4) and (3.6), respectively. For $x \in \mathbb{M}_1$, consider

$$\begin{aligned} |(Hx)(t)| &= \left| \xi(t_0) e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(\tau)) ([a(\tau) - p(\tau)] x(\tau) + f(\tau, x(\vartheta(\tau)))) \Delta\tau \right| \\ &\leq |\xi(t_0) e_p(t, t_0)| + \int_{t_0}^{t} |e_p(t, \sigma(\tau))| \left|[a(\tau) - p(\tau)] x(\tau) + f(\tau, x(\vartheta(\tau)))\right| \Delta\tau \\ &\leq lU + k \int_{t_0}^{t} |e_p(t, \sigma(\tau))| (|a(\tau) - p(\tau)| + g(\tau)) \Delta\tau \\ &\leq lU + \alpha k \leq k \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \end{aligned}$$

which obviously implies $H:\mathbb{M}_1\to\mathbb{M}_1$. As the next step, we prove the mapping H is contraction. Let $x_1,x_2\in\mathbb{M}_1$. Then, we have

$$\begin{aligned}\left|\left(Hx_1\right)(t)-\left(Hx_2\right)(t)\right| &\leq \int_{t_0}^{t}\left[\left|e_p\left(t,\sigma\left(\tau\right)\right)\right|\left|a\left(\tau\right)-p\left(\tau\right)\right|\left\|x_1-x_2\right\|\right.\\ &\quad\left.+\left|e_p\left(t,\sigma\left(\tau\right)\right)\right|\left|f\left(\tau,x_1\left(\vartheta\left(\tau\right)\right)\right)-f\left(\tau,x_2\left(\vartheta\left(\tau\right)\right)\right)\right|\right]\Delta\tau\\ &\leq \left\|x_1-x_2\right\|\int_{t_0}^{t}\left|e_p\left(t,\sigma\left(\tau\right)\right)\right|\left(\left|a\left(\tau\right)-p\left(\tau\right)\right|+g\left(\tau\right)\right)\Delta\tau\\ &\leq \alpha\left\|x_1-x_2\right\|,\end{aligned}$$

which indicates H is a contraction. By contraction mapping principle, we deduce that H has a unique fixed point which solves (3.1) and consequentially, (3.1) has an equi-bounded solution. □

Remark 3.4 *In [18], authors consider the nonlinear delay difference equation*

$$x\left(t+1\right)=a\left(t\right)x\left(t\right)+b\left(t\right)x\left(t-g\left(t\right)\right)+c\left(t\right)\Delta x\left(t-g\left(t\right)\right), \tag{3.7}$$

where Δ stands for the forward difference operator, $a,c:\mathbb{Z}\to\mathbb{R}$ and g is an appropriate delay function, and they studied the existence of equi-bounded solutions for (3.7) under sufficient conditions. The assumption $a\left(t\right)\neq 0$ for all $t\in\mathbb{Z}$ is made in [18, Theorem 3.1], which implies the regressivity of the equation if we rewrite (3.7) as

$$\Delta x\left(t\right)=\left(a\left(t\right)-1\right)x\left(t\right)+b\left(t\right)x\left(t-g\left(t\right)\right)+c\left(t\right)\Delta x\left(t-g\left(t\right)\right). \tag{3.8}$$

If one sets $c=0$, $\vartheta\left(t\right)=t-g\left(t\right)$, and $f\left(t,x\left(\vartheta\left(t\right)\right)\right)=b\left(t\right)x\left(t-g\left(t\right)\right)$, then the relationship between (3.8) and (3.1) is obvious when $\mathbb{T}=\mathbb{Z}$. We shall emphasize that as a consequence of using a new variation of parameters formula for (3.1), Theorem 3.3 does not involve regressivity condition on the coefficient of the delay dynamic equation unlike [18, Theorem 3.1]. Thus, we do not only unify but also improve the exisiting literature by providing less conditions.

3.1.3. Asymptotic stability

This part is devoted to obtain necessary and sufficient conditions for the asymptotic stability of the zero solution of (3.1) due to fixed point theory. First, we provide the following definitions due to [16].

Definition 3.5 *The zero solution of (3.1) is said to be stable, if for any $\varepsilon>0$ and $t_0\in\mathbb{T}$, there exists a $\gamma\left(t_0,\varepsilon\right)>0$ such that $\left|x_0\right|\leq\gamma$ implies $\left|x\left(t,x_0,t_0\right)\right|<\varepsilon$ for all $t\geq t_0$.*

Definition 3.6 *The zero solution of (3.1) is said to be asymptotically stable, if it is stable and $x\left(t,x_0,t_0\right)\to 0$ as $t\to\infty$.*

For the construction of the stability theorem, we shall write the following condition in addition to **C1** and **C2**:

C3 For $t_1 > t_0$ $(t_1 \in \mathbb{T})$, there exists a $t_2 > t_1$ $(t_2 \in \mathbb{T})$ so that for $t > t_2$ and $x \in B_k$

$$|f(t, x(\vartheta(t)))| \leq g(t) \|x\|_{t_1}^{t}, \tag{3.9}$$

where B_k is as in (3.3) and then norm $\|x\|_s^t$ is given by

$$\|x\|_s^t := \sup_{h\in[s,t]} |x(h)|$$

similar to [22].

Next, we present our stability result for the delay dynamic equation (3.1) without assuming the function a is regressive, and instead of making the assumption $e_a(t,t_0) \to 0$ as $t \to \infty$ we ask the asymptotic property $e_p(t,t_0) \to 0$ as $t \to \infty$. We shall point out that the following result is comparable with [22, Theorem 2] when $\mathbb{T} = \mathbb{Z}$, and it relaxes the first and the last condition of [22, Theorem 2] due to the utilization of new variation of parameters formula.

Theorem 3.7 *Assume that the conditions **C1-C3** are satisfied. Then, the zero solution of (3.1) is asymptotically stable if and only if*

$$e_p(t,t_0) \to 0 \text{ as } t \to \infty. \tag{3.10}$$

Proof Suppose that **C1-C3** and (3.10) hold and set

$$S := \sup_{t\in[t_0,\infty)_{\mathbb{T}}} |e_p(t,t_0)|. \tag{3.11}$$

First of all, we show that if $x \in \mathbb{M}_2$, then $(Hx) \in B_k$, where H is defined in (3.6). Choose a positive constant γ so that the inequality $\gamma S + \alpha k \leq k$ is satisfied. We assume $\xi \in B_\gamma$ and $x \in \mathbb{M}_2$, where the set $\mathbb{M}_2$ is given in (3.5). Note that $x \in B_k$ since $x \in \mathbb{M}_2$. In the sequel, we obtain the following inequality

$$\begin{aligned}
|(Hx)(t)| &\leq |\xi(t_0)| |e_p(t,t_0)| + \int_{t_0}^{t} |e_p(t,\sigma(\tau))| |[a(\tau) - p(\tau)] x(\tau) + f(\tau, x(\vartheta(\tau)))| \Delta\tau \\
&\leq \gamma S + \|x\| \int_{t_0}^{t} |e_p(t,\sigma(\tau))| (|a(\tau) - p(\tau)| + g(\tau)) \Delta\tau \\
&\leq \gamma S + \alpha k \leq k,
\end{aligned}$$

which shows $(Hx) \in B_k$.

As the next step, we prove that $(Hx)(t) \to 0$ as $t \to \infty$. It is clear that $x(t) \to 0$ as $t \to \infty$ since $x \in \mathbb{M}_2$. That is, there exists $t_1 > t_0$ $(t_1 \in \mathbb{T})$ such that $|x(t)| < \frac{\varepsilon}{3}$ whenever $t \geq t_1$. On the other hand, there exists $t_2 > t_1$ $(t_2 \in \mathbb{T})$ so that the inequality (3.9) holds by **C3** for $t > t_2 > t_1$. In the sequel, we focus on the integral term of the mapping H and obtain

$$\begin{aligned}
&\left|\int_{t_0}^{t} e_p\left(t,\sigma\left(\tau\right)\right)\left(\left[a\left(\tau\right)-p\left(\tau\right)\right]x\left(\tau\right)+f\left(\tau,x\left(\vartheta\left(\tau\right)\right)\right)\right)\Delta\tau\right| \\
&\leq \int_{t_0}^{t} \left|e_p\left(t,\sigma\left(\tau\right)\right)\right|\left|\left[a\left(\tau\right)-p\left(\tau\right)\right]x\left(\tau\right)+f\left(\tau,x\left(\vartheta\left(\tau\right)\right)\right)\right|\Delta\tau \\
&= \int_{t_0}^{t_2} \left|e_p\left(t,\sigma\left(\tau\right)\right)\right|\left|\left[a\left(\tau\right)-p\left(\tau\right)\right]x\left(\tau\right)+f\left(\tau,x\left(\vartheta\left(\tau\right)\right)\right)\right|\Delta\tau \\
&+ \int_{t_2}^{t} \left|e_p\left(t,\sigma\left(\tau\right)\right)\right|\left|\left[a\left(\tau\right)-p\left(\tau\right)\right]x\left(\tau\right)+f\left(\tau,x\left(\vartheta\left(\tau\right)\right)\right)\right|\Delta\tau.
\end{aligned}$$

This yields

$$\begin{aligned}
&\left|\int_{t_0}^{t} e_p\left(t,\sigma\left(\tau\right)\right)\left(\left[a\left(\tau\right)-p\left(\tau\right)\right]x\left(\tau\right)+f\left(\tau,x\left(\vartheta\left(\tau\right)\right)\right)\right)\Delta\tau\right| \\
&\leq \|x\| \int_{t_0}^{t_2} \left|e_p\left(t,t_2\right)\right|\left|e_p\left(t_2,\sigma\left(\tau\right)\right)\right|\left(\left|a\left(\tau\right)-p\left(\tau\right)\right|+g\left(\tau\right)\right)\Delta\tau \\
&+ \int_{t_2}^{t} \left|e_p\left(t,\sigma\left(\tau\right)\right)\right|\left(\left|a\left(\tau\right)-p\left(\tau\right)\right|\left|x\left(\tau\right)\right|+g\left(\tau\right)\|x\|_{t_1}^{\tau}\right)\Delta\tau \\
&\leq \|x\|\left|e_p\left(t,t_2\right)\right| \int_{t_0}^{t_2} \left|e_p\left(t_2,\sigma\left(\tau\right)\right)\right|\left(\left|a\left(\tau\right)-p\left(\tau\right)\right|+g\left(\tau\right)\right)\Delta\tau \\
&+ \varepsilon\int_{t_2}^{t} \left|e_p\left(t,\sigma\left(\tau\right)\right)\right|\left(\left|a\left(\tau\right)-p\left(\tau\right)\right|+g\left(\tau\right)\right)\Delta\tau \\
&\leq \alpha k\left|e_p\left(t,t_2\right)\right|+\alpha\frac{\varepsilon}{3}. \qquad (3.12)
\end{aligned}$$

Notice that we have $\gamma\left|e_p\left(t,t_0\right)\right| < \frac{\varepsilon}{3}$ and $\alpha k\left|e_p\left(t,t_2\right)\right| < \frac{\varepsilon}{3}$ for $t > t_2 > t_1$ due to the condition (3.10). By considering (3.12), we get

$$|(Hx)(t)| \leq |\xi(t_0)| |e_p(t,t_0)| + \int_{t_0}^{t_2} |e_p(t,\sigma(\tau))| |[a(\tau)-p(\tau)]x(\tau) + f(\tau, x(\vartheta(\tau)))| \Delta\tau$$

$$+ \int_{t_2}^{t} |e_p(t,\sigma(\tau))| |[a(\tau)-p(\tau)]x(\tau) + f(\tau, x(\vartheta(\tau)))| \Delta\tau$$

$$\leq \gamma |e_p(t,t_0)| + \alpha k |e_p(t,t_2)| + \alpha\frac{\varepsilon}{3}$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \alpha\frac{\varepsilon}{3} < \varepsilon,$$

which indicates $(Hx)(t) \to 0$ as $t \to \infty$. Thus, H maps $\mathbb{M}_2$ into $\mathbb{M}_2$.

Besides, one may easily obtain the inequality

$$|(Hx_1)(t) - (Hx_2)(t)| \leq \alpha \|x_1 - x_2\|$$

for an arbitrary pair $x_1, x_2 \in \mathbb{M}_2$ as it is done in the proof of Theorem 3.3. Consequently, H has a unique fixed point in $\mathbb{M}_2$ which solves (3.1) for $t \in [t_0,\infty)_{\mathbb{T}}$.

As the concluding step, we show that the zero solution of (3.1) is stable. For $\varepsilon > 0$ and $\varepsilon < k$ are given, we choose $\gamma < \varepsilon$ which satisfies $\gamma S + \alpha\varepsilon < \varepsilon$, where $\xi \in B_\gamma$ and S is as in (3.11). Let $\tilde{t} > t_0$ $\left(\tilde{t} \in \mathbb{T}\right)$ such that $\left|x\left(\tilde{t}\right)\right| > \varepsilon$ and $|x(u)| < \varepsilon$ for $u \in \left[t_0, \tilde{t}\right)_{\mathbb{T}}$. If x is a solution of (3.1), then

$$\left|x\left(\tilde{t}\right)\right| \leq \gamma \left|e_p\left(\tilde{t},t_0\right)\right| + \int_{t_0}^{\tilde{t}} \left|e_p\left(\tilde{t},\sigma(\tau)\right)\right| |[a(\tau)-p(\tau)]x(\tau) + f(\tau, x(\vartheta(\tau)))| \Delta\tau$$

$$\leq \gamma S + \alpha \|x\|$$

$$< \gamma S + \alpha\varepsilon < \varepsilon,$$

which yields a contradiction with $\left|x\left(\tilde{t}\right)\right| > \varepsilon$ when $\tilde{t} \geq t_0$. That means $|x(t)| < \varepsilon$ for all $t \in [t_0,\infty)_{\mathbb{T}}$ and consequently the zero solution of (3.1) is asymptotically stable.

On the contrary**,** suppose that x is asymptotically stable and the condition (3.10) does not hold; that is there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} |e_p(t_0,t_n)| = w \neq 0. \tag{3.13}$$

By using **C1** together with (3.13), we get the following inequalities

$$|e_p(t_0,t_n)| \int_{t_0}^{t_n} |e_p(t_n,\sigma(\tau))| (|a(\tau)-p(\tau)| + g(\tau)) \Delta\tau \leq \alpha |e_p(t_0,t_n)|,$$

and

$$\int_{t_0}^{t_n} |e_p(t_0,\sigma(\tau))| (|a(\tau)-p(\tau)| + g(\tau)) \Delta\tau \leq \alpha w. \tag{3.14}$$

It is clear that the sequence represented by (3.14) is bounded, and consequentially, it has a convergent subsequence. For the readership, we shall not change the notation used in (3.14). Thus, we assume that

$$\lim_{n\to\infty}\int_{t_0}^{t_n} |e_p(t_0,\sigma(\tau))|\left(|a(\tau)-p(\tau)|+g(\tau)\right)\Delta\tau = w^*.$$

To obtain a contradiction to for the conclusion of the proof, we also suppose that

$$\int_{t_{\tilde{n}}}^{t_n} |e_p(t_0,\sigma(\tau))|\left(|a(\tau)-p(\tau)|+g(\tau)\right)\Delta\tau < \frac{1-\alpha}{2S^2}, \tag{3.15}$$

where S is as in (3.11) and $\tilde{n}$ is sufficiently large.

Next, we consider the solution $x^{\xi} := x(t,t_{\tilde{n}},\xi)$ of (3.1) with $x^{\xi} = \xi(t)$ on $[\vartheta(t_{\tilde{n}}), t_{\tilde{n}}]_{\mathbb{T}}$ and $\xi(t_{\tilde{n}}) = \gamma$. Since the solution is assumed to be asymptotically stable, one may write $|x(t)| < Q$ for $t > t_{\tilde{n}}$ with $x(t_{\tilde{n}}) = \xi(t_{\tilde{n}}) = \gamma$. For all $t \geq t_{\tilde{n}}$, we have

$$\begin{aligned} |x(t)| &\leq \gamma|e_p(t,t_{\tilde{n}})| + \int_{t_{\tilde{n}}}^{t} |e_p(t,\sigma(\tau))|\left(|a(\tau)-p(\tau)|\,|x(t)| + |f(\tau,x(\vartheta(\tau)))|\right)\Delta\tau \\ &\leq \gamma S + \alpha\sup_{t\geq t_{\tilde{n}}}|x(t)|, \end{aligned}$$

which provides

$$|x(t)| \leq \frac{\gamma S}{1-\alpha} \text{ for all } t \geq t_{\tilde{n}}. \tag{3.16}$$

As the following step, we write

$$\begin{aligned} |x(t_n)| &\geq \gamma|e_p(t_n,t_{\tilde{n}})| - \int_{t_{\tilde{n}}}^{t_n} |e_p(t_n,\sigma(\tau))|\left(|a(\tau)-p(\tau)|\,|x(t)| + |f(\tau,x(\vartheta(\tau)))|\right)\Delta\tau \\ &\geq \gamma|e_p(t_n,t_{\tilde{n}})| - \frac{\gamma S}{1-\alpha}\int_{t_{\tilde{n}}}^{t_n} |e_p(t_n,\sigma(\tau))|\left(|a(\tau)-p(\tau)|+g(t)\right)\Delta\tau \\ &= \gamma|e_p(t_n,t_{\tilde{n}})| - \frac{\gamma S}{1-\alpha}|e_p(t_n,t_{\tilde{n}})|\,|e_p(t_{\tilde{n}},t_0)|\int_{t_{\tilde{n}}}^{t_n} |e_p(t_0,\sigma(\tau))|\left(|a(\tau)-p(\tau)|+g(t)\right)\Delta\tau \\ &\geq |e_p(t_n,t_{\tilde{n}})|\left[\gamma - \frac{\gamma S}{1-\alpha}S\frac{1-\alpha}{2S^2}\right] = \frac{\gamma}{2}|e_p(t_n,t_{\tilde{n}})|, \end{aligned}$$

where we utilized (3.16) together with (3.11) and (3.15). By using (3.13), we deduce that

$$|x(t_n)| \geq \frac{\gamma}{2}|e_p(t_n,t_{\tilde{n}})| = \frac{\gamma}{2}|e_p(t_n,t_0)|\,|e_p(t_0,t_{\tilde{n}})| \to \frac{\gamma}{2} \text{ as } n\to\infty$$

which contradicts with the asymptotic stability of the solution. This finalizes the proof. □

Remark 3.8 *The condition (3.10) of Theorem 3.7 may look as a very restrictive condition for the regressive function* p*. However, the limit result*

$$e_p(t, t_0) \to 0 \text{ as } t \to \infty$$

holds for any negative-valued function p *satisfying* $|p(t)| \leq \omega$ *for all* $t \in \mathbb{T}$*, where* $\omega > 0$ *and* $-\omega \in \mathcal{R}^+$*. We refer to [1, Example 1] for the proof of this fact.*

3.2. Periodicity

We indicate the property

$$f(t \pm T) = f(t) \text{ for a fixed } T > 0 \text{ and for all } t \in \mathbb{T}$$

as the conventional periodicity notion for a function defined on a time scale. Notice that, in order to define a periodic function on a time scale $\mathbb{T}$, one has to ensure that $\mathbb{T}$ is translation invariant (additively periodic) that is there exists a $P > 0$ such that $t \pm P \in \mathbb{T}$ for all $t \in \mathbb{T}$ (see [19]). We shall emphasize that addition is not the only way to step forward and backward on a time scale. Additive periodicity condition is a very restrictive condition for time scales since it rules out several time scales involving $q^{\mathbb{Z}} \cup \{0\}$ on which q-difference equations are constructed. Thus, we may use shift operators $\delta_\pm$ to define a backward and forward motion on time scales, and this approach sparks off studying periodicity notion for time scales which are not necessarily additively periodic.

Before obtaining sufficient conditions for the existence of a periodic solution in shifts $\delta_\pm$ for delay dynamic equations on time scales, we provide the following introductory part for the readership.

3.2.1. New periodicity concept on time scales

In this section, we introduce basics of the shift operators and the new periodicity concept on time scales under the guidance of [2] and [3]. The definitions, and results given in this part of the paper can be found in [3].

Definition 3.9 *Let* $\mathbb{T}^*$ *be a nonempty subset of the time scale* $\mathbb{T}$ *including a fixed number* $t_0 \in \mathbb{T}^*$ *such that there exists operators* $\delta_\pm : [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ *satisfying the following properties:*

1. The function $\delta_\pm$ are strictly increasing with respect to their second arguments, if

   $$(T_0, t), (T_0, u) \in \mathcal{D}_\pm := \{(s, t) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_\pm(s, t) \in \mathbb{T}^*\},$$

   then

   $$T_0 \leq t \leq u \text{ implies } \delta_\pm(T_0, t) \leq \delta_\pm(T_0, u);$$

2. If $(T_1, u), (T_2, u) \in \mathcal{D}_-$ with $T_1 < T_2$, then $\delta_-(T_1, u) > \delta_-(T_2, u)$ and if $(T_1, u), (T_2, u) \in \mathcal{D}_+$ with $T_1 < T_2$, then $\delta_+(T_1, u) < \delta_+(T_2, u)$;

3. If $t \in [t_0, \infty)_{\mathbb{T}}$, then $(t, t_0) \in \mathcal{D}_+$ and $\delta_+(t, t_0) = t$. Moreover, if $t \in \mathbb{T}^*$, then $(t_0, t) \in \mathcal{D}_+$ and $\delta_+(t_0, t) = t$;

4. If $(s, t) \in \mathcal{D}_\pm$, then $(s, \delta_\pm(s, t)) \in \mathcal{D}_\mp$ and $\delta_\mp(s, \delta_\pm(s, t)) = t$;

5. If $(s, t) \in \mathcal{D}_\pm$ and $(u, \delta_\pm(s, t)) \in \mathcal{D}_\mp$, then $(s, \delta_\mp(u, t)) \in \mathcal{D}_\pm$ and $\delta_\mp(u, \delta_\pm(s, t)) = \delta_\pm(s, \delta_\mp(u, t))$.

Then the operators δ_+ and δ_- are called forward and backward shift operators associated with the initial point t_0 on $\mathbb{T}^*$ and the sets $\mathcal{D}_+$ and $\mathcal{D}_-$ are domain of the operators, respectively.

Example 3.10 *Table 4 shows examples of shift operators* $\delta_\pm(s,t)$ *on particular time scales:*

Table 4.

$\mathbb{T}$	t_0	$\mathbb{T}^*$	$\delta_-(s,t)$	$\delta_+(s,t)$
$\mathbb{R}$	0	$\mathbb{R}$	$t-s$	$t+s$
$\mathbb{Z}$	0	$\mathbb{Z}$	$t-s$	$t+s$
$q^{\mathbb{Z}}\cup\{0\}$	1	$q^{\mathbb{Z}}$	$\frac{t}{s}$	st
$\mathbb{N}^{1/2}$	0	$\mathbb{N}^{1/2}$	$\left(t^2-s^2\right)^{1/2}$	$\left(t^2+s^2\right)^{1/2}$

Lemma 3.11 *Let* $\delta_\pm$ *be the shift operators associated with the initial point* t_0*. Then we have the following:*

1. $\delta_-(t,t)=t_0$ for all $t\in[t_0,\infty)_{\mathbb{T}}$;
2. $\delta_-(t_0,t)=t$ for all $t\in\mathbb{T}^*$;
3. If $(s,t)\in\mathcal{D}_+$, then $\delta_+(s,t)=u$ implies $\delta_-(s,u)=t$ and if $(s,u)\in\mathcal{D}_-$, then $\delta_-(s,u)=t$ implies $\delta_+(s,t)=u$;
4. $\delta_+(t,\delta_-(s,t_0))=\delta_-(s,t)$ for all $(s,t)\in\mathcal{D}_+$ with $t\geq t_0$;
5. $\delta_+(u,t)=\delta_+(t,u)$ for all $(u,t)\in([t_0,\infty)_{\mathbb{T}}\times[t_0,\infty)_{\mathbb{T}})\cap\mathcal{D}_+$;
6. $\delta_+(s,t)\in[t_0,\infty)_{\mathbb{T}}$ for all $(s,t)\in\mathcal{D}_+$ with $t\geq t_0$;
7. $\delta_-(s,t)\in[t_0,\infty)_{\mathbb{T}}$ for all $(s,t)\in([t_0,\infty)_{\mathbb{T}}\times[s,\infty)_{\mathbb{T}})\cap\mathcal{D}_-$;
8. If $\delta_+(s,.)$ is Δ-differentiable in its second variable, then $\delta_+^{\Delta_t}(s,.)>0$;
9. $\delta_+(\delta_-(u,s),\delta_-(s,v))=\delta_-(u,v)$ for all $(s,v)\in([t_0,\infty)_{\mathbb{T}}\times[s,\infty)_{\mathbb{T}})\cap\mathcal{D}_-$ and $(u,s)\in([t_0,\infty)_{\mathbb{T}}\times[u,\infty)_{\mathbb{T}})\cap\mathcal{D}_-$;
10. If $(s,t)\in\mathcal{D}_-$ and $\delta_-(s,t)=t_0$, then $s=t$.

Definition 3.12 (Periodicity in shifts) *Let* $\mathbb{T}$ *be a time scale with the shift operators* $\delta_\pm$ *associated with the initial point* $t_0\in\mathbb{T}^*$*, then* $\mathbb{T}$ *is said to be periodic in shifts* $\delta_\pm$*, if there exists a* $p\in(t_0,\infty)_{\mathbb{T}^*}$ *such that* $(p,t)\in\mathcal{D}_\mp$ *for all* $t\in\mathbb{T}^*$*.* P *is called the period of* $\mathbb{T}$ *if*

$$P=\inf\left\{p\in(t_0,\infty)_{\mathbb{T}^*}:(p,t)\in\mathcal{D}_\mp \textit{ for all } t\in\mathbb{T}^*\right\}>t_0.$$

Observe that an additive periodic time scale must be unbounded. However, unlike additively periodic time scales a time scale, periodic in shifts, may be bounded.

Example 3.13 *The following time scales are not additive periodic but periodic in shifts* $\delta_\pm$*.*

1. $\mathbb{T}_1=\left\{\pm n^2 : n\in\mathbb{Z}\right\},\ \delta_{\pm}(P,t)=\begin{cases}\left(\sqrt{t}\pm\sqrt{P}\right)^2 & if\ t>0\\ \pm P & if\ t=0\\ -\left(\sqrt{-t}\pm\sqrt{P}\right)^2 & if\ t<0\end{cases},\ P=1,\ t_0=0,$

2. $\mathbb{T}_2=\overline{q^{\mathbb{Z}}},\ \delta_{\pm}(P,t)=P^{\pm1}t,\ P=q,\ t_0=1,$

3. $\mathbb{T}_3=\overline{\cup_{n\in\mathbb{Z}}\left[2^{2n},2^{2n+1}\right]},\ \delta_{\pm}(P,t)=P^{\pm1}t,\ P=4,\ t_0=1,$

4. $\mathbb{T}_4=\left\{\frac{q^n}{1+q^n} : q>1\ \textit{is constant and}\ n\in\mathbb{Z}\right\}\cup\{0,1\},$

$$\delta_{\pm}(P,t)=\frac{q^{\left(\frac{\ln\left(\frac{t}{1-t}\right)\pm\ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}}{1+q^{\left(\frac{\ln\left(\frac{t}{1-t}\right)\pm\ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}},\quad P=\frac{q}{1+q}.$$

Notice that the time scale $\mathbb{T}_4$ in Example 3.13 is bounded above and below and

$$\mathbb{T}_4^*=\left\{\frac{q^n}{1+q^n} : q>1 \text{ is constant and } n\in\mathbb{Z}\right\}.$$

Corollary 3.14 *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with the period P. Then we have*

$$\delta_{\pm}(P,\sigma(t))=\sigma(\delta_{\pm}(P,t))\ \textit{for all}\ t\in\mathbb{T}^*. \tag{3.17}$$

Definition 3.15 (Periodic function in shifts $\delta_{\pm}$) *Let $\mathbb{T}$ be a time scale P-periodic in shifts. We say that a real valued function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_{\pm}$ if there exists a $T\in[P,\infty)_{\mathbb{T}^*}$ such that*

$$(T,t)\in\mathcal{D}_{\pm}\ \textit{and}\ f\left(\delta_{\pm}^T(t)\right)=f(t)\ \textit{for all}\ t\in\mathbb{T}^*, \tag{3.18}$$

where $\delta_{\pm}^T(t)=\delta_{\pm}(T,t)$. T is called period of f, if it is the smallest number satisfying (3.18).

Definition 3.16 (Δ-periodic function in shifts $\delta_{\pm}$) *Let $\mathbb{T}$ be a time scale P-periodic in shifts. A real valued function f defined on $\mathbb{T}^*$ is Δ-periodic function in shifts if there exists a $T\in[P,\infty)_{\mathbb{T}^*}$ such that*

$$(T,t)\in\mathcal{D}_{\pm}\ \textit{for all}\ t\in\mathbb{T}^* \tag{3.19}$$

$$\textit{the shifts}\ \delta_{\pm}^T\ \textit{are}\ \Delta\textit{-differentiable with rd-continuous derivatives} \tag{3.20}$$

and

$$f\left(\delta_{\pm}^T(t)\right)\delta_{\pm}^{\Delta T}(t)=f(t) \tag{3.21}$$

for all $t\in\mathbb{T}^*$, where $\delta_{\pm}^T(t)=\delta_{\pm}(T,t)$. The smallest number T satisfying (3.19-3.21) is called period of f.

Example 3.17 *Let $g(t)$ be a T-periodic function in shifts defined on $q^{\mathbb{Z}}$. Then the function*

$$f(t) = \frac{g(t)}{t}$$

is Δ-periodic function in shifts with period T.

Theorem 3.18 *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with period $P \in (t_0, \infty)_{\mathbb{T}^*}$ and f a Δ-periodic function in shifts $\delta_{\pm}$ with period $T \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $f \in C_{rd}(\mathbb{T})$, then*

$$\int_{t_0}^{t} f(s)\Delta s = \int_{\delta_{\pm}^{T}(t_0)}^{\delta_{\pm}^{T}(t)} f(s)\Delta s.$$

As an implementation of Theorem 3.18 (see [3, Theorem 2]), the following result is proved in [11]

Lemma 3.19 ([11]) *Let $\mathbb{T}$ be a time scale P-periodic in shifts and the shift operators $\delta_{\pm}^{T}$ are Δ-differentiable on $t \in \mathbb{T}^*$ where $T \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $p \in \mathcal{R}$ is a Δ-periodic function in shifts $\delta_{\pm}$ with period $T \in [P, \infty)_{\mathbb{T}^*}$. Then*

$$e_p\left(\delta_{\pm}^{T}(t), \delta_{\pm}^{T}(t_0)\right) = e_p(t, t_0)$$

for all $t, t_0 \in \mathbb{T}^*$.

3.2.2. Periodic solution

In this part, we suppose that $\mathbb{T}$ is a P-periodic time scale in shifts $\delta_{\pm}$, which is not necessarily unbounded from above. We concentrate on the following delay dynamic equation

$$x^{\Delta}(t) = a(t)x(t) + f(t, x(\delta_{-}(s,t))), \quad t \in \mathbb{T}, \tag{3.22}$$

where a and f are rd-continuous functions. Clearly, the equation (3.22) is derived from (3.1) by setting $\vartheta(t) = \delta_{-}(s,t)$. The use of backward shift operator as the delay enables us to pursue our analysis without assuming that the delay function is periodic in shifts $\delta_{\pm}$. We refer to [2, 4, 11] for a similar approach. Since we focus on the existence of a periodic solution of (3.22), it is natural to make the following assumptions

C4 a is Δ-periodic in shifts, i.e. $a\left(\delta_{\pm}^{T}(t)\right)\delta_{\pm}^{\Delta T}(t) = a(t)$ for all $t \in \mathbb{T}^*$,

C5 f is Δ-periodic in shifts on its first argument, that is $f\left(\delta_{\pm}^{T}(t), .\right)\delta_{\pm}^{\Delta T}(t) = f(t, .)$ for all $t \in \mathbb{T}^*$.

We shall point out that we utilize the backward shift operator δ_{-} as a delay term in (3.22) by setting $\vartheta(t) = \delta_{-}(s,t)$, where ϑ is the delay function used in (3.1). Consequently, we pursue our analysis without an additional periodicity assumption, namely periodicity of the delay function.

Next, we define the set $\mathbb{M}_T$ as the set of functions which are T-periodic in shifts $\delta_{\pm}$. Then, $\mathbb{M}_T$ is a Banach space when it is endowed by the norm

$$\|x\| = \max_{t \in \left[t_0, \delta_{+}^{T}(t_0)\right]} |x(t)|.$$

In the sequel, we give the following lemma which is crucial for proving the existence of a periodic solution of (3.22).

Lemma 3.20 *Assume* ***C4*** *and* ***C5****. If* $x \in \mathbb{M}_T$*, then* $x(t)$ *is a solution of (3.22) if and only if*

$$x(t) = \int_t^{\delta_+^T(t)} \frac{e_{\widehat{p}}(t, \sigma(\tau))}{e_{\ominus\widehat{p}}\left(\delta_+^T(t), t\right) - 1} \left([a(\tau) - \widehat{p}(\tau)]\, x(\tau) + f(\tau, x(\delta_-(s, \tau)))\right) \Delta\tau,$$

where $\widehat{p} \in \mathcal{R}$ *and* $\widehat{p}$ *is* Δ*-periodic in shifts with period* T*.*

Proof Let $x \in \mathbb{M}_T$ be a solution of (3.22). The necessity part of the proof is straightforward. For the sufficiency part, we multiply both sides of (3.22) with $e_{\ominus\widehat{p}}(t, t_0)$ and get

$$[x(t)\, e_{\ominus\widehat{p}}(t, t_0)]^{\Delta} = \frac{e_{\ominus\widehat{p}}(t, t_0)}{1 + \mu(t)\,\widehat{p}(t)} \left([a(t) - \widehat{p}(t)]\, x(t) + f(t, x(\delta_-(s, t)))\right).$$

Integrating the above equality from t to $\delta_+^T(t)$, we arrive at

$$x(t)\left[e_{\widehat{p}}\left(t_0, \delta_+^T(t)\right) - e_{\widehat{p}}(t_0, t)\right] = \int_t^{\delta_+^T(t)} e_{\widehat{p}}(t_0, \sigma(\tau)) \left([a(\tau) - \widehat{p}(\tau)]\, x(\tau) + f(\tau, x(\delta_-(s, \tau)))\right) \Delta\tau. \tag{3.23}$$

We reach the desired equality

$$x(t) = \int_t^{\delta_+^T(t)} \frac{e_{\widehat{p}}(t, \sigma(\tau))}{e_{\ominus\widehat{p}}\left(\delta_+^T(t), t\right) - 1} \left([a(\tau) - \widehat{p}(\tau)]\, x(\tau) + f(\tau, x(\delta_-(s, \tau)))\right) \Delta\tau,$$

by multiplying both sides of (3.23) with $e_{\widehat{p}}(t, t_0)$. □

In preparation for the next result, we introduce the mapping $\widehat{H}$ by

$$\left(\widehat{H}\varphi\right)(t) = \int_t^{\delta_+^T(t)} \frac{e_{\widehat{p}}(t, \sigma(\tau))}{e_{\ominus\widehat{p}}\left(\delta_+^T(t), t\right) - 1} \left([a(\tau) - \widehat{p}(\tau)]\, \varphi(\tau) + f(\tau, \varphi(\delta_-(s, \tau)))\right) \Delta\tau. \tag{3.24}$$

Lemma 3.21 *Let* $\widehat{H}$ *is as in (3.24). If* ***C4*** *and* ***C5*** *hold, then* $\widehat{H} : \mathbb{M}_T \to \mathbb{M}_T$*.*

Proof Suppose that $\varphi \in \mathbb{M}_T$, and the conditions **C4** and **C5** are satisfied. Then, we consider

$$\left(\widehat{H}\varphi\right)\left(\delta_{\pm}^{T}(t)\right) = \int_{\delta_{\pm}^{T}(t)}^{\delta_{+}^{T}\left(\delta_{\pm}^{T}(t)\right)} \frac{e_{\widehat{p}}\left(\delta_{+}^{T}(t), \sigma(\tau)\right)}{e_{\ominus\widehat{p}}\left(\delta_{+}^{T}\left(\delta_{+}^{T}(t)\right), \delta_{+}^{T}(t)\right) - 1}\left(\left[a(\tau) - \widehat{p}(\tau)\right]\varphi(\tau)\right.$$

$$\left. + f\left(\tau, \varphi\left(\delta_{-}(s,\tau)\right)\right)\right)\Delta\tau$$

$$= \int_{t}^{\delta_{+}^{T}(t)} \frac{e_{\widehat{p}}\left(\delta_{+}^{T}\left(\delta_{\pm}^{T}(t)\right), \sigma\left(\delta_{\pm}^{T}(\tau)\right)\right)}{e_{\ominus\widehat{p}}\left(\delta_{+}^{T}\left(\delta_{\pm}^{T}(t)\right), \delta_{\pm}^{T}(t)\right) - 1}\left(\left[a\left(\delta_{\pm}^{T}(\tau)\right) - \widehat{p}\left(\delta_{\pm}^{T}(\tau)\right)\right]\varphi\left(\delta_{\pm}^{T}(\tau)\right)\right.$$

$$\left. + f\left(\delta_{\pm}^{T}(\tau), \varphi\left(\delta_{-}\left(s, \delta_{\pm}^{T}(\tau)\right)\right)\right)\right)\delta_{\pm}^{\Delta T}(\tau)\,\Delta\tau$$

$$= \int_{t}^{\delta_{+}^{T}(t)} \frac{e_{\widehat{p}}\left(t, \sigma(\tau)\right)}{e_{\ominus\widehat{p}}\left(\delta_{+}^{T}(t), t\right) - 1}\left(\left[a(\tau) - \widehat{p}(\tau)\right]\varphi(\tau) + f\left(\tau, \varphi\left(\delta_{-}(s,\tau)\right)\right)\right)\Delta\tau,$$

where we used Lemma 3.19, Corollary 3.14 and Definition 3.9. This proves the assertion. □

We omit the proof of the next result since it is trivial.

Lemma 3.22 *Assume **C2** and also suppose that the condition*

C6 $$\int_{t}^{\delta_{+}^{T}(t)} \left|\frac{e_{\widehat{p}}(t,\sigma(\tau))}{e_{\ominus\widehat{p}}\left(\delta_{+}^{T}(t),t\right)-1}\right|\left(\left|a(\tau) - p(\tau)\right| + g(\tau)\right)\Delta\tau \le \beta < 1 \text{ for all } t \in \mathbb{T}^{*}$$

holds. Then the mapping $\widehat{H}$ given in (3.24) is a contraction.

Due to Lemma 3.21 and Lemma 3.22, we present the following theorem which provides the sufficient conditions for the existence of a unique periodic solution in shifts $\delta_{\pm}$ for the delay dynamic equation (3.22) by contraction mapping principle.

Theorem 3.23 *Suppose that the conditions **C2**, **C4**-**C6** are satisfied. Then, the delay dynamic equation (3.22) has a unique periodic solution in shifts $\delta_{\pm}$.*

4. Applications

We present the following examples on a large class of time scales to illustrate the application potential of our results on several mathematical models.

Example 4.1 *Consider the following delay dynamic equation*

$$x^{\Delta}(t) = -a(t)\,x(t) + c(t)\tanh\left(x\left(t - \tau(t)\right)\right) + I(t),\ \ t \in \mathbb{T}, \tag{4.1}$$

which is utilized to model the dynamics of a single artificial effective neuron with time varying processing delay on time scales. If we set $\mathbb{T} = \mathbb{R}$, then the equation turns to the original model

$$x'(t) = -a(t)\,x(t) + c(t)\tanh\left(x\left(t - \tau(t)\right)\right) + I(t),\ \ t \in \mathbb{R} \tag{4.2}$$

given in [15]. In this example, we concentrate on (4.2) in order to highlight controllability of our conditions. If we compare (4.2) with (3.1), then we observe that

$$f\left(t, x\left(\vartheta\left(t\right)\right)\right) = c\left(t\right) \tanh\left(x\left(t - \tau\left(t\right)\right)\right) + I\left(t\right),$$

*where c and I are continuous and bounded. As the first task, one may easily verify the condition **C2**. For any $x, y \in B_k$ we get*

$$\begin{aligned} \left|f\left(t, x\right) - f\left(t, y\right)\right| &= \left|c\left(t\right) \tanh x - c\left(t\right) \tanh y\right| \\ &= \left|c\left(t\right)\right| \left|\tanh x - \tanh y\right| \\ &\leq \left|c\left(t\right)\right| \left\|x - y\right\|. \end{aligned}$$

*On the other hand, we amend the condition **C1** for the delay differential equation (4.2) as follows:*

C1* $\displaystyle\int_0^t e^{-\int_u^t p(s)ds} \left(\left|p\left(u\right) - a\left(u\right)\right| + \left|c\left(u\right)\right|\right) du \leq \alpha_1^* < 1.$

Now, we set $p\left(t\right) = \frac{1}{1+t^2}$, and consequentially $e_p\left(t, 0\right) = e^{\int_0^t p(s)ds} = e^{\arctan t}$ is bounded. Next, we assume that the functions a and c satisfy the following inequality

$$\left|a\left(t\right)\right| + \left|c\left(t\right)\right| \leq \frac{1}{10} \frac{1}{1+t^2} \text{ for all } t \in [0, \infty).$$

*Then, we verify the condition **C1*** as follows*

$$\begin{aligned} \int_0^t e^{-\int_u^t \frac{1}{1+s^2} ds} \left(\left| \frac{1}{1+u^2} - a\left(u\right) \right| + \left|c\left(u\right)\right| \right) du &\leq \frac{11}{10} \int_0^t e^{-\int_u^t \frac{1}{1+s^2} ds} \frac{1}{1+u^2} du \\ &= \frac{11}{10} \left(e^{-\int_u^t \frac{1}{1+s^2} ds} \Big|_{u=0}^{u=t} \right) \\ &= \frac{11}{10} \left(1 - e^{\arctan t}\right) \leq \frac{11}{10} \left(1 - e^{-\pi/2}\right) = \alpha_1^*. \end{aligned}$$

It is obvious that $x = 0$ is not a solution of the equation (4.2) since the stimulus input I cannot be zero; that is $f\left(t, 0\right) = I\left(t\right)$. Thus, we deduce that the delay differential equation (4.2) has an equi-bounded solution $x \in B_k$ due to the Theorem 3.3, for any k satisfying the inequality

$$le^{\pi/2} + \alpha_1^* k + \alpha_2^* \leq k,$$

where $\left|\xi\left(t\right)\right| \leq l$ for all $t \in \left[\tau\left(0\right), 0\right]$, and

$$\int_0^t e^{-\int_u^t \frac{1}{1+s^2} ds} I\left(u\right) du \leq \alpha_2^*.$$

Example 4.2 *We focus on the following delay dynamic equation*

$$x^{\Delta}(t) = \lambda x(t) + F(x(\vartheta(t))), \; t \in \mathbb{T}, \tag{4.3}$$

which might be used to model dynamics of a population. If we set $\mathbb{T} = \mathbb{N}_0$, *and assume* $\vartheta(t) = t - \tau$, *then (4.3) is equal to the Clark's equation (see [10])*

$$\Delta x_n = (\gamma - 1) x_n + F(x_{n-\tau}), \tag{4.4}$$

where $0 \leq \gamma \leq 1$ *and* $F : [0, \infty) \to [0, \infty)$ *is continuous. In (4.4), it is assumed that a proportion* γx_n *of adults survives the next generation and the new borns admit a constant maturation delay. We suppose that* $F(0) = 0$, *and* F *is a Lipschitz function with constant* ϖ *so that the conditions **C2** and **C3** hold. Moreover, we set* $p_n = -\frac{1}{2+n}$ *for* $n = 0, 1, \ldots$ *which satisfies the condition (3.10) due to the Remark 3.8. If we adapt the condition **C1** for the equation (4.4), then we write its discrete counterpart as*

$$\sum_{j=0}^{n-1} \left(\prod_{r=j+1}^{n-1} \left(1 - \frac{1}{2+r} \right) \right) \left(\left| \gamma - 1 + \frac{1}{2+j} \right| + \varpi \right) \leq \alpha < 1. \tag{4.5}$$

Therefore, we observe that the zero solution of Clark's equation given in (4.4) is asymptotically stable as a result of the Theorem 3.7 whenever (4.5) holds.

Example 4.3 *Let* $\mathbb{T} = 2^{\mathbb{Z}} \cup \{0\}$ *with the shift operators* $\delta_{\pm}(s, t) = s^{\pm 1} t$, $t \in \mathbb{T}^* = 2^{\mathbb{Z}}$, *and consider the following* q*-difference equation*

$$D_q x(t) = \frac{-1 + \sin\left(\frac{\ln t}{\ln 2}\pi\right)}{4t} x(t) + \frac{1}{32t} x(\delta_{-}(s, t)), \; t \in 2^{\mathbb{Z}}. \tag{4.6}$$

If we compare (4.6) with (3.22), then we get

$$a(t) = \frac{-1 + \sin\left(\frac{\ln t}{\ln 2}\pi\right)}{4t} \text{ and } f(t, x) = \frac{1}{32t} x.$$

It is trivial to show that a *is* Δ 4*-periodic in shifts, and also* $f(\delta_{\pm}(4, t), x)\, \delta_{\pm}^{\Delta}(4, t) = f(t, x)$. *Therefore, the conditions **C4** and **C5** hold. Besides, the condition **C2** can be verified with* $g(t) = \frac{1}{32t}$. *We shall recall that, when* $\mathbb{T} = 2^{\mathbb{Z}} \cup \{0\}$, *we have the following representations*

$$e_p(2^m, 2^{m_0}) = \prod_{k \in [m_0, m)} \left[1 + 2^k p\left(2^k\right) \right], \tag{4.7}$$

and

$$\int_{2^m}^{2^n} p(s)\, d_2 s = \sum_{k=m}^{n-1} 2^k p\left(2^k\right) \tag{4.8}$$

for exponential function and definite integral, respectively. Next, we set $\widehat{p}(t) = -\frac{7}{8t}$ *which is* Δ 4*-periodic in shifts, and it remains to prove that the condition **C6** is satisfied in order to employ Theorem 3.23. In our case,*

C6 *can be rewritten as follows*

$$\sum_{s=n}^{n+1} 2^s \left| \frac{e_{\widehat{p}}\left(2^n, 2^{s+1}\right)}{\left(e_{\widehat{p}}\left(2^{n+2}, 2^n\right)\right)^{-1} - 1} \right| \left(\left| \frac{-1+\sin(s\pi)}{2^{s+2}} + \frac{7}{2^{s+3}} \right| + \frac{1}{2^{s+5}} \right) \leq \beta < 1 \text{ for all } t \in 2^{\mathbb{Z}} \tag{4.9}$$

due to (4.8). We obtain $e_{\widehat{p}}\left(2^n, 2^{s+1}\right) = 8^{-n+s+1}$ *and* $\left(e_{\widehat{p}}\left(2^{n+2}, 2^n\right)\right)^{-1} - 1 = 63$ *by utilizing (4.7). Thus, (4.9) turns to*

$$\begin{aligned} \frac{8^{-n+1}}{63} \sum_{s=n}^{n+1} 2^s 8^s \left(\left| \frac{-1+\sin(s\pi)}{2^{s+2}} + \frac{7}{2^{s+3}} \right| + \frac{1}{2^{s+5}} \right) &\leq \frac{8^{-n+1}}{63} \sum_{s=n}^{n+1} 2^s 8^s \left(\frac{7}{2^{s+3}} + \frac{1}{2^{s+5}} \right) \\ &= \frac{8^{-n+1}}{63} \frac{15}{32} \sum_{s=n}^{n+1} 8^s < 1 \end{aligned}$$

and it is satisfied. Consequentially, all conditions of Theorem 3.23 hold, and the delayed q*-difference equation (4.6) has a unique* 4*-periodic solution in shifts.*

5. Conclusion

This manuscript handles the delay dynamic equation (3.1) and provides an elaborative analysis regarding boundedness, stability, and periodicity of its solutions. The function $\vartheta : [t_0, \infty)_{\mathbb{T}} \to [\vartheta(t_0), \infty)_{\mathbb{T}}$ is used as an abstract delay function, and ϑ is specifically chosen as backward shift operator, i.e. $\vartheta(t) = \delta_-(s,t)$, in the part on which the existence of periodic solutions is studied. By appealing fixed point theory, some conditions are obtained which guarantees the existence of an equi-bounded solution and the asymptotic stability of zero solution for (3.1). Additionally, the new periodicity concept on time scales is utilized to discuss the existence of periodic solutions in shifts for the delay dynamic equation (3.22). In our analysis, we proposed a particular variation of parameters formula by using an auxiliary function to invert a mapping for the utilization of contraction mapping principle. This method does not only relax some conditions on the equation but also provides an alternative approach for the qualitative analysis of dynamic equation on time scales. Although our results coincide with some existing results in particular cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, this study is not only a unification but also a significant extension for the existing literature since it allows to construct delay dynamic equation (3.1) on more general domains, such as $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ and $\mathbb{T} = \mathbb{P}_{a,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$, which are also important in modelling real life processes.

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