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Bilinear multipliers of small Lebesgue spaces

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Abstract: Let G be a compact abelian metric group with Haar measure λ and $\hat{G}$ its dual with Haar measure μ. Assume that $1 < p_i < \infty$, $p_i' = \frac{p_i}{p_i - 1}$, $(i = 1, 2, 3)$ and $\theta \geq 0$. Let $L^{(p_i', \theta}(G)$, $(i = 1, 2, 3)$ be small Lebesgue spaces. A bounded sequence $m(\xi, \eta)$ defined on $\hat{G} \times \hat{G}$ is said to be a bilinear multiplier on G of type $[(p_1'; (p_2'; (p_3']_\theta$ if the bilinear operator B_m associated with the symbol m

$$B_m(f, g)(x) = \sum_{s \in \hat{G}} \sum_{t \in \hat{G}} \hat{f}(s) \hat{g}(t) m(s, t) \langle s + t, x \rangle$$

defines a bounded bilinear operator from $L^{(p_1', \theta}(G) \times L^{(p_2', \theta}(G)$ into $L^{(p_3', \theta}(G)$. We denote by $BM_\theta[(p_1'; (p_2'; (p_3']$ the space of all bilinear multipliers of type $[(p_1'; (p_2'; (p_3']_\theta$. In this paper, we discuss some basic properties of the space $BM_\theta[(p_1'; (p_2'; (p_3']$ and give examples of bilinear multipliers.

Key words: Bilinear multipliers, grand Lebesgue spaces, small Lebesgue spaces

1. Introduction

Let Ω be locally compact Hausdorff space and let (Ω, B, μ) be finite Borel measure space. The grand Lebesgue space $L^{p)}(\Omega), (1 < p < \infty)$ is defined by the norm

$$\|f\|_{p)} = \sup_{0 < \varepsilon \leq p-1} \left(\varepsilon \fint_\Omega |f|^{p-\varepsilon} d\mu \right)^{\frac{1}{p-\varepsilon}},$$

where by $\fint_\Omega$ we denote $\frac{1}{\mu(\Omega)} \int_\Omega$. (see [10]). A generalization of the grand Lebesgue spaces are the spaces $L^{p),\theta}(\Omega)$, $\theta \geq 0$, defined by the norm (see [3])

$$\|f\|_{p),\theta} = \sup_{0 < \varepsilon \leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\fint_\Omega |f|^{p-\varepsilon} d\mu \right)^{\frac{1}{p-\varepsilon}} = \sup_{0 < \varepsilon \leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon};$$

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when $\theta = 0$ the space $L^{p),0}(\Omega)$ reduces to Lebesgue space $L^{p}(\Omega)$ and when $\theta = 1$ the space $L^{p),1}(\Omega)$ reduces to grand Lebesgue space $L^{p)}(\Omega)$, (see [6]). For $0 < \varepsilon \leq p-1$,

$$L^{p}(\Omega) \subset L^{p),\theta}(\Omega) \subset L^{p-\varepsilon}(\Omega)$$

hold. It is known that the subspace $C_c^{\infty}(\Omega)$ is not dense in $L^{p),\theta}(\Omega)$, where $C_c^{\infty}(\Omega)$ is the space of infinitely differentiable complex valued functions defined on Ω with compact support. Its closure consists of functions $f \in L^{p}(\Omega)$ such that (see [6])

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon} = 0.$$

For some properties and applications of $L^{p)}(\Omega)$, we refer to [4, 5, 7–9].

Let $p' = \frac{p}{p-1}$, $1 < p < \infty$. First, consider an auxiliary space namely $L^{(p',\theta}(\Omega)$, $\theta \geq 0$, defined by

$$\|g\|_{(p',\theta} = \inf_{g=\sum_{k=1}^{\infty} g_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p-\varepsilon}} \left(\fint_{\Omega} |g_k(x)|^{(p-\varepsilon)'} d(x) \right)^{\frac{1}{(p-\varepsilon)'}} \right\},$$

where the functions g_k, $k \in \mathbb{N}$, being in M_0, the set of all real valued measurable functions, finite a.e. in Ω. After this definition, the generalized small Lebesgue spaces have been defined by

$$L^{p)',\theta}(\Omega) = \left\{ g \in M_0 \colon \ \|g\|_{p)',\theta} < \infty \right\},$$

where

$$\|g\|_{p)',\theta} = \sup_{\substack{0 \leq \psi \leq |g| \\ \psi \in L^{(p',\theta}(\Omega)}} \|\psi\|_{(p',\theta}.$$

For $\theta = 0$, it is $\|f\|_{(p',0} = \|f\|_{p'}$, (see [1, 3, 5].

Let G be a locally compact abelian metric group with Haar measure λ and let $\hat{G}$ be dual group with Haar measure μ. The translation and modulation operators are given by

$$T_x f(t) = f(t-x), \ M_{\xi} f(t) = \langle t, \xi \rangle f(t), \ t, x \in G, \ \xi \in \hat{G}. \tag{1.1}$$

For a function $f \in L^{1}(G)$, the function $\hat{f}$ defined on $\hat{G}$ by

$$\hat{f}(\gamma) = \int_{G} f(x) \langle \gamma, -x \rangle d\lambda(x), \ \gamma \in \hat{G} \tag{1.2}$$

is called the Fourier transform of f, (see [13]). The behaviors of the translation and modulation operators under the Fourier transform are

$$(M_{-s_0} f)\,\hat{} = T_{-s_0} \hat{f}, \quad (T_{-t_0} f)\,\hat{} = M_{t_0} \hat{f}, \tag{1.3}$$

where $s_0 \in \hat{G}$, $t_0 \in G$, (see [13]).

2. Main results

Let G be a locally compact abelian metric group and $\hat{G}$ its dual with Haar measures λ and μ, respectively. Before giving the definition of bilinear multiplier on G of type $[(p_1'; (p_2'; (p_3']_\theta$, we remember that if $\lambda(G)$ is finite, then G is compact. Thus, the $\hat{G}$ dual of G (Pontryagin dual) is a discrete group, and the dual measure on this group is the counting measure. Also since G is compact abelian metric group, then $\hat{G}$ is countable (see [13]).

Definition 2.1 *Let G be a compact abelian metric group with Haar measure λ and $\hat{G}$ its dual with Haar measure μ. Assume that $1 < p_i < \infty$, $p_i' = \frac{p_i}{p_i - 1}$, $(i = 1, 2, 3)$ and $\theta \geq 0$. We also assume that $m(s, t)$ is a bounded sequence on $\hat{G} \times \hat{G}$. Consider the bilinear operator B_m associated with the symbol m*

$$B_m(f, g)(x) = \sum_{s \in \hat{G}} \sum_{t \in \hat{G}} \hat{f}(s)\,\hat{g}(t)\,m(s, t)\,\langle s + t, x\rangle,$$

defined for functions $f, g \in C^\infty(G)$. m is said to be a bilinear multiplier on G of type $[(p_1'; (p_2'; (p_3']_\theta$, if there exists $C > 0$ such that

$$\|B_m(f, g)\|_{(p_3', \theta} \leq C \|f\|_{(p_1', \theta} \|g\|_{(p_2', \theta} \tag{2.1}$$

for all $f, g \in C^\infty(G)$. That means B_m extends to a bounded bilinear operator from $L^{(p_1', \theta}(G) \times L^{(p_2', \theta}(G)$ into $L^{(p_3', \theta}(G)$. We denote by $BM_\theta[(p_1'; (p_2'; (p_3']$ the space of all bilinear multipliers of type $[(p_1'; (p_2'; (p_3']_\theta$ and

$$\|m\|_{[(p_1'; (p_2'; (p_3']_\theta} = \|B_m\|. \tag{2.2}$$

Lemma 2.2 *(Hölder-type inequality for generalized small Lebesgue spaces)*

Let $\frac{1}{p'} = \frac{1}{p_1'} + \frac{1}{p_2'}$ and $r.r' < p' + 1$. If $f \in L^{(p_1', \theta}(G)$ and $g \in L^{(p_2', \theta}(G)$, then $fg \in L^{(r', \theta}(G)$. Furthermore,

$$\|fg\|_{(r', \theta} \leq C \|f\|_{(p_1', \theta} \|g\|_{(p_2', \theta}$$

for some $C > 0$.

Proof Take any $f \in C^\infty(G) \subset L^{(p_1', \theta}(G)$ and $g \in C^\infty(G) \subset L^{(p_2', \theta}(G)$. Let $\frac{1}{p'} = \frac{1}{p_1'} + \frac{1}{p_2'}$ and $r.r' < p' + 1$. Since $1 = \frac{1}{r} + \frac{1}{r'}$, we have $r + r' = r.r'$. Then using the assumption $r.r' < p' + 1$, we write $r + r' < p' + 1$ and so $r + r' - 1 < p'$. For a fixed $0 < \varepsilon \leq r - 1$, we have $r' + \varepsilon \leq r' + r - 1 < p'$. Therefore, since $\mu(G) < \infty$, we obtain $L^{p'}(G) \subset L^{r' + \varepsilon}(G)$. Moreover, we know that $L^{r' + \varepsilon}(G) \subset L^{(r', \theta}(G)$, (see [3]). Then we have the inclusion $L^{p'}(G) \subset L^{r' + \varepsilon}(G) \subset L^{(r', \theta}(G)$. That means, there exists $C_1 > 0$ such that

$$\|fg\|_{(r', \theta} \leq C_1 \|fg\|_{p'}. \tag{2.3}$$

If we apply the Hölder inequality to the right side of (2.3), there exists $C_2 > 0$ such that

$$\|fg\|_{p'} \leq C_2 \|f\|_{p_1'} \|g\|_{p_2'}. \tag{2.4}$$

On the other hand, since $L^{(p_1',\theta}(G) \subset L^{p_1'}(G)$ and $L^{(p_2',\theta}(G) \subset L^{p_2'}(G)$, (see [3]), we have

$$\|f\|_{p_1'} \leq C_3 \|f\|_{(p_1',\theta} \tag{2.5}$$

and

$$\|g\|_{p_2'} \leq C_4 \|g\|_{(p_2',\theta} \tag{2.6}$$

for some C_3, $C_4 > 0$. Combining the inequalities (2.3)–(2.6), we obtain

$$\|fg\|_{(r',\theta} \leq C \|f\|_{(p_1',\theta} \|g\|_{(p_2',\theta}, \tag{2.7}$$

where $C = C_1C_2C_3C_4$. Now define the bilinear mapping $F((f,g)) = fg$, from $(C^\infty \times C^\infty)(G)$ to $L^{(r',\theta}(G)$. By (2.7), it is continuous. Since $(C^\infty \times C^\infty)(G)$ is dense in $L^{(p_1',\theta}(G) \times L^{(p_2',\theta}(G)$, then there exists a unique continuous bilinear extension of F denoted $F^\sim$ from $L^{(p_1',\theta}(G) \times L^{(p_2',\theta}(G)$ to $L^{(r',\theta}(G)$. Furthermore, the norm of $F^\sim$ is equal to the norm of F. Therefore, for all $f \in L^{(p_1',\theta}(G)$ and $g \in L^{(p_2',\theta}(G)$, the inequality

$$\|fg\|_{(r',\theta} = \|F((f,g))\|_{(r',\theta} \leq C \|f\|_{(p_1',\theta} \|g\|_{(p_2',\theta}$$

is achieved. □

Example 2.3 *Let $f \in L^{(9,\theta}(G)$ and $g \in L^{(10,\theta}(G)$. Since $p_1' = 9$ and $p_2' = 10$, then $\frac{1}{p'} = \frac{1}{p_1'} + \frac{1}{p_2'} = \frac{1}{9} + \frac{1}{10}$, and so $p' = \frac{90}{19}$. Also, let $r = 3$ and $r' = \frac{3}{2}$. Then $\frac{1}{r} + \frac{1}{r'} = 1$. Hence, we have $r.r' = 3.\frac{3}{2} = \frac{9}{2} < \frac{90}{19} + 1 = p' + 1$. Therefore, from the Lemma 2.2, we obtain that $fg \in L^{(\frac{3}{2},\theta}(G)$ and*

$$\|fg\|_{(\frac{3}{2},\theta} \leq C \|f\|_{(9,\theta} \|g\|_{(10,\theta}$$

for some $C > 0$.

Theorem 2.4 *Let $1 < p_i < \infty$, $p_i' = \frac{p_i}{p_i - 1}$, $(i = 1,2,3)$ and $\theta > 0$. Then $m \in BM_\theta[(p_1'; (p_2'; (p_3']$ if and only if there exists $C > 0$ such that*

$$\left|\sum_{s\in\hat{G}}\sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,\hat{h}(s+t)\,m(s,t)\right| \leq C \|f\|_{(p_1',\theta} \|g\|_{(p_2',\theta} \|h\|_{p_3),\theta}$$

for all $f \in L^{(p_1',\theta}(G)$, $g \in L^{(p_2',\theta}(G)$ and $h \in [L^{p_3}]_{p_3),\theta}$, where $[L^{p_3}]_{p_3),\theta}$ is the closure of $C^\infty(G)$ in $L^{p_3),\theta}(G)$.

Proof Assume that $m \in BM_\theta[(p_1'; (p_2'; (p_3']$. Let f, $g \in C^\infty(G) \subset L^1(G)$ and $h \in C^\infty(G) \cap L^{p_3),\theta}(G)$. Since $\mu(G) < \infty$, we have $h \in L^1(G)$. Thus, we write

$$\left|\sum_{s\in\hat{G}}\sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,\hat{h}(s+t)\,m(s,t)\right| =$$

$$= \left| \sum_{s\in\hat{G}} \sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t) \left\{ \int_G h(y) \langle s+t, -y\rangle \, d\lambda(y) \right\} m(s,t) \right|$$

$$= \left| \int_G h(y)\, \tilde{B}_m(f,g)(y)\, d\lambda(y) \right| \le \int_G |h(y)| \left| \tilde{B}_m(f,g)(y) \right| d\lambda(y) \tag{2.8}$$

where λ is Haar measure on G and $\tilde{B}_m(f,g)(y) = B_m(f,g)(-y)$. From the assumption $m \in BM_\theta\left[(p_1'; (p_2'; (p_3'\right]$, we have $B_m(f,g) \in L^{(p_3',\theta}(G)$. Since G is group, we obtain $\tilde{B}_m(f,g) \in L^{(p_3',\theta}(G)$. By using the Hölder inequality for generalized small Lebesgue spaces (see [3]) and the inequality (2.8), we write

$$\left| \sum_{s\in\hat{G}} \sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,\hat{h}(s+t)\, m(s,t) \right| \le \int_G |h(y)| \left| \tilde{B}_m(f,g)(y) \right| d\lambda(y)$$

$$\le \left\| \tilde{B}_m(f,g) \right\|_{(p_3',\theta} \|h\|_{p_3),\theta} = \|B_m(f,g)\|_{(p_3',\theta} \|h\|_{p_3),\theta}. \tag{2.9}$$

Also since $m \in BM_\theta\left[(p_1'; (p_2'; (p_3'\right]$, there exists $C > 0$ such that

$$\|B_m(f,g)\|_{(p_3',\theta} \le C \|f\|_{(p_1',\theta} \|g\|_{(p_2',\theta}. \tag{2.10}$$

Combining (2.9) and (2.10), we find

$$\left| \sum_{s\in\hat{G}} \sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,\hat{h}(s+t)\, m(s,t) \right| \le C \|f\|_{(p_1',\theta} \|g\|_{(p_2',\theta} \|h\|_{p_3),\theta}.$$

For the proof of converse, assume that there exists a constant $C > 0$ such that

$$\left| \sum_{s\in\hat{G}} \sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,\hat{h}(s+t)\, m(s,t) \right| \le C \|f\|_{(p_1',\theta} \|g\|_{(p_2',\theta} \|h\|_{p_3),\theta}$$

for all f, $g \in C^\infty(G)$ and , $h \in C^\infty(G) \cap L^{p_3),\theta}(G)$. From this inequality and (2.8), we write

$$\left| \int_G h(y)\, \tilde{B}_m(f,g)(y)\, d\lambda(y) \right| \le C \|f\|_{(p_1',\theta} \|g\|_{(p_2',\theta} \|h\|_{p_3),\theta}. \tag{2.11}$$

Define a function ℓ from $C^\infty(G) \cap L^{p_3),\theta}(G)$ to $\mathbb{C}$ such that

$$\ell(h) = \int_G h(y)\, \tilde{B}_m(f,g)(y)\, d\lambda(y).$$

This function ℓ is well defined and linear. Moreover, it is bounded by (2.11). Since $\overline{C^\infty(G) \cap L^{p_3),\theta}(G)} = \left[L^{p_3}\right]_{p_3),\theta}$, ℓ extends to a bounded function from $\left[L^{p_3}\right]_{p_3),\theta}$ to $\mathbb{C}$. Then $\ell \in \left(\left[L^{p_3}\right]_{p_3),\theta}\right)^* = L^{p_3)',\theta}(G) \simeq$

$L^{(p_3{}',\theta}(G)$ and by (2.11), we have

$$\|B_m(f,g)\|_{(p_3',\theta} = \|\ell\| = \sup_{\|h\|_{p_3),\theta}\le 1} \frac{|l(h)|}{\|h\|_{p_3),\theta}}$$

$$\le C\,\|f\|_{(p_1',\theta}\,\|g\|_{(p_2',\theta}\,.$$

Hence, we obtain $m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$. □

Theorem 2.5 *Let* $m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$. *Then*

a) $T_{(s_0,t_0)}m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$ *for each* $(s_0,t_0) \in \hat{G}\times\hat{G}$ *and*

$$\left\|T_{(s_0,t_0)}m\right\|_{\left[(p_1';(p_2';(p_3'\right]_\theta} = \|m\|_{\left[(p_1';(p_2';(p_3'\right]_\theta}\,.$$

b) $M_{t_0}^2 M_{s_0}^1 m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$ *for each* $(s_0,t_0)\in G\times G$ *and*

$$\left\|M_{t_0}^2 M_{s_0}^1 m\right\|_{\left[(p_1';(p_2';(p_3'\right]_\theta} = \|m\|_{\left[(p_1';(p_2';(p_3'\right]_\theta}\,,$$

where $M_{s_0}^1 m(s,t) = \langle s,s_0\rangle\, m(s,t)$ *and* $M_{t_0}^2 m(s,t) = \langle t,t_0\rangle\, m(s,t)$.

Proof **a)** Let $f,\ g\in C^\infty(G)$. Let us say $s-s_0=u$ and $t-t_0=v$. Then

$$\begin{aligned} B_{T_{(s_0,t_0)}m}(f,g)(x) &= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,T_{(s_0,t_0)}m(s,t)\,\langle s+t,x\rangle \\ &= \sum_{u\in\hat{G}}\sum_{v\in\hat{G}} \hat{f}(u+s_0)\,\hat{g}(v+t_0)\,m(u,v)\,\langle (s_0+t_0)+(u+v),x\rangle \\ &= \sum_{u\in\hat{G}}\sum_{v\in\hat{G}} T_{-s_0}\hat{f}(u)\,T_{-t_0}\hat{g}(v)\,m(u,v)\,\langle s_0+t_0,x\rangle\,\langle u+v,x\rangle\,. \end{aligned} \tag{2.12}$$

Then by using (1.3) and (2.12), we have

$$\begin{aligned} B_{T_{(s_0,t_0)}m}(f,g)(x) &= \sum_{u\in\hat{G}}\sum_{v\in\hat{G}} T_{-s_0}\hat{f}(u)\,T_{-t_0}\hat{g}(v)\,m(u,v)\,\langle s_0+t_0,x\rangle\,\langle s+t,x\rangle \\ &= \langle s_0+t_0,x\rangle\, B_m(M_{-s_0}f, M_{-t_0}g)(x)\,. \end{aligned} \tag{2.13}$$

Using the assumption $m\in BM_\theta[(p_1';(p_2';(p_3']$ and the equality (2.13), we have

$$\begin{aligned} \left\|B_{T_{(\xi_0,\eta_0)}m}(f,g)\right\|_{(p_3',\theta} &= \left\|\langle s_0+t_0,x\rangle\, B_m(M_{-s_0}f, M_{-t_0}g)\right\|_{(p_3',\theta} \\ &= \left\|B_m(M_{-s_0}f, M_{-t_0}g)\right\|_{(p_3',\theta} \\ &\le C\,\|M_{-s_0}f\|_{(p_1',\theta}\,\|M_{-t_0}g\|_{(p_2',\theta} \\ &= C\,\|f\|_{(p_1',\theta}\,\|g\|_{(p_2',\theta} \end{aligned}$$

for some $C>0$. Thus, $T_{(s_0,t_0)}m \in BM_\theta[(p_1';(p_2';(p_3']$. Then by (2.2),

$$\left\|T_{(s_0,t_0)}m\right\|_{\left[(p_1';(p_2';(p_3'\right]_\theta} = \left\|B_{T_{(s_0,t_0)}m}\right\|.$$

This implies

$$\left\|T_{(s_0,t_0)}m\right\|_{[(p_1';(p_2';(p_3']_\theta} = \left\|B_{T_{(s_0,t_0)}m}\right\|$$

$$= \sup\left\{\frac{\|B_m(M_{-s_0}f, M_{-t_0}g)\|_{(p_3',\theta}}{\|M_{-s_0}f\|_{(p_1',\theta}\|M_{-t_0}g\|_{(p_2',\theta}} : \|M_{-s_0}f\|_{(p_1',\theta} \leq 1, \ \|M_{-t_0}g\|_{(p_2',\theta} \leq 1\right\}$$

$$= \|B_m\| = \|m\|_{[(p_1';(p_2';(p_3']_\theta}.$$

b) Let $f, g \in C^\infty(G)$. By definition of modulation operators (1.1), we have

$$B_{M_{t_0}^2 M_{s_0}^1 m}(f,g)(x) = \sum_{s\in\hat{G}}\sum_{t\in\hat{G}} \hat{f}(s)\hat{g}(t) M_{t_0}^2 M_{s_0}^1(m(s,t))\langle s+t, x\rangle$$

$$= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}} \langle s, s_0\rangle \hat{f}(s)\langle t, t_0\rangle \hat{g}(t) m(s,t)\langle s+t, x\rangle$$

$$= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}} M_{s_0}\hat{f}(s) M_{t_0}\hat{g}(t) m(s,t)\langle s+t, x\rangle. \tag{2.14}$$

Then by using (1.3) and (2.14)

$$B_{M_{t_0}^2 M_{s_0}^1 m}(f,g)(x) = \sum_{s\in\hat{G}}\sum_{t\in\hat{G}} (T_{-s_0}f)\hat{\ }(s)(T_{-t_0}g)\hat{\ }(t) m(s,t)\langle s+t, x\rangle = B_m(T_{-s_0}f, T_{-t_0}g)(x).$$

Since $m \in BM_\theta[(p_1';(p_2';(p_3']$,

$$\left\|B_{M_{t_0}^2 M_{s_0}^1 m}(f,g)\right\|_{(p_3',\theta} = \|B_m(T_{-s_0}f, T_{-t_0}g)\|_{(p_3',\theta} \leq \|B_m\|\|T_{-s_0}f\|_{(p_1',\theta}\|T_{-t_0}g\|_{(p_2',\theta}$$

$$= \|B_m\|\|f\|_{(p_1',\theta}\|g\|_{(p_2',\theta} \tag{2.15}$$

and so $M_{t_0}^2 M_{s_0}^1 m \in BM_\theta[(p_1';(p_2';(p_3']$. Finally, by (2.15) we achieve that

$$\left\|M_{t_0}^2 M_{s_0}^1 m\right\|_{[(p_1';(p_2';(p_3']_\theta} = \sup\left\{\frac{\|B_m(T_{-s_0}f, T_{-t_0}g)\|_{(p_3',\theta}}{\|T_{-s_0}f\|_{(p_1',\theta}\|T_{-t_0}g\|_{(p_2',\theta}} : \|T_{-s_0}f\|_{(p_1',\theta} \leq 1, \ \|T_{-t_0}g\|_{(p_2',\theta} \leq 1\right\}$$

$$= \|m\|_{[(p_1';(p_2';(p_3']_\theta}.$$

□

Let A be an automorphism of G. The $\lambda \circ A$ is a nontrival Haar measure on G. For any Borel set $U \subseteq G$, the modules of A is defined by $|A| = \lambda(AU)$ such that $d\lambda(Ax) = |A|\,d\lambda(x)$. Moreover, the adjoint $A^*$of A is an automorphism of $\hat{G}$. The adjoint operator A^* is defined by $\langle Ax, s\rangle = \langle x, A^*s\rangle$ for $x \in G$ and $s \in \hat{G}$. Furthermore, the $\mu \circ A^*$ is a nontrival Haar measure on $\hat{G}$ such that $d\mu(A^*s) = |A^*|\,d\mu(s)$. It is known that $|A| = |A^*|$, $(A^*)^{-1} = \left(A^{-1}\right)^*$and $|A|^{-1} = \left|A^{-1}\right|$, (see [2]).

Definition 2.6 *Let A be an automorphism of G. The dilation operator $D_A^{p'}$ on $L^{(p',\theta}(G)$ is defined by*

$$D_A^{p'}f(x) = |A|^{\frac{1}{p'}} f(Ax).$$

Lemma 2.7 *Let A be an automorphism of G and $f \in L^{(p',\theta}(G)$. Then $D_A^{p'} f \in L^{(p',\theta}(G)$. Moreover,*

$$\left\|D_A^{p'} f\right\|_{(p',\theta} = |A|^{\frac{1}{p'}} \|f\|_{(p',\theta} \leq \|f\|_{(p',\theta}, \ \text{if} \quad |A| < 1$$
$$\left\|D_A^{p'} f\right\|_{(p',\theta} = \|f\|_{(p',\theta}, \qquad \text{if} \quad |A| \geq 1.$$

Proof Let A be an automorphism of G and $f \in L^{(p',\theta}(G)$. If we say $Ax = u$ and use the equality $|A|^{-1} = \left|A^{-1}\right|$, then

$$\begin{aligned}
\left\|D_A^{p'} f\right\|_{(p',\theta} &= \inf_{D_A^{p'} f = \sum_{k=1}^{\infty} D_A^{p'} f_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p-\varepsilon}} \left(\fint_G \left| D_A^{p'} f_k(x) \right|^{(p-\varepsilon)'} d\lambda(x) \right)^{\frac{1}{(p-\varepsilon)'}} \right\} \\
&= \inf_{f = \sum_{k=1}^{\infty} f_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p-\varepsilon}} \left(\fint_G \left| |A|^{\frac{1}{p'}} f_k(u) \right|^{(p-\varepsilon)'} d\lambda\left(A^{-1}u\right) \right)^{\frac{1}{(p-\varepsilon)'}} \right\} \\
&= \inf_{f = \sum_{k=1}^{\infty} f_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p-\varepsilon}} |A|^{\frac{1}{p'}} |A|^{-\frac{1}{(p-\varepsilon)'}} \left(\fint_G |f_k(u)|^{(p-\varepsilon)'} d\lambda(u) \right)^{\frac{1}{(p-\varepsilon)'}} \right\} \\
&= \inf_{f = \sum_{k=1}^{\infty} f_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p-\varepsilon}} |A|^{1-\frac{1}{p}} |A|^{\frac{1}{(p-\varepsilon)}-1} \left(\fint_G |f_k(u)|^{(p-\varepsilon)'} d\lambda(u) \right)^{\frac{1}{(p-\varepsilon)'}} \right\} \\
&= \inf_{f = \sum_{k=1}^{\infty} f_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} |A|^{\frac{1}{(p-\varepsilon)}-\frac{1}{p}} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p-\varepsilon}} \left(\fint_G |f_k(u)|^{(p-\varepsilon)'} d\lambda(u) \right)^{\frac{1}{(p-\varepsilon)'}} \right\}. \qquad (2.16)
\end{aligned}$$

Assume that $|A| < 1$ and $0 < \varepsilon < p-1$. Since $\inf_{0<\varepsilon<p-1} |A|^{\frac{1}{(p-\varepsilon)}-\frac{1}{p}} = |A|^{1-\frac{1}{p}} = |A|^{\frac{1}{p'}}$, then by the last inequality and (2.16), we have

$$\begin{aligned}
\left\|D_A^{p'} f\right\|_{(p',\theta} &= \inf_{f = \sum_{k=1}^{\infty} f_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} |A|^{\frac{1}{(p-\varepsilon)}-\frac{1}{p}} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p-\varepsilon}} \left(\fint_G |f_k(u)|^{(p-\varepsilon)'} d\lambda(u) \right)^{\frac{1}{(p-\varepsilon)'}} \right\} \\
&= |A|^{\frac{1}{p'}} \inf_{f = \sum_{k=1}^{\infty} f_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p-\varepsilon}} \left(\fint_G |f_k(u)|^{(p-\varepsilon)'} d\lambda(u) \right)^{\frac{1}{(p-\varepsilon)'}} \right\} = |A|^{\frac{1}{p'}} \|f\|_{(p',\theta}.
\end{aligned}$$

Thus, $D_A^{p'} f \in L^{(p',\theta}(G)$.

Let $|A| \geq 1$, and let $0 < \varepsilon < p - 1$. Since $\inf\limits_{0<\varepsilon<p-1} |A|^{\frac{1}{(p-\varepsilon)} - \frac{1}{p}} = 1$, by (2.16), we have

$$\left\| D_A^{p'} f \right\|_{(p',\theta} = \inf_{f=\sum\limits_{k=1}^{\infty} f_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} |A|^{\frac{1}{(p-\varepsilon)} - \frac{1}{p}} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p-\varepsilon}} \left(\fint_G |f_k(u)|^{(p-\varepsilon)'} d\lambda(u) \right)^{\frac{1}{(p-\varepsilon)'}} \right\}$$

$$= \inf_{f=\sum\limits_{k=1}^{\infty} f_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p-\varepsilon}} \left(\fint_G |f_k(u)|^{(p-\varepsilon)'} d\lambda(u) \right)^{\frac{1}{(p-\varepsilon)'}} \right\} = \|f\|_{(p',\theta}.$$

Thus, $D_A^{p'} f \in L^{(p',\theta}(G)$. □

Theorem 2.8 *Let A be an automorphism of G and $m \in BM_\theta\left[(p'_1; (p'_2; (p'_3\right]$. If $\frac{1}{q} = \frac{1}{p'_1} + \frac{1}{p'_2} - \frac{1}{p'_3}$, then $\tilde{D}_{A^*}^q m \in BM_\theta\left[(p'_1; (p'_2; (p'_3\right]$, where $\tilde{D}_{A^*}^q m(s,t) = |A^*|^{\frac{1}{q}}\ m(A^*s, A^*t)$. Furthermore,*

$$\left\| \tilde{D}_{A^*}^q m \right\|_{\left[(p'_1;(p'_2;(p'_3\right]_\theta} \leq \|m\|_{\left[(p'_1;(p'_2;(p'_3\right]_\theta}.$$

Proof Take any $f \in L^{(p'_1,\theta}(G)$ and $g \in L^{(p'_2,\theta}(G)$. We know by Lemma 2.7 that $D_A^{p_1} f \in L^{(p'_1,\theta}(G)$ and $D_A^{p_2} g \in L^{(p'_2,\theta}(G)$. If we put $A^*s = u$ and $A^*t = v$, then $d\mu(u) = |A^*|\, d\mu(s)$ and $d\mu(v) = |A^*|\, d\mu(t)$. From the assumption $\frac{1}{q} = \frac{1}{p'_1} + \frac{1}{p'_2} - \frac{1}{p'_3}$, we have

$$\begin{aligned} B_{\tilde{D}_{A^*}^q m}(f,g)(x) &= \sum_{s\in\hat{G}} \sum_{t\in\hat{G}} \hat{f}(s)\, \hat{g}(t)\, \tilde{D}_{A^*}^q m(s,t) \langle s+t, x\rangle \\ &= |A^*|^{-2} \sum_{u\in\hat{G}} \sum_{v\in\hat{G}} \hat{f}\left(A^{*-1}u\right) \hat{g}\left(A^{*-1}v\right) |A^*|^{\frac{1}{q}}\, m(u,v) \left\langle \left(A^{-1}\right)^*(u+v), x \right\rangle \\ &= |A^*|^{-2} \sum_{u\in\hat{G}} \sum_{v\in\hat{G}} \hat{f}\left(A^{*-1}u\right) \hat{g}\left(A^{*-1}v\right) |A^*|^{\frac{1}{p'_1} + \frac{1}{p'_2} - \frac{1}{p'_3}}\, m(u,v) \left\langle u+v, A^{-1}x \right\rangle \\ &= |A^*|^{-\frac{1}{p'_3}} \sum_{u\in\hat{G}} \sum_{v\in\hat{G}} D_{A^{*-1}}^{p_1} \hat{f}(u)\, D_{A^{*-1}}^{p_2} \hat{g}(v)\, m(u,v) \left\langle u+v, A^{-1}x \right\rangle. \end{aligned} \tag{2.17}$$

On the other hand, if we say that $Ay = s$, then we have

$$\begin{aligned} \left(D_A^{p'_1} f \right)\hat{}\,(u) &= \int_G D_A^{p'_1} f(y) \langle u, -y\rangle\, d\lambda(y) \\ &= \int_G |A|^{\frac{1}{p'_1}} f(s) \left\langle u, -A^{-1}s \right\rangle d\lambda\left(A^{-1}s\right) = \int_G |A|^{\frac{1}{p'_1} - 1} f(s) \left\langle u, -A^{-1}s \right\rangle d\lambda(s) \\ &= |A^*|^{-\frac{1}{p_1}} \int_G f(s) \left\langle A^{*-1}u, -s \right\rangle d\lambda(s) = |A^*|^{-\frac{1}{p_1}} \hat{f}\left(A^{*-1}u\right) = D_{A^{*-1}}^{p_1} \hat{f}(u). \end{aligned}$$

Similarly, we achieve $\left(D_A^{p_2'}g\right)\hat{} = D_{A^{*-1}}^{p_2}\hat{g}$. Then from (2.17), we obtain

$$B_{\tilde{D}_{A^*}^q m}(f,g)(x) = |A^*|^{-\frac{1}{p_3}} \sum_{u\in\hat{G}}\sum_{v\in\hat{G}} D_{A^{*-1}}^{p_1}\hat{f}(u)\, D_{A^{*-1}}^{p_2}\hat{g}(u)\, m(u,v)\left\langle u+v, A^{-1}x\right\rangle$$

$$= |A|^{-\frac{1}{p_3'}} B_m\left(D_A^{p_1'}f, D_A^{p_2'}g\right)\left(A^{-1}x\right) = D_{A^{-1}}^{p_3'}B_m\left(D_A^{p_1'}f, D_A^{p_2'}g\right)(x). \tag{2.18}$$

Since $m\in BM_\theta\left[(p_1';(p_2';(p_3'\right]$, from Lemma 2.7 and (2.18), we have

$$\left\|B_{\tilde{D}_{A^*}^q m}(f,g)\right\|_{(p_3',\theta} = \left\|D_{A^{-1}}^{p_3'}B_m\left(D_A^{p_1'}f, D_A^{p_2'}g\right)\right\|_{(p_3',\theta} \le \left\|B_m\left(D_A^{p_1'}f, D_A^{p_2'}g\right)\right\|_{(p_3',\theta}$$

$$\le \|B_m\| \left\|D_A^{p_1'}f\right\|_{(p_1',\theta}\left\|D_A^{p_2'}g\right\|_{(p_2',\theta}$$

$$= \|m\|_{\left[(p_1';(p_2';(p_3'\right]_\theta} \|f\|_{(p_1',\theta}\|g\|_{(p_2',\theta}. \tag{2.19}$$

Thus, we obtain $\tilde{D}_{A^*}^q m\in BM_\theta\left[(p_1';(p_2';(p_3'\right]$. Moreover, by (2.19),

$$\left\|\tilde{D}_{A^*}^q m\right\|_{\left[(p_1';(p_2';(p_3'\right]_\theta} \le \|m\|_{\left[(p_1';(p_2';(p_3'\right]_\theta}.$$

□

Theorem 2.9 *Let A be an automorphism of G and $m\in BM_\theta\left[(p_1';(p_2';(p_3'\right]$ such that $m(A^*s, A^*t) = m(s,t)$, where $\frac{1}{q} = \frac{1}{p_1'}+\frac{1}{p_2'}-\frac{1}{p_3'}$. Then*

$$\frac{1}{p_1'}+\frac{1}{p_2'}=\frac{1}{p_3'}.$$

Proof Assume that $f\in L^{(p_1',\theta}(G)$ and $g\in L^{(p_2',\theta}(G)$. Since $\frac{1}{q} = \frac{1}{p_1'}+\frac{1}{p_2'}-\frac{1}{p_3'}$, then by (2.18)

$$B_{\tilde{D}_{A^*}^q m}(f,g)(x) = D_{A^{-1}}^{p_3'}B_m\left(D_A^{p_1'}f, D_A^{p_2'}g\right)(x),\ x\in G. \tag{2.20}$$

On the other hand, we write

$$D_{A^{-1}}^{p_3'}B_m\left(D_A^{p_1'}f, D_A^{p_2'}g\right)(x) =$$

$$= |A|^{-\frac{1}{p_3'}}\sum_{u\in\hat{G}}\sum_{v\in\hat{G}}\left(D_A^{p_1'}f\right)\hat{}(u)\left(D_A^{p_2'}g\right)\hat{}(v)\, m(u,v)\left\langle u+v, A^{-1}x\right\rangle$$

$$= |A|^{-\frac{1}{p_3'}}\sum_{u\in\hat{G}}\sum_{v\in\hat{G}} D_{A^{*-1}}^{p_1}\hat{f}(u)\, D_{A^{*-1}}^{p_2}\hat{g}(u)\, m(u,v)\left\langle u+v, A^{-1}x\right\rangle$$

$$= |A|^{-\frac{1}{p_3'}}\sum_{u\in\hat{G}}\sum_{v\in\hat{G}} |A^*|^{-\frac{1}{p_1}}\hat{f}\left(A^{*-1}u\right)|A^*|^{-\frac{1}{p_2}}\times$$

$$\times\,\hat{g}\left(A^{*-1}v\right)m(u,v)\left\langle A^{*-1}(u+v), x\right\rangle. \tag{2.21}$$

We make the substitution $A^{*-1}u = s$, $A^{*-1}v = t$ in (2.21). Using $\mu\left(A^{*}\hat{G}\right) = |A^{*}|\,\mu\left(\hat{G}\right)$, $|A| = |A^{*}|$, $(A^{*})^{-1} = \left(A^{-1}\right)^{*}$, (see [2]) and the assumption $m\left(A^{*}s, A^{*}t\right) = m\left(s,t\right)$, we have

$$\begin{aligned} &D_{A^{-1}}^{p_3'} B_m\left(D_A^{p_1'} f, D_A^{p_2'} g\right)(x) = \\ &= |A|^{-\frac{1}{p_3'}-\frac{1}{p_1}-\frac{1}{p_2}} |A^{*}| \sum_{s\in\hat{G}} |A^{*}| \sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\, m\left(A^{*}s, A^{*}t\right)\langle s+t, x\rangle \\ &= |A|^{\frac{1}{p_1'}+\frac{1}{p_2'}-\frac{1}{p_3'}} B_m(f,g)(x). \end{aligned} \tag{2.22}$$

Hence by (2.20) and (2.22), we have

$$B_m(f,g)(x) = |A|^{-\left(\frac{1}{p_1'}+\frac{1}{p_2'}-\frac{1}{p_3'}\right)} B_{\tilde{D}_{A^{*}}^{q} m}(f,g)(x). \tag{2.23}$$

Since $m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$, by Theorem 2.8, we have $\tilde{D}_{A^{*}}^{q} m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$ and

$$\left\|\tilde{D}_{A^{*}}^{q} m\right\|_{\left[(p_1';(p_2';(p_3'\right]_\theta} \le \|m\|_{\left[(p_1';(p_2';(p_3'\right]_\theta}.$$

Then, by (2.23) and Theorem 2.8,

$$\begin{aligned} \|B_m(f,g)\|_{(p_3',\theta} &= |A|^{-\left(\frac{1}{p_1'}+\frac{1}{p_2'}-\frac{1}{p_3'}\right)} \left\|B_{\tilde{D}_{A^{*}}^{q} m}(f,g)\right\|_{(p_3',\theta} \\ &\le |A|^{-\left(\frac{1}{p_1'}+\frac{1}{p_2'}-\frac{1}{p_3'}\right)} \left\|\tilde{D}_{A^{*}}^{q} m\right\|_{\left[(p_1';(p_2';(p_3'\right]_\theta} \|f\|_{(p_1',\theta} \|g\|_{(p_2',\theta} \\ &\le |A|^{-\left(\frac{1}{p_1'}+\frac{1}{p_2'}-\frac{1}{p_3'}\right)} \|B_m\|\, \|f\|_{(p_1',\theta} \|g\|_{(p_2',\theta}. \end{aligned}$$

Since this inequality holds for any $|A|$, one needs $\frac{1}{p_1'} + \frac{1}{p_2'} = \frac{1}{p_3'}$. □

Theorem 2.10 *Let* $m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$.

a) *If* $\Phi \in \ell^1\left(\hat{G}\times\hat{G}\right)$, *then* $\Phi * m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$ *and*

$$\|\Phi * m\|_{\left[(p_1';(p_2';(p_3'\right]_\theta} \le \|\Phi\|_{\ell^1} \|m\|_{\left[(p_1';(p_2';(p_3'\right]_\theta},$$

where $\Phi * m$ *is convolution of* Φ *and* m.

b) *If* $\Phi \in L^1(G\times G)$, *then* $\Phi\hat{\ } m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$ *and*

$$\|\Phi\hat{\ } m\|_{\left[(p_1';(p_2';(p_3'\right]_\theta} \le \|\Phi\|_1 \|m\|_{\left[(p_1';(p_2';(p_3'\right]_\theta},$$

where $\Phi\hat{\ } m$ *is the multiplication of the Fourier transform of* Φ *and the function* m.

c) *Let* $\frac{1}{p'} = \frac{1}{p_1'} + \frac{1}{p_2'}$, $p_3.p_3' < p' + 1$ *and* $m(s,t) = a$. *Then* $m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$.

Proof **a)** Take any $f,\ g \in C^{\infty}(G)$. Then

$$\begin{aligned} B_{\Phi * m}(f,g)(x) &= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,(\Phi * m)(s,t)\,\langle s+t,x\rangle \\ &= \sum_{u\in\hat{G}}\sum_{v\in\hat{G}} \left(\sum_{s\in\hat{G}}\sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,m(s-u,t-v)\,\langle s+t,x\rangle\right)\Phi(u,v) \\ &= \sum_{u\in\hat{G}}\sum_{v\in\hat{G}} B_{T_{(u,v)}m}(f,g)(x)\,\Phi(u,v)\,. \end{aligned} \tag{2.24}$$

Since $m \in BM_{\theta}\left[(p_1'; (p_2'; (p_3'\right]$, by Theorem 2.5, we have $T_{(u,v)}m \in BM_{\theta}\left[(p_1'; (p_2'; (p_3'\right]$. Using the equality (2.24), we write

$$\begin{aligned} \left\|B_{\Phi * m}(f,g)\right\|_{(p_3',\theta} &\le \sum_{u\in\hat{G}}\sum_{v\in\hat{G}} |\Phi(u,v)| \left\|T_{(u,v)}m\right\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}} \|f\|_{(p_1',\theta}\|g\|_{(p_2',\theta} \\ &= \|m\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}} \|\Phi\|_{\ell^1}\|f\|_{(p_1',\theta}\|g\|_{(p_2',\theta} < \infty. \end{aligned} \tag{2.25}$$

Hence, $\Phi * m \in BM_{\theta}\left[(p_1'; (p_2'; (p_3'\right]$, and by (2.25)

$$\|\Phi * m\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}} \le \|\Phi\|_{\ell^1}\|m\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}}\,.$$

b) Let $\Phi \in L^1(G\times G)$. Take any $f,\ g \in C^{\infty}(G)$. Then we have

$$\begin{aligned} B_{\hat{\Phi} m}(f,g)(x) &= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,\Phi\hat{\ }m(s,t)\,\langle s+t,x\rangle \\ &= \int_G\int_G \Phi(u,v)\left(\sum_{s\in\hat{G}}\sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,M_{-v}^2M_{-u}^1 m(s,t)\,\langle s+t,x\rangle\right) d\lambda(u)\,d\lambda(v) \\ &= \int_G\int_G \Phi(u,v)\,B_{M_{-v}^2M_{-u}^1}(f,g)(x)\,d\lambda(u)\,d\lambda(v)\,, \end{aligned} \tag{2.26}$$

where M_{-v}^2 and M_{-u}^1 are modulation operators. Since $m \in BM_{\theta}\left[(p_1'; (p_2'; (p_3'\right]$, by Theorem 2.5, we have $M_{-v}^2M_{-u}^1 m \in BM_{\theta}\left[(p_1'; (p_2'; (p_3'\right]$. Then by the equality (2.26), we obtain

$$\begin{aligned} \left\|B_{\Phi\hat{\ }m}(f,g)\right\|_{(p_3',\theta} &\le \int_G\int_G |\Phi(u,v)| \left\|M_{-v}^2M_{-u}^1 m\right\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}} \|f\|_{(p_1',\theta}\|g\|_{(p_2',\theta}\,d\lambda(u)\,d\lambda(v) \\ &= \|m\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}} \|\Phi\|_1\|f\|_{(p_1',\theta}\|g\|_{(p_2',\theta}\,. \end{aligned} \tag{2.27}$$

Thus, $\Phi\hat{\ }m \in BM_{\theta}\left[(p_1'; (p_2'; (p_3'\right]$. By (2.27), we achieve that

$$\left\|\Phi\hat{\ }m\right\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}} \le \|\Phi\|_1\|m\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}}\,.$$

c) Take any $f \in L^{(p_1',\theta}(G)$ and $g \in L^{(p_2',\theta}(G)$. Then by Lemma 2.2, we have

$$\left\|B_m(f,g)\right\|_{(p_3',\theta} = |a| \left\| \sum_{s\in\hat{G}} \sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,\langle s,x\rangle\,\langle t,x\rangle \right\|_{(p_3',\theta}$$

$$= |a| \left\| \left(\sum_{s\in\hat{G}} \hat{f}(s)\,\langle s,x\rangle \right) \left(\sum_{t\in\hat{G}} \hat{g}(t)\,\langle t,x\rangle \right) \right\|_{(p_3',\theta} = |a|\,\|fg\|_{(p_3',\theta}\,. \tag{2.28}$$

Then by Lemma 2.2 and (2.28), we obtain

$$\left\|B_m(f,g)\right\|_{(p_3',\theta} \le C\,|a|\,\|f\|_{(p_1',\theta}\,\|g\|_{(p_2',\theta}\,.$$

Thus, $m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$. □

Corollary 2.11 *Let* $\frac{1}{p'} = \frac{1}{p_1'} + \frac{1}{p_2'}$, $p_3.p_3' < p' + 1$. *If* $\Phi \in L^1(G\times G)$, *then* $\Phi\hat{} \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$.

Proof Let $\frac{1}{p'} = \frac{1}{p_1'} + \frac{1}{p_2'}$, $p_3.p_3' < p' + 1$. If we take $m(s,t) = 1$ in Theorem 2.10 (c), we have $m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$. Since $\Phi \in L^1(G\times G)$, by Theorem 2.10 (b), we obtain $\Phi\hat{} = \Phi\hat{}m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$. □

The following Propositions 2.12 and 2.13 are proved as in [7, 11, 12].

Proposition 2.12 *Let* A *be an automorphism of* G *and let* $m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$. *If* $\Psi \in \ell^1\left(\hat{G}, |A^*|^{-\frac{1}{q}}\,d\mu\right)$ *such that* $\frac{1}{q} = \frac{1}{p_1'} + \frac{1}{p_2'} - \frac{1}{p_3'}$, *then*

$$m_\Psi(s,t) = \sum_{u\in A^*\hat{G}} m\left(A^*s, A^*t\right)\Psi(u) \in BM_\theta\left[(p_1';(p_2';(p_3'\right].$$

Moreover,

$$\|m_\Psi\|_{\left[(p_1';(p_2';(p_3'\right]_\theta} \le \|\Psi\|_{\ell^1\left(\hat{G},|A^*|^{-\frac{1}{q}}\,d\mu\right)}\,\|m\|_{\left[(p_1';(p_2';(p_3'\right]_\theta}\,.$$

Proposition 2.13 *Let* $m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$. *If* $U_1,\ U_2$ *are bounded measurable sets in* $\hat{G}$, *then*

$$h(s,t) = \frac{1}{\mu(U_1\times U_2)} \sum_{u\in U_1} \sum_{v\in U_2} m(s+u, t+v) \in BM_\theta\left[(p_1';(p_2';(p_3'\right].$$

Proposition 2.14 *Let* $\frac{1}{p'} = \frac{1}{p_1'} + \frac{1}{p_2'}$, $p_3.p_3' < p' + 1$ *and let* $A,\ B$ *be automorphisms of* G. *If* $\lambda \in M(G)$ *and* $m(s,t) = \hat{\lambda}\left(A^*s + B^*t\right)$, *then* $m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$. *Moreover,*

$$\|m\|_{\left[(p_1';(p_2';(p_3'\right]_\theta} \le C\,\|\lambda\|$$

for some $C > 0$.

Proof Let $f, g, \in C^{\infty}(G)$. Since

$$f(x - Ay) = \sum_{s \in \hat{G}} \hat{f}(s) \langle s, A(-y) + x \rangle$$

and

$$g(x - Ay) = \sum_{t \in \hat{G}} \hat{g}(t) \langle t, A(-y) + x \rangle,$$

then

$$\begin{aligned} B_m(f,g)(x) &= \sum_{s \in \hat{G}} \sum_{t \in \hat{G}} \hat{f}(s) \hat{g}(t) m(s,t) \langle s+t, x \rangle \\ &= \sum_{s \in \hat{G}} \sum_{t \in \hat{G}} \hat{f}(s) \hat{g}(t) \left\{ \int_G \langle A^* s + B^* t, -y \rangle d\lambda(y) \right\} \langle s+t, x \rangle \end{aligned}$$

$$\begin{aligned} &= \sum_{s \in \hat{G}} \sum_{t \in \hat{G}} \hat{f}(s) \hat{g}(t) \left\{ \int_G \langle s, A(-y) \rangle \langle t, A(-y) \rangle d\lambda(y) \right\} \langle s, x \rangle \langle t, x \rangle \\ &= \int_G \left\{ \sum_{s \in \hat{G}} \hat{f}(s) \langle s, A(-y) + x \rangle \right\} \left\{ \sum_{t \in \hat{G}} \hat{g}(t) \langle t, A(-y) + x \rangle \right\} d\lambda(y) \\ &= \int_G f(x - Ay) g(x - Ay) d\lambda(y). \end{aligned} \tag{2.29}$$

By (2.29) and Lemma 2.2, we have

$$\begin{aligned} \|B_m(f,g)\|_{(p_3',\theta} &\le \int_G C \|f(. - Ay)\|_{(p_1',\theta} \|g(. - Ay)\|_{(p_2',\theta} d|\lambda|(y) \\ &= C \int_G \|f\|_{(p_1',\theta} \|g\|_{(p_2',\theta} d|\lambda|(y) = C \|f\|_{(p_1',\theta} \|g\|_{(p_2',\theta} \|\lambda\|. \end{aligned} \tag{2.30}$$

Since $\lambda \in M(G)$, then by (2.30) $m \in BM_\theta[(p_1'; (p_2'; (p_3']$. Thus, we have

$$\|m\|_{[(p_1';(p_2';(p_3']_\theta} \le C \|\lambda\|.$$

□

It is known that the unit operator I is an automorphism of G. It is easy to see the conjugate I^* of I is a unit operator from $\hat{G}$ into itself. It is continuous, one-to-one and onto. Thus, I^* becomes an automorphism of $\hat{G}$. Similarly one can easily show that $-I$ and its conjugate $-I^*$ are authomorphisms of G and $\hat{G}$ respectively. Since $\hat{\lambda}(A^* s + B^* t) = \hat{\lambda}(s \mp t)$, in Proposition 2.14, one can get $m(s,t) = \hat{\lambda}(s \mp t)$. As an application of this result we can give the following example.

Example 2.15 *If* $\lambda \in M(G)$ *and* $m(s,t) = \hat{\lambda}(s \mp t)$*, then* $\hat{\lambda} \in \tilde{M}_\theta[(p_1'; (p_2'; (p_3']$ *and*

$$\left\| \hat{\lambda} \right\|_{[(p_1'; (p_2'; (p_3']_\theta} \leq C \left\| \lambda \right\|_1, \ C > 0$$

for $\frac{1}{p_1'} + \frac{1}{p_2'} = \frac{1}{p'}$ *and* $p_3' p_3 < p' + 1$*.*

Theorem 2.16 *Let* $\frac{1}{p'} = \frac{1}{p_1'} + \frac{1}{p_2'}$ *,* $p_3 . p_3' < p' + 1$*. If* $m(s,t) = \hat{\Psi}_1(s)\hat{\Phi}(s,t)\hat{\Psi}_2(t)$ *such that* $\Phi \in L^1(G \times G)$ *and* $\Psi_1, \ \Psi_2 \in L^1(G)$*, then* $m \in BM_\theta[(p_1'; (p_2'; (p_3']$*.*

Proof Let $f, g \in C^\infty(G)$ and $h \in C^\infty(G) \cap L^{p_3),\theta}(G)$. If we use the assumption $m(s,t) = \hat{\Psi}_1(s)\hat{\Phi}(s,t)\hat{\Psi}_2(t)$, we get

$$\left| \sum_{s \in \hat{G}} \sum_{t \in \hat{G}} \hat{f}(s)\hat{g}(t)\hat{h}(s+t)m(s,t) \right| =$$

$$= \left| \sum_{s \in \hat{G}} \sum_{t \in \hat{G}} \hat{f}(s)\hat{g}(t) \left\{ \int_G h(y) \langle s+t, -y \rangle \, d\lambda(y) \right\} m(s,t) \right|$$

$$= \left| \int_G h(y) \left\{ \sum_{s \in \hat{G}} \sum_{t \in \hat{G}} \hat{f}(s)\hat{g}(t)\hat{\Psi}_1(s)\hat{\Phi}(s,t)\hat{\Psi}_2(t) \langle s+t, -y \rangle \right\} d\lambda(y) \right|$$

$$= \left| \int_G h(y) \left\{ \sum_{s \in \hat{G}} \sum_{t \in \hat{G}} (f * \Psi_1)\hat{\ }(s)(g * \Psi_2)\hat{\ }(t)\hat{\Phi}(s,t) \langle s+t, -y \rangle \right\} d\lambda(y) \right|$$

$$\leq \int_G \left| h(y) \tilde{B}_{\hat{\Phi}}(f * \Psi_1, g * \Psi_2)(y) \right| d\lambda(y). \tag{2.31}$$

Now let $f = \sum_{k=1}^{\infty} f_k$. Then we have $f * \Psi_1 = \sum_{k=1}^{\infty} f_k * \Psi_1$. On the other hand, since $L^{(p_1-\varepsilon)'}(G)$ is Banach convolution module over $L^1(G)$, we find

$$\left\| f * \Psi_1 \right\|_{(p_1',\theta} = \inf_{f * \Psi_1 = \sum_{k=1}^{\infty} f_k * \Psi_1} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p_1-\varepsilon}} \left(\fint_G |f_k * \Psi_1(x)|^{(p_1-\varepsilon)'} d\lambda(x) \right)^{\frac{1}{(p_1-\varepsilon)'}} \right\}$$

$$= \inf_{f * \Psi_1 = \sum_{k=1}^{\infty} f_k * \Psi_1} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p_1-\varepsilon}} \left\| f_k * \Psi_1 \right\|_{(p_1-\varepsilon)'} \right\}$$

$$\leq \inf_{f = \sum_{k=1}^{\infty} f_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p_1-\varepsilon}} \left\| f_k \right\|_{(p_1-\varepsilon)'} \left\| \Psi_1 \right\|_1 \right\}$$

$$= \left\| \Psi_1 \right\|_1 \inf_{f = \sum_{k=1}^{\infty} f_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p_1-\varepsilon}} \left\| f_k \right\|_{(p_1-\varepsilon)'} \right\} = \left\| f \right\|_{(p_1',\theta} \left\| \Psi_1 \right\|_1. \tag{2.32}$$

Similarly, we write

$$\|g * \Psi_2\|_{(p_2',\theta} \leq \|g\|_{(p_2',\theta} \|\Psi_2\|_1 . \tag{2.33}$$

Thus, we have $f*\Psi_1 \in L^{(p_1',\theta}(G)$ and $g*\Psi_2 \in L^{(p_2',\theta}(G)$. Moreover, by Corollary 2.11, $\hat{\Phi} \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$. Then we achieve $B_{\hat{\Phi}}(f*\Psi_1, g*\Psi_2) \in L^{(p_3',\theta}(G)$. By using the Hölder inequality for generalized small Lebesgue spaces and the inequalities (2.31)–(2.33), we have

$$\begin{aligned}\left|\sum_{s\in\hat{G}}\sum_{t\in\hat{G}}\hat{f}(s)\,\hat{g}(t)\,\hat{h}(s+t)\,m(s,t)\right| &\leq \|h\|_{p_3),\theta}\left\|B_{\hat{\Phi}}(f*\Psi_1, g*\Psi_2)\right\|_{(p_3',\theta}\\ &\leq \|h\|_{p_3),\theta}\left\|B_{\hat{\Phi}}\right\|\|f*\Psi_1\|_{(p_1',\theta}\|g*\Psi_2\|_{(p_2',\theta}\\ &\leq \|h\|_{p_3),\theta}\left\|B_{\hat{\Phi}}\right\|\|\Psi_1\|_1\|f\|_{(p_1',\theta}\|\Psi_2\|_1\|g\|_{(p_2',\theta}.\end{aligned}$$

Then

$$\left|\sum_{s\in\hat{G}}\sum_{t\in\hat{G}}\hat{f}(s)\,\hat{g}(t)\,\hat{h}(s+t)\,m(s,t)\right| \leq C\|f\|_{(p_1',\theta}\|g\|_{(p_2',\theta}\|h\|_{p_3),\theta},$$

where $C = \left\|B_{\hat{\Phi}}\right\|\|\Psi_1\|_1\|\Psi_2\|_1$. Hence, by Theorem 2.4, we obtain $m \in BM_\theta\left[(p_1';(p_2';(p_3'\right]$. □

Example 2.17 *If $K \in L^1(G)$ then $m(s,t) = \hat{K}(s-t)$ defines a bilinear multiplier in $BM_\theta[(p_1';(p_2';(p_3']$ and*

$$\|m\|_{\left[(p_1';(p_2';(p_3'\right]_\theta} \leq C\|K\|_1, \ C > 0$$

for $\frac{1}{p'} = \frac{1}{p_1'} + \frac{1}{p_2'}$, $p_3.p_3' < p' + 1$.

Indeed for $f,\ g \in C^\infty(G) \subset L^1(G)$, one has

$$\begin{aligned}B_m(f,g)(x) &= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}}\hat{f}(s)\,\hat{g}(t)\,m(s,t)\langle s+t, x\rangle\\ &= \int_G\sum_{s\in\hat{G}}\sum_{t\in\hat{G}}\hat{f}(s)\,\hat{g}(t)\,K(y)\langle s-t,-y\rangle\langle s+t,x\rangle\,d\lambda(y)\\ &= \int_G\sum_{s\in\hat{G}}\sum_{t\in\hat{G}}\hat{f}(s)\,\hat{g}(t)\,K(y)\langle s,x-y\rangle\langle t,x+y\rangle\,d\lambda(y)\\ &= \int_G f(x-y)\,g(x+y)\,K(y)\,d\lambda(y).\end{aligned} \tag{2.34}$$

Then from (2.34) and by Lemma 2.2,

$$\|B_m(f,g)\|_{(p_3',\theta} \leq C\int_G\|f(x-y)\|_{(p_1',\theta}\|g(x+y)\|_{(p_2',\theta}|K(y)|\,d\lambda(y),$$

for some $C > 0$. Since $\|T_{-x} f(-y)\|_{(p_1',\theta} = \|f(y)\|_{(p_1',\theta}$ and $\|T_{-x} g(y)\|_{(p_2',\theta} = \|g(y)\|_{(p_2',\theta}$ by Theorem 2.5, then

$$\|B_m(f,g)\|_{(p_3',\theta} \le C \int_G \|T_{-x} f(-y)\|_{(p_1',\theta} \|T_{-x} g(y)\|_{(p_2',\theta} |K(y)|\, d\lambda(y)$$

$$= C \int_G \left\|\tilde{f}\right\|_{(p_1',\theta} \|g\|_{(p_2',\theta} |K(y)|\, d\lambda(y) = C \|f\|_{(p_1',\theta} \|g\|_{(p_2',\theta} \|K\|_1 \tag{2.35}$$

$$= C_1 \|f\|_{(p_1',\theta} \|g\|_{(p_2',\theta},$$

where $C_1 = C\|K\|_1$ and $\tilde{f}(y) = f(-y)$. Thus, $m \in BM_\theta[(p_1'; (p_2'; (p_3']$. Finally, by using (2.35), we obtain

$$\|m\|_{[(p_1';(p_2';(p_3']_\theta} = \sup\left\{ \frac{\|B_m(f,g)\|_{(p_3',\theta}}{\|f\|_{(p_1',\theta}\|g\|_{(p_2',\theta}} : \|f\|_{(p_1',\theta} \le 1,\ \|g\|_{(p_2',\theta} \le 1 \right\} \le C\|K\|_1.$$

Definition 2.18 *Let $1 < p_i < \infty$, $p_i' = \frac{p_i}{p_i - 1}$, $(i = 1,2,3)$ and $\theta > 0$. We denote by $\tilde{M}_\theta[(p_1'; (p_2'; (p_3']$ the space of measurable functions $M : \hat{G} \to \mathbb{C}$ such that $m(s,t) = M(s-t) \in BM_\theta[(p_1'; (p_2'; (p_3']$, that is to say*

$$B_M(f,g)(x) = \sum_{s\in\hat{G}} \sum_{t\in\hat{G}} \hat{f}(s)\, \hat{g}(t)\, M(s-t) \langle s+t, x\rangle$$

extends to bounded bilinear map from $L^{(p_1',\theta}(G) \times L^{(p_2',\theta}(G)$ to $L^{(p_3',\theta}(G)$. We denote $\|M\|_{[(p_1';(p_2';(p_3']_\theta} = \|B_M\|$.

Proposition 2.19 *Let $M \in \ell^1\left(\hat{G}\right)$. Then for all $f \in L^{(p_1',\theta}(G)$ and $g \in L^{(p_2',\theta}(G)$*

$$B_M(f,g)(x) = \int_G f(x-y)\, g(x+y)\, M^\vee(y)\, dy,$$

where $M^\vee$ is the inverse Fourier transform of the function M.

Proof Let $f,\ g \in C^\infty(G)$. Then

$$B_M(f,g)(x) = \sum_{s\in\hat{G}} \sum_{t\in\hat{G}} \hat{f}(s)\, \hat{g}(t)\, M(s-t) \langle s+t, x\rangle$$

$$= \sum_{s\in\hat{G}} \sum_{t\in\hat{G}} \hat{f}(s)\, \hat{g}(t) \left(\int_G M^\vee(y) \langle s-t, -y\rangle \langle s+t, x\rangle\, d\lambda(y) \right)$$

$$= \int_G \check{M}(y) \left(\sum_{s\in\hat{G}} \hat{f}(s) \langle s, x-y\rangle \right) \left(\sum_{t\in\hat{G}} \hat{g}(t) \langle t, x+y\rangle \right) d\lambda(y)$$

$$= \int_G f(x-y)\, g(x+y)\, M^\vee(y)\, d\lambda(y).$$

□

Example 2.20 *Let G be a locally compact abelian metric group and let 0_G be the unit of G. Take the bilinear Hardy–Littlewood maximal function on G;*

$$M(f,g)(x)=\sup_{r>0}\frac{1}{\lambda(B(0_G,r))}\int\limits_{B(0_G,r)}|f(x-y)\,g(x+y)|\,d\lambda(y)$$

for all $f,g\in L^1_{loc}(G)$, where $B(0_G,r)$ is open ball in G. The Hardy-Littlewood maximal function is bounded from $L^{(p_1',\theta}(G)\times L^{(p_2',\theta}(G)$ to $L^{(p_3',\theta}(G)$ whenever $\frac{1}{p'}=\frac{1}{p_1'}+\frac{1}{p_2'}$ and $p_3.p_3'<p'+1$.

Take the function

$$M(y)=\frac{1}{\lambda(B(0_G,r))}\chi_{B(0_G,r)}(y).$$

Since $M\in L^1(G)$, by Proposition 2.19, M defines a bilinear multiplier in $\tilde{M}_\theta[(p_1';(p_2';(p_3']$ and

$$\|B_M(f,g)\|_{(p_3',\theta}\leq C\,\|f\|_{(p_1',\theta}\,\|g\|_{(p_2',\theta}, \tag{2.36}$$

where

$$B_M(f,g)(x)=\frac{1}{\lambda(B(0_G,r))}\int\limits_{B(0_G,r)}f(x-y)\,g(x+y)\,d\lambda(y) \tag{2.37}$$

for all $r>0$, by (2.37) we get

$$\begin{aligned}M(f,g)(x)&=\sup_{r>0}\frac{1}{\lambda(B(0_G,r))}\int\limits_{B(0_G,r)}|f(x-y)\,g(x+y)|\,d\lambda(y)\\&=\sup_{r>0}B_M(|f|,|g|)\end{aligned} \tag{2.38}$$

which, together with (2.36) implies

$$\|M(f,g)\|_{(p_3',\theta}\leq C\,\|f\|_{(p_1',\theta}\,\|g\|_{(p_2',\theta}.$$

Therefore, $M(f,g)$ is bounded from $L^{(p_1',\theta}(G)\times L^{(p_2',\theta}(G)$ to $L^{(p_3',\theta}(G)$.

Proposition 2.21 *Let $K\in\ell^1\left(\hat{G}\right)$. Then the following equalities are satisfied;*

a) $B_{T_{y_1+y_2}K}(f,g)=M_{y_1-y_2}B_K(M_{-y_1}f,M_{y_2}g)$, $y_1,\ y_2\in\hat{G}$,

b) $B_{M_yK}(f,g)=B_K(T_{-y}f,T_yg)$, $y\in\hat{G}$.

Proof **a)** Let $f,\ g\in C^\infty(G)$ and let $y_1,\ y_2\in\hat{G}$. If we make the substitutions $s-y_1=u$ and $t+y_2=v$, then we have

$$\begin{aligned}B_{T_{y_1+y_2}K}(f,g)(x) &= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}}\hat{f}(s)\,\hat{g}(t)\,T_{y_1+y_2}K(s-t)\langle s+t,x\rangle \\ &= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}}\hat{f}(s)\,\hat{g}(t)\,K(s-t-y_1-y_2)\langle s+t,x\rangle \\ &= \sum_{u\in\hat{G}}\sum_{v\in\hat{G}}T_{-y_1}\hat{f}(u)\,T_{y_2}\hat{g}(v)\,K(u-v)\langle u+v,x\rangle\langle y_1-y_2,x\rangle \\ &= \langle y_1-y_2,x\rangle\sum_{u\in\hat{G}}\sum_{v\in\hat{G}}(M_{-y_1}f)\,\hat{}\,(u)\,(M_{y_2}g)\,\hat{}\,(v)\,K(u-v)\langle u+v,x\rangle \\ &= M_{y_1-y_2}B_K(M_{-y_1}f,M_{y_2}g)(x)\,.\end{aligned}$$

b) Let $f,\ g\in C^{\infty}(G)$ and let $y\in\hat{G}$. Then

$$\begin{aligned}B_{M_yK}(f,g)(x) &= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}}\hat{f}(s)\,\hat{g}(t)\,M_yK(s-t)\langle s+t,x\rangle \\ &= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}}\hat{f}(s)\,\hat{g}(t)\langle s-t,y\rangle K(s-t)\langle s+t,x\rangle \\ &= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}}\langle s,y\rangle\hat{f}(s)\langle t,-y\rangle\hat{g}(t)\,K(s-t)\langle s+t,x\rangle\end{aligned}$$

$$\begin{aligned}&= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}}M_y\hat{f}(s)\,M_{-y}\hat{g}(t)\,K(s-t)\langle s+t,x\rangle \\ &= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}}(T_{-y}f)\,\hat{}\,(s)\,(T_yg)\,\hat{}\,(t)\,K(s-t)\langle s+t,x\rangle = B_K(T_{-y}f,T_yg)(x)\,,\end{aligned}$$

where we have used the formulas in (1.3). □

Theorem 2.22 *Let* $K\in\tilde{M}_\theta[(p_1';(p_2';(p_3']$.

a) *If* $\Phi\in\ell^1\left(\hat{G}\right)$, *then* $\Phi*K\in\tilde{M}_\theta[(p_1';(p_2';(p_3']$ *and*

$$\|\Phi*K\|_{[(p_1';(p_2';(p_3']_\theta}\le\|\Phi\|_{\ell^1}\|K\|_{[(p_1';(p_2';(p_3']_\theta}\,.$$

b) *If* $\Phi\in L^1(G)$, *then* $\hat{\Phi}K\in\tilde{M}_\theta[(p_1';(p_2';(p_3']$ *and*

$$\left\|\hat{\Phi}K\right\|_{[(p_1';(p_2';(p_3']_\theta}\le\|\Phi\|_1\|K\|_{[(p_1';(p_2';(p_3']_\theta}\,.$$

Proof **a)** Take any $f,\ g\in C^{\infty}(G)$, by Proposition 2.21, we get

$$
\begin{aligned}
B_{\Phi*K}(f,g)(x) &= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,(\Phi*K)(s-t)\,\langle s+t,x\rangle \\
&= \sum_{u\in\hat{G}} \left(\sum_{s\in\hat{G}}\sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,K(s-t-u)\,\langle s+t,x\rangle\right)\Phi(u) \\
&= \sum_{u\in\hat{G}} B_{T_uK}(f,g)(x)\,\Phi(u) = \sum_{u\in\hat{G}} B_{T_{u+0_{\hat{G}}}K}(f,g)(x)\,\Phi(u) \\
&= \sum_{u\in\hat{G}} M_{u-0_{\hat{G}}} B_K\left(M_{-u}f, M_{0_{\hat{G}}}g\right)(x)\,\Phi(u)\,. \qquad (2.39)
\end{aligned}
$$

Since $K\in\tilde{M}_{\theta}[(p_1';(p_2';(p_3']$, by (2.39), we have

$$
\begin{aligned}
\|B_{\Phi*K}(f,g)\|_{(p_3',\theta} &\le \sum_{u\in\hat{G}} \left\|M_{u-0_{\hat{G}}} B_K\left(M_{-u}f, M_{0_{\hat{G}}}g\right)(x)\,\Phi(u)\right\|_{(p_3',\theta} \\
&\le \sum_{u\in\hat{G}} |\Phi(u)|\,\|K\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}} \|M_{-u}f\|_{(p_1',\theta}\,\|g\|_{(p_2',\theta} \\
&= \|K\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}} \|\Phi\|_{\ell^1}\,\|f\|_{(p_1',\theta}\,\|g\|_{(p_2',\theta} < \infty.
\end{aligned}
$$

Hence, $\Phi*\ K\in\tilde{M}_{\theta}[(p_1';(p_2';(p_3']$ and

$$
\|\Phi*K\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}} \le \|\Phi\|_{\ell^1}\,\|K\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}}\,.
$$

b) Let $f,\ g\in C^{\infty}(G)$. Then by Propositon 2.21

$$
\begin{aligned}
B_{\hat{\Phi}K}(f,g)(x) &= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,\hat{\Phi}K(s-t)\,\langle s+t,x\rangle \\
&= \int_G \Phi(u)\left(\sum_{s\in\hat{G}}\sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,M_{-u}K(s-t)\,\langle s+t,x\rangle\right)d\lambda(u) \\
&= \int_G \Phi(u)\,B_{M_{-u}K}(f,g)(x)\,d\lambda(u) = \int_G \Phi(u)\,B_K(T_uf,T_{-u}g)(x)\,d\lambda(u)\,. \qquad (2.40)
\end{aligned}
$$

Since $K\in\tilde{M}_{\theta}[(p_1';(p_2';(p_3']$, by (2.40), we obtain

$$
\begin{aligned}
\left\|B_{\hat{\Phi}K}(f,g)\right\|_{(p_3',\theta} &\le \int_G |\Phi(u)|\,\|K\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}} \|T_uf\|_{(p_1',\theta}\,\|T_{-u}g\|_{(p_2',\theta}\,d\lambda(u) \\
&= \|K\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}} \|\Phi\|_1\,\|f\|_{(p_1',\theta}\,\|g\|_{(p_2',\theta} < \infty. \qquad (2.41)
\end{aligned}
$$

Finally, $\hat{\Phi}K\in\tilde{M}_{\theta}[(p_1';(p_2';(p_3']$ and by (2.41)

$$
\left\|\hat{\Phi}K\right\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}} \le \|\Phi\|_1\,\|K\|_{\left[(p_1';(p_2';(p_3'\right]_{\theta}}\,.
$$

□

Proposition 2.23 *Let* $\Phi \in L^1(G)$ *and* $M \in \tilde{M}_\theta[(p_1';(p_2';(p_3']$. *Then* $m(s,t) = M(s-t)\hat{\Phi}(s+t) \in BM_\theta[(p_1';(p_2';(p_3']$ *and*

$$\|m\|_{[(p_1';(p_2';(p_3']_\theta} \leq \|\Phi\|_1 \|M\|_{[(p_1';(p_2';(p_3']_\theta}.$$

Proof Let $f,\ g \in C^\infty(G)$. Then for all $x \in G$, we have

$$\begin{aligned} B_m(f,g)(x) &= \sum_{s\in\hat{G}}\sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,M(s-t)\,\hat{\Phi}(s+t)\,\langle s+t,x\rangle \\ &= \int_G \Phi(u)\left(\sum_{s\in\hat{G}}\sum_{t\in\hat{G}} \hat{f}(s)\,\hat{g}(t)\,M(s-t)\,\langle s+t,x-u\rangle\right) d\lambda(u) \\ &= \int_G \Phi(u)\,B_M(f,g)(x-u)\,d\lambda(u) = \Phi * B_M(f,g)(x). \end{aligned} \tag{2.42}$$

If we use the proof technique in (2.32), by the hypothesis and (2.42), we get

$$\begin{aligned} \|B_m(f,g)\|_{(p_3',\theta} &= \|\Phi * B_M(f,g)\|_{(p_3',\theta} \leq \|B_M(f,g)\|_{(p_3',\theta}\|\Phi\|_1 \\ &\leq \|\Phi\|_1 \|M\|_{[(p_1';(p_2';(p_3']_\theta}\|f\|_{(p_1',\theta}\|g\|_{(p_2',\theta} < \infty. \end{aligned}$$

Hence, $m \in BM_\theta[(p_1';(p_2';(p_3']$ and

$$\|m\|_{[(p_1';(p_2';(p_3']_\theta} \leq \|\Phi\|_1 \|M\|_{[(p_1';(p_2';(p_3']_\theta}.$$

□

Proposition 2.24 *Let* $K \in \ell^1(\hat{G})$ *be nonzero function and let* $K \in \tilde{M}_\theta[(p_1';(p_2';(p_3']$. *If* A *is an automorphism of* G *and if* $\frac{1}{q} = \frac{1}{p_1'} + \frac{1}{p_2'} - \frac{1}{p_3'}$, *then there exists* $C > 0$ *such that*

$$\left|\int_G K^\vee(u)\,d\lambda(u)\right| \leq C\,|A|^{\frac{1}{p_3'}}\|K\|_{[(p_1';(p_2';(p_3']_\theta},\quad |A| < 1$$

and

$$\left|\int_G K^\vee(u)\,d\lambda(u)\right| \leq C\,|A|^{-\frac{1}{q}}\|K\|_{[(p_1';(p_2';(p_3']_\theta},\quad |A| \geq 1.$$

Proof Let A be any automorphism of G. Define a function $f : G \to \mathbb{C}$ by $f(x) = \langle \gamma, Ax\rangle$ for fixed $\gamma \in \hat{G}$.

By Proposition 2.19, we write

$$B_K(f,f)(x) = \int_G f(x-y) f(x+y) K^{\vee}(y)\,dy$$
$$= \int_G \langle \gamma, A(x-y)\rangle \langle \gamma, A(x+y)\rangle K^{\vee}(y)\,dy$$
$$= \int_G \langle \gamma, Ax\rangle \langle \gamma, Ax\rangle K^{\vee}(y)\,dy = \langle \gamma, Ax\rangle \langle \gamma, Ax\rangle \int_G K^{\vee}(y)\,dy. \tag{2.43}$$

Using (2.43) and making the substitution $Ax = u$, we have

$$\|B_K(f,f)\|_{p_3'} = \left(\int_G |B_K(f,f)(x)|^{p_3'}\,d\lambda(x)\right)^{\frac{1}{p_3'}}$$
$$= \left|\int_G K^{\vee}(y)\,dy\right| \left(\int_G |\langle \gamma, Ax\rangle \langle \gamma, Ax\rangle|^{p_3'}\,d\lambda(x)\right)^{\frac{1}{p_3'}}$$
$$= \left|\int_G K^{\vee}(y)\,dy\right| \left(\int_G |\langle \gamma, u\rangle|^{p_3'} |\langle \gamma, u\rangle|^{p_3'} |A|^{-1}\,d\lambda(u)\right)^{\frac{1}{p_3'}}.$$

Since $|\langle \gamma, u\rangle| = 1$, we achieve

$$\|B_K(f,f)\|_{p_3'} = |A|^{-\frac{1}{p_3'}} \lambda(G)^{\frac{1}{p_3'}} \left|\int_G K^{\vee}(y)\,dy\right|. \tag{2.44}$$

On the other hand, we can write

$$f(x) = \langle \gamma, Ax\rangle = |A|^{-\frac{1}{p_1'}} |A|^{\frac{1}{p_1'}} \langle \gamma, Ax\rangle = |A|^{-\frac{1}{p_1'}} D_A^{p_1'}\gamma(x).$$

Let $|A| \geq 1$. By Lemma 2.7, we obtain

$$\|f\|_{(p_1',\theta} = \left\| |A|^{-\frac{1}{p_1'}} D_A^{p_1'}\gamma \right\|_{(p_1',\theta} = |A|^{-\frac{1}{p_1'}} \left\| D_A^{p_1'}\gamma \right\|_{(p_1',\theta}$$
$$= |A|^{-\frac{1}{p_1'}} \|\gamma\|_{(p_1',\theta}.$$

Since $L^{p_1'+\varepsilon}(G) \subset L^{(p_1',\theta}(G)$, (see [3]), there exists $C_1 > 0$ such that $\|\gamma\|_{(p_1',\theta} \leq C_1 \|\gamma\|_{p_1'+\varepsilon}$. Then

$$\|f\|_{(p_1',\theta} = |A|^{-\frac{1}{p_1'}} \|\gamma\|_{(p_1',\theta} \leq C_1 |A|^{-\frac{1}{p_1'}} \|\gamma\|_{p_1'+\varepsilon}$$
$$= C_1 |A|^{-\frac{1}{p_1'}} \left(\int_G |\langle \gamma, x\rangle|^{p_1'+\varepsilon}\,d\lambda(x)\right)^{\frac{1}{p_1'+\varepsilon}}$$
$$= C_1 |A|^{-\frac{1}{p_1'}} \lambda(G)^{\frac{1}{p_1'+\varepsilon}} < \infty. \tag{2.45}$$

Similarly, there exists $C_2 > 0$ such that

$$\|f\|_{(p_2',\theta} \leq C_2 \, |A|^{-\frac{1}{p_2'}} \, \lambda\,(G)^{\frac{1}{p_2'+\varepsilon}} < \infty. \tag{2.46}$$

Using the assumption $K \in \tilde{M}_\theta[(p_1'; (p_2'; (p_3']$ and the inequalities (2.45) and (2.46), we obtain

$$\begin{aligned}\|B_K\,(f,f)\|_{(p_3',\theta} &\leq \|K\|_{[(p_1';(p_2';(p_3']_\theta} \, \|f\|_{(p_1',\theta} \, \|f\|_{(p_2',\theta} \\ &\leq C_1 C_2 \, |A|^{-\frac{1}{p_1'}-\frac{1}{p_2'}} \, \lambda\,(G)^{\frac{1}{p_1'+\varepsilon}+\frac{1}{p_2'+\varepsilon}} \, \|K\|_{[(p_1';(p_2';(p_3']_\theta}\,. \end{aligned} \tag{2.47}$$

Since $L^{(p_3',\theta}\,(G) \subset L^{p_3'}\,(G)$, there exists $C_3 > 0$ such that

$$\|B_K\,(f,f)\|_{p_3'} \leq C_3 \, \|B_K\,(f,f)\|_{(p_3',\theta}\,. \tag{2.48}$$

Then by (2.47) and (2.48), we have

$$\|B_K\,(f,f)\|_{p_3'} \leq C_1 C_2 C_3 \, |A|^{-\frac{1}{p_1'}-\frac{1}{p_2'}} \, \lambda\,(G)^{\frac{1}{p_1'+\varepsilon}+\frac{1}{p_2'+\varepsilon}} \, \|K\|_{[(p_1';(p_2';(p_3']_\theta}\,. \tag{2.49}$$

By (2.44) and (2.49), we achieve

$$\begin{aligned}\lambda\,(G)^{\frac{1}{p_3'}} \, |A|^{-\frac{1}{p_3'}} \left| \int_G K^\vee\,(y)\,dy \right| &= \|B_K\,(f,f)\|_{p_3'} \leq \\ &\leq C_1 C_2 C_3 \, |A|^{-\frac{1}{p_1'}-\frac{1}{p_2'}} \, \lambda\,(G)^{\frac{1}{p_1'+\varepsilon}+\frac{1}{p_2'+\varepsilon}} \, \|K\|_{[(p_1';(p_2';(p_3']_\theta}\,.\end{aligned}$$

This implies

$$\left| \int_G K^\vee\,(y)\,dy \right| \leq C \, |A|^{\frac{1}{p_3'}-\frac{1}{p_1'}-\frac{1}{p_2'}} \, \|K\|_{[(p_1';(p_2';(p_3']_\theta} = C \, |A|^{-\frac{1}{q}} \, \|K\|_{[(p_1';(p_2';(p_3']_\theta}\,,$$

where $C = C_1 C_2 C_3 \lambda\,(G)^{\frac{1}{p_1'+\varepsilon}+\frac{1}{p_2'+\varepsilon}-\frac{1}{p_3'}}$.

Now let $|A| < 1$. By Lemma 2.7, we have

$$\begin{aligned}\|f\|_{(p_1',\theta} &= \left\| |A|^{-\frac{1}{p_1'}} \, D_A^{p_1'}\gamma \right\|_{(p_1',\theta} = |A|^{-\frac{1}{p_1'}} \left\| D_A^{p_1'}\gamma \right\|_{(p_1',\theta} \\ &= |A|^{-\frac{1}{p_1'}} \, |A|^{\frac{1}{p_1'}} \, \|\gamma\|_{(p_1',\theta} = \|\gamma\|_{(p_1',\theta}\,.\end{aligned}$$

Then, since $L^{p_1'+\varepsilon}\,(G) \subset L^{(p_1',\theta}\,(G)$, we achieve

$$\begin{aligned}\|f\|_{(p_1',\theta} &= \|\gamma\|_{(p_1',\theta} \leq C_1 \, \|\gamma\|_{p_1'+\varepsilon} \\ &= C_1 \lambda\,(G)^{\frac{1}{p_1'+\varepsilon}} < \infty,\end{aligned} \tag{2.50}$$

for some $C_1 > 0$. Similarly, we have

$$\|f\|_{(p_2',\theta} \leq C_2 \lambda\,(G)^{\frac{1}{p_2'+\varepsilon}} < \infty, \tag{2.51}$$

for some $C_2 > 0$. Again using the assumption $K \in \tilde{M}_\theta[(p_1'; (p_2'; (p_3']$ and the inequalities (2.50) and (2.51), we obtain

$$\|B_K(f,f)\|_{(p_3',\theta} \leq \|K\|_{[(p_1';(p_2';(p_3']_\theta} \|f\|_{(p_1',\theta} \|f\|_{(p_2',\theta}$$
$$\leq C_1 C_2 \lambda(G)^{\frac{1}{p_1'+\varepsilon}+\frac{1}{p_2'+\varepsilon}} \|K\|_{[(p_1';(p_2';(p_3']_\theta} . \tag{2.52}$$

Thus, by (2.48) and (2.52), we have

$$\|B_K(f,f)\|_{p_3'} \leq C_1 C_2 C_3 \lambda(G)^{\frac{1}{p_1'+\varepsilon}+\frac{1}{p_2'+\varepsilon}} \|K\|_{[(p_1';(p_2';(p_3']_\theta} . \tag{2.53}$$

Using (2.44) and (2.53), we achieve

$$\lambda(G)^{\frac{1}{p_3'}} |A|^{-\frac{1}{p_3'}} \left| \int_G K^\vee(y)\, dy \right| = \|B_K(f,f)\|_{p_3'} \leq$$
$$\leq C_1 C_2 C_3 \lambda(G)^{\frac{1}{p_1'+\varepsilon}+\frac{1}{p_2'+\varepsilon}} \|K\|_{[(p_1';(p_2';(p_3']_\theta} .$$

Then

$$\left| \int_G K^\vee(y)\, dy \right| \leq C |A|^{\frac{1}{p_3'}} \|K\|_{[(p_1';(p_2';(p_3']_\theta} ,$$

where $C = C_1 C_2 C_3 \lambda(G)^{\frac{1}{p_1'+\varepsilon}+\frac{1}{p_2'+\varepsilon}-\frac{1}{p_3'}}$. □

Proposition 2.25 *Let $K \in \ell^1\left(\hat{G}\right)$ be nonzero function and let $K \in \tilde{M}_\theta[(p_1'; (p_2'; (p_3']$. If A is an automorphism of G satisfying $|A| > 1$, then*

$$\frac{1}{p_3'} \geq \frac{1}{p_1'} + \frac{1}{p_2'}.$$

Proof Assume that $\frac{1}{p_3'} < \frac{1}{p_1'} + \frac{1}{p_2'}$. By Proposition 2.21,

$$\left\|B_{T_{-u}K}(f,g)\right\|_{(p_3',\theta} = \left\|B_{T_{-u+0_{\hat{G}}}K}(f,g)\right\|_{(p_3',\theta}$$
$$= \left\|M_{-u-0_{\hat{G}}} B_K\left(M_u f, M_{0_{\hat{G}}} g\right)\right\|_{(p_3',\theta} = \|B_K(M_u f, g)\|_{(p_3',\theta}$$
$$\leq \|K\|_{[(p_1';(p_2';(p_3']_\theta} \|M_u f\|_{(p_1',\theta} \|g\|_{(p_2',\theta}$$
$$= \|K\|_{[(p_1';(p_2';(p_3']_\theta} \|f\|_{(p_1',\theta} \|g\|_{(p_2',\theta} ,$$

so $T_{-u}K \in \tilde{M}_\theta[(p_1'; (p_2'; (p_3']$ for all $u \in \hat{G}$. On the other hand,

$$\begin{aligned}
\|T_{-u}K\|_{[(p_1';(p_2';(p_3']_\theta} &= \left\|B_{T_{-u}K}\right\| \\
&= \sup\left\{ \frac{\left\|B_{T_{-u}K}(f,g)\right\|_{(p_3',\theta}}{\|f\|_{(p_1',\theta}\|g\|_{(p_2',\theta}} : \|f\|_{(p_1',\theta} \le 1,\ \|g\|_{(p_2',\theta} \le 1 \right\} \\
&= \sup\left\{ \frac{\left\|M_{-u}B_K\left(M_u f, M_{0_{\hat{G}}} g\right)\right\|_{(p_3',\theta}}{\|M_u f\|_{(p_1',\theta}\left\|M_{0_{\hat{G}}} g\right\|_{(p_2',\theta}} : \|M_u f\|_{(p_1',\theta} \le 1,\ \left\|M_{0_{\hat{G}}} g\right\|_{(p_2',\theta} \le 1 \right\} \\
&= \|B_K\| = \|K\|_{[(p_1';(p_2';(p_3']_\theta}.
\end{aligned}$$

Thus, by Proposition 2.24, there exists $C > 0$ such that

$$\begin{aligned}
\left| \int_G (T_{-u}K)^{\vee}(y)\, d\lambda(y) \right| &\le C\, |A|^{-\frac{1}{q}} \|T_{-u}K\|_{[(p_1';(p_2';(p_3']_\theta} \\
&= C\, |A|^{-\frac{1}{q}} \|K\|_{[(p_1';(p_2';(p_3']_\theta}. \qquad (2.54)
\end{aligned}$$

Since $T_{-u}K \in \ell^1\left(\hat{G}\right)$, from (2.54), we write

$$\begin{aligned}
|K(u)| = \left|T_{-u}K\left(0_{\hat{G}}\right)\right| &= \left|\left((T_{-u}K)^{\vee}\right)^{\wedge}\left(0_{\hat{G}}\right)\right| = \left| \int_G (T_{-u}K)^{\vee}(y) \left\langle 0_{\hat{G}}, -y \right\rangle d\lambda(y) \right| \\
&= \left| \int_G (T_{-u}K)^{\vee}(y)\, d\lambda(y) \right| \le C\, |A|^{-\frac{1}{q}} \|K\|_{[(p_1';(p_2';(p_3']_\theta} \qquad (2.55)
\end{aligned}$$

for all $u \in \hat{G}$. Since $\frac{1}{p_3'} < \frac{1}{p_1'} + \frac{1}{p_2'}$, then $-\frac{1}{q} < 0$. Thus, the right side of (2.55) approaches zero for $|A| \to \infty$. This implies $K = 0$. But this is a contradiction with the assumption $K \neq 0$. Then the assumption $\frac{1}{p_3'} < \frac{1}{p_1'} + \frac{1}{p_2'}$ is not true. Therefore, we conclude $\frac{1}{p_3'} \ge \frac{1}{p_1'} + \frac{1}{p_2'}$. □

Corollary 2.26 *Let $\frac{1}{p_3'} < \frac{1}{p_1'} + \frac{1}{p_2'}$. If A is an automorphism of G satisfying $|A| > 1$, then $\tilde{M}_\theta[(p_1';(p_2';(p_3'] = \{0\}$.*

Proof Take any $K \in \tilde{M}_\theta[(p_1';(p_2';(p_3']$. Let $\Psi \in L^1(G)$ such that $0 \neq \Psi^{\wedge} \in \ell^1\left(\hat{G}\right)$. By Theorem 2.22, we have $\Psi^{\wedge}K \in \tilde{M}_\theta[(p_1';(p_2';(p_3']$. On the other hand, since $K \in \tilde{M}_\theta[(p_1';(p_2';(p_3']$, K is bounded function. Then we have $\Psi^{\wedge}K \in \ell^1\left(\hat{G}\right)$. Since $\Psi^{\wedge}K \in \ell^1\left(\hat{G}\right) \cap \tilde{M}_\theta[(p_1';(p_2';(p_3']$ and $\frac{1}{p_3'} < \frac{1}{p_1'} + \frac{1}{p_2'}$, by Proposition 2.25, we have $\Psi^{\wedge}K = 0$. Moreover, since $\Psi^{\wedge}$ is a nonzero function, we obtain $K = 0$. This completes the proof. □

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