Diameter estimate for a class of compact generalized quasi-Einstein manifolds

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Abstract: In this paper, we discuss the lower diameter estimate for a class of compact generalized quasi-Einstein manifolds which are closely related to the conformal geometry. Using the Bochner formula and the Hopf maximum principle, we get a gradient estimate for the potential function of the manifold. Based on the gradient estimate, we get the lower diameter estimate for this class of generalized quasi-Einstein manifolds.

Key words: Generalized quasi-Einstein manifolds, lower diameter estimate, the Bochner formula, maximum principle

1. Introduction
To extend the notion of quasi-Einstein, Catino [3] introduced the concept of generalized quasi-Einstein manifold. Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold with \(n \geq 3\). If there exist three smooth functions \(f\), \(\beta\) and \(\lambda\) on \((M, g)\) such that the Ricci tensor satisfy

\[
\text{Ric} + \nabla^2 f - \beta df \otimes df = \lambda g,
\]

then \((M, g)\) is called a generalized quasi-Einstein manifold, where \(\nabla^2\) and \(\otimes\) denote the Hessian and the tensorial product, respectively. The function \(f\) in (1.1) is usually called potential function. If \(m\) is a positive integer and \(\beta = m^{-1}\), then \((M, g)\) is called generalized \(m\)-quasi-Einstein manifold (see [2]). Natural examples of generalized quasi-Einstein manifolds are given by Einstein manifolds, gradient Ricci solitons, gradient Ricci almost solitons and quasi-Einstein manifolds.

The classification of generalized \(m\)-quasi-Einstein manifolds is extensively studied, see for example [1, 2, 7, 9, 10, 11, 14]. Nowadays, the study of diameter estimate is an attractive topic in Riemannian geometry. Wei and Wylie [19] studied the upper diameter estimate and extended the Bonnet-Myers theorem to the Riemannian manifold with Bakry-Emery curvature bounded from below. Limoncu [12, 13], Soylu [16] and Tadano [17] improved the upper diameter estimate in [19]. Futaki and Sano [5] obtained a lower diameter bound for compact shrinking Ricci soliton. Futaki and Li [4] improved the diameter estimate in [5]. Wang [18] got a lower diameter bound for compact \(\tau\)-quasi-Einstein manifold. Hu, Mao and Wang [8] got a lower diameter estimate for compact generalized quasi-Einstein manifold satisfying \(\lambda = \lambda(f)\), \(\lambda'(t) \geq 0\) and \([f^2 + \frac{2}{\lambda(t)}] \lambda'(t)^2 \geq 0\).

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold with \(n \geq 3\). If there exist smooth functions \(f\)
and \( \lambda \) on \((M, g)\) such that the Ricci tensor satisfy
\[
\text{Ric} + \nabla^2 f - \frac{1}{2 - n} df \otimes df = \lambda g,
\] (1.2)
then \( M \) is called generalized \((2 - n)\)-quasi-Einstein manifold in this paper. Generalized \((2 - n)\)-quasi-Einstein manifolds are closely related to the conformal metric \( \bar{g} = e^{-\frac{2}{n} f} g \) which is important in conformal geometry. Jauregui and Wylie [11] got the classification of generalized quasi-Einstein manifolds admitting a conformal diffeomorphism using the metric \( \bar{g} = e^{-\frac{2}{n} f} g \). Remark 3.3 in [11] shows that the conformal metric \( \bar{g} = e^{-\frac{2}{n} f} g \) is an Einstein metric if and only if \((M, g)\) is a generalized \((2 - n)\)-quasi-Einstein manifold. Catino [3] gave a local characterization of generalized quasi-Einstein manifolds with harmonic weyl tensor using \( \bar{g} = e^{-\frac{2}{n} f} g \).

Ribeiro and Tenenblat [15] provided a complete classification of generalized \((2 - n)\)-quasi-Einstein manifolds satisfying (1.2) with \( \lambda = 0 \).

Motivated by [8, 10, 11, 15, 18], we study the lower diameter estimate for generalized \((2 - n)\)-quasi-Einstein manifolds satisfying (1.2). As far as we know, the study of the diameter estimate for nontrivial generalized quasi-Einstein manifolds is very few up to now. Since \( \lambda \) is a function in generalized quasi-Einstein manifolds, the diameter estimate of generalized quasi-Einstein manifolds is much more difficult than that of quasi-Einstein manifolds. To overcome this difficult, we need to use some new skills.

2. Some basic lemmas for generalized \((2 - n)\)-quasi-Einstein manifolds

Generalized \((2 - n)\)-quasi-Einstein manifolds are closely related to the conformal metric \( \bar{g} = e^{-\frac{2}{n} f} g \) which is important in conformal geometry. In this section, we give some basic lemmas for generalized \((2 - n)\)-quasi-Einstein manifolds.

**Lemma 2.1** If \((M, g)\) is a generalized \((2 - n)\)-quasi-Einstein manifold satisfying (1.2), then there exists a constant \( \beta \) such that the following equality holds
\[
\Delta f - |\nabla f|^2 + (n - 2)\lambda - \beta e^{\frac{2}{n} f} = 0.
\] (2.1)

**Proof** Similar to Lemma 2 in [2], we have
\[
\nabla R = \frac{2(1 - n)}{2 - n} \text{Ric}(\nabla f) + \frac{2}{2 - n} (R - (n - 1)\lambda) \nabla f + 2(n - 1) \nabla \lambda
\] (2.2)
and
\[
R + \Delta f - \frac{1}{2 - n} |\nabla f|^2 = n\lambda.
\] (2.3)
It follows from (1.2) that
\[
\text{Ric}(\nabla f) = \lambda \nabla f - \frac{1}{2} \nabla |\nabla f|^2 + \frac{1}{2 - n} |\nabla f|^2 \nabla f.
\] (2.4)

Putting (2.4) into (2.2), we obtain
\[
\nabla R = \frac{2(1 - n)}{2 - n} \left[ \lambda \nabla f - \frac{1}{2} \nabla |\nabla f|^2 + \frac{1}{2 - n} |\nabla f|^2 \nabla f \right] + \frac{2}{2 - n} (R - (n - 1)\lambda) \nabla f + 2(n - 1) \nabla \lambda
\]
\[ \frac{4(1-n)}{2-n} \lambda \nabla f - \frac{1-n}{2-n} \nabla |\nabla f|^2 + 2(1-n) |\nabla f|^2 \nabla f + \frac{2}{2-n} R \nabla f + 2(n-1) \nabla \lambda. \]  

(2.5)

By (2.5), we have

\[ \nabla R - \frac{2}{2-n} R \nabla f = 2(n-1)(\nabla \lambda - \frac{1-n}{2-n} \lambda \nabla f) - \frac{1-n}{2-n} (\nabla |\nabla f|^2 - \frac{2}{2-n} |\nabla f|^2 \nabla f). \]  

(2.6)

According to (2.6), we arrive at

\[ \nabla (Re^{\frac{2}{n-2}f}) = 2(n-1) \nabla (\lambda e^{\frac{2}{n-2}f}) - \frac{1-n}{2-n} \nabla(|\nabla f|^2 e^{\frac{2}{n-2}f}). \]

Therefore, we conclude that there exists a constant \( \beta \) such that

\[ Re^{\frac{2}{n-2}f} - 2(n-1)(\lambda e^{\frac{2}{n-2}f}) + \frac{1-n}{2-n} (|\nabla f|^2 e^{\frac{2}{n-2}f}) = -\beta. \]

Thus

\[ R - 2(n-1) \lambda + \frac{1-n}{2-n} |\nabla f|^2 + \beta e^{\frac{2}{n-2}f} = 0. \]  

(2.7)

On the other hand, by (2.3) we have

\[ R = -\Delta f + \frac{1}{2-n} |\nabla f|^2 + n \lambda. \]  

(2.8)

Putting (2.8) into (2.7), we conclude that (2.1) is true.

\[ \square \]

**Lemma 2.2** Suppose that \((M, g)\) is a compact generalized \((2-n)\)-quasi-Einstein manifold satisfying (1.2), \(\lambda_{\text{max}} = \max_{x \in M} \lambda(x)\). If \(\beta > 0\), then \(\lambda_{\text{max}} > 0\).

**Proof** Suppose that \(x_0 \in M\) is the maximum point of \(f\). Then \(\Delta f(x_0) \leq 0\) and \(\nabla f(x_0) = 0\). Since \(\beta > 0\), by (2.1) we have

\[ (n-2) \lambda_{\text{max}} \geq (n-2) \lambda(x_0) \geq \beta e^{\frac{2}{n-2}f(x_0)} > 0. \]

The proof of Lemma 2.2 is complete.

\[ \square \]

### 3. Gradient estimate

To consider the lower diameter estimate for compact generalized \((2-n)\)-quasi-Einstein manifold \(M\) satisfying (1.2), we need to get a gradient estimate for \(h = e^{\alpha f}\).

**Lemma 3.1** Let \(h = e^{\alpha f}\). If \(\beta\) is the constant in Lemma 2.1, then the following equality holds

\[ \triangle |\nabla h|^2 = 2|\nabla^2 h|^2 + (2 + \frac{1}{\alpha}) \frac{\nabla |\nabla h|^2 \nabla h}{h} - (2 + 2 \alpha) \frac{|\nabla h|^4}{h^2} - 2(n-2) \alpha \lambda |\nabla h|^2 + 2 \lambda |\nabla h|^2 \]

\[ + 2|1 + \frac{2}{(2-n)\alpha} \alpha^{\frac{1}{2}} \nabla h \nabla |\nabla h|^2 + 2 \alpha (2-n) \alpha |\nabla h|^4}{h^2} - 2 \alpha (2-n) \lambda \nabla h \nabla \lambda h \]  

(3.1)

on generalized \((2-n)\)-quasi-Einstein manifolds.
Proof Direct calculation shows that
\[ \Delta h = \alpha^2 e^{\alpha f} |\nabla f|^2 + \alpha e^{\alpha f} \Delta f = h(\alpha^2 |\nabla f|^2 + \alpha \Delta f). \] (3.2)

Since \( \nabla h = ah \nabla f \), by (2.1) and (3.2) we conclude that
\[ \Delta h = \left(1 + \frac{1}{\alpha}\right)|\nabla h|^2 h - (n - 2)\alpha \lambda h + \alpha \beta h^{\left(1 + \frac{2}{n-\alpha}\right)}. \] (3.3)

Therefore
\[ 2\nabla \Delta h \cdot \nabla h = 2\nabla \left[ \left(1 + \frac{1}{\alpha}\right)|\nabla h|^2 h - (n - 2)\alpha \lambda h + \alpha \beta h^{\left(1 + \frac{2}{n-\alpha}\right)} \right] \nabla h \]
\[ = \left(2 + \frac{2}{\alpha}\right) \frac{\nabla |\nabla h|^2 \nabla h}{h} - \left(2 + \frac{2}{\alpha}\right) \frac{|\nabla h|^4}{h^2} - 2(n - 2)\alpha \lambda |\nabla h|^2 \]
\[ + 2\left(1 + \frac{2}{(2 - n)\alpha}\right) \alpha \beta h^{\left(1 + \frac{2}{n-\alpha}\right)} |\nabla h|^2 - 2\alpha(n - 2)h \nabla \lambda \nabla h. \] (3.4)

Since \( \nabla h = ah \nabla f \), by (1.2) we have
\[ 2\text{Ric}(\nabla h, \nabla h) = 2\lambda |\nabla h|^2 + \frac{2 + 2(2 - n)\alpha}{\alpha^2(2 - n)} \frac{|\nabla h|^4}{h^2} - \frac{\nabla |\nabla h|^2 \nabla h}{ah}. \] (3.5)

On the other hand, according to the Bochner formula we have
\[ \triangle |\nabla h|^2 = 2|\nabla^2 h|^2 + 2\nabla \Delta h \cdot \nabla h + 2\text{Ric}(\nabla h, \nabla h). \] (3.6)

Putting (3.4) and (3.5) into (3.6), we conclude that (3.1) is true. \( \square \)

In the following, we always suppose that \( M \) is a compact generalized \((2 - n)\)-quasi-Einstein manifold, \( \lambda_{\text{max}} = \max_{x \in M} \lambda(x) \), \( h = e^{\alpha f} \), \( D = \{ x \in M; \nabla h(x) \neq 0 \} \), \( \beta \) is the constant in Lemma 2.1 and
\[ F(h) = |\nabla h|^2 + (n - 2)\alpha \lambda_{\text{max}} h^2 - \frac{(2 - n)\alpha^2}{1 + (2 - n)\alpha} \beta h^{\left(1 + \frac{2}{n-\alpha}\right)}. \] (3.7)

Lemma 3.2 If \( \alpha > 0 \) and \( \beta \geq 0 \), then there exists a smooth vector field \( X \) on \( M \) such that
\[ \Delta F \geq \nabla F \cdot X + \frac{[\alpha(n - 2) - 1]^2}{\alpha^2(2 - n)(n - 1)} \frac{|\nabla h|^4}{h^2} + 2\lambda |\nabla h|^2 - 2\alpha(n - 2)\lambda |\nabla h|^2 \]
\[ - 2\alpha(n - 2)h \nabla \lambda \nabla h + \left[\frac{2}{n - 1} \left(1 + \frac{1}{\alpha}\right)^2 - (2 + \frac{2}{\alpha}) + \frac{2 + 2(2 - n)\alpha}{\alpha^2(2 - n)}\right] \frac{|\nabla h|^4}{h^2}. \] (3.8)

holds on \( D \).

Proof Direct calculation shows that
\[ \nabla F = \nabla |\nabla h|^2 + 2(n - 2)\alpha \lambda_{\text{max}} h \nabla h - 2\alpha \beta h^{\left(1 + \frac{2}{n-\alpha}\right)} \nabla h. \] (3.9)
Therefore, we have
\[ \Delta F = \Delta |\nabla h|^2 + 2(n - 2)\alpha \lambda_{\text{max}} h |\nabla h|^2 + 2(n - 2)\alpha \lambda_{\text{max}} h \triangle h - 2\alpha \beta h^{(1 + \frac{2}{\alpha - 1})} \triangle h - 2[1 + \frac{2}{(2-n)\alpha}] \alpha \beta h^{\frac{2}{\alpha - 1}} |\nabla h|^2. \] (3.10)

For the purpose of convenience, we let
\[ G(h) = (n - 2)\alpha \lambda_{\text{max}} h - \alpha \beta h^{(1 + \frac{2}{\alpha - 1})}, \quad L(h) = (n - 2)\alpha \lambda h - \alpha \beta h^{(1 + \frac{2}{\alpha - 1})}. \] (3.11)

Putting (3.1) into (3.10), we obtain
\[ \Delta F = 2|\nabla^2 h|^2 + (2 + \frac{1}{\alpha}) \frac{\nabla |\nabla h|^2 \nabla h}{h} - (2 + \frac{2}{\alpha}) \frac{|\nabla h|^4}{h^2} - 2(n - 2)\alpha |\nabla h|^2 + 2\alpha |\nabla h|^2 \]
\[ + \frac{2 + 2(2-n)\alpha}{\alpha^2(2-n)} \frac{|\nabla h|^4}{h^2} - 2\alpha(2-n)h \nabla \nabla h + 2(n - 2)\alpha \lambda_{\text{max}} h |\nabla h|^2 + 2G(h) \triangle h. \] (3.12)

Consider a point \( O \in D \). Rotating the orthonormal frame at \( O \) so that \( |\nabla h|(O) = h_1(O) \neq 0 \). According to (2.10) in [18], we have
\[ 2|\nabla^2 h|^2 + (2 + \frac{1}{\alpha}) \frac{\nabla |\nabla h|^2 \nabla h}{h} \geq \frac{2n}{n - 1} h_1^2 \triangle h - \frac{4}{n - 1} h_1 \triangle h + \frac{2}{n - 1} (\triangle h)^2 + \frac{4\alpha + 2}{\alpha} h_1 |\nabla h|^2. \] (3.13)

According to (3.9), we get
\[ h_1 = \frac{\nabla F \nabla h}{2 |\nabla h|^2} - (n - 2)\alpha \lambda_{\text{max}} h + \alpha \beta h^{(1 + \frac{2}{\alpha - 1})} = \frac{\nabla F \nabla h}{2 |\nabla h|^2} - G(h). \] (3.14)

Putting (3.14) into (3.13), we conclude that there exists a smooth vector field \( X \) on \( M \) such that
\[ 2|\nabla^2 h|^2 + 2 + \frac{1}{\alpha} \frac{\nabla |\nabla h|^2 \nabla h}{h} \geq \nabla F \cdot X + \frac{2n}{n - 1} [G(h)]^2 + \frac{4}{n - 1} G(h) \triangle h + \frac{2}{n - 1} (\triangle h)^2 - \frac{4\alpha + 2}{\alpha} |\nabla h|^2 G(h). \] (3.15)

On the other hand, by (3.3) we have
\[ 2G(h) \triangle h + \frac{2n}{n - 1} [G(h)]^2 + \frac{4}{n - 1} G(h) \triangle h + \frac{2}{n - 1} (\triangle h)^2 \]
\[ = \frac{2n + 2}{n - 1} G(h) [(1 + \frac{1}{\alpha}) \frac{|\nabla h|^2}{h} - L(h)] + \frac{2n}{n - 1} [G(h)]^2 + \frac{2}{n - 1} [(1 + \frac{1}{\alpha}) \frac{|\nabla h|^2}{h} - L(h)]^2 \]
\[ = \frac{2n + 2}{n - 1} G(h) (1 + \frac{1}{\alpha}) \frac{|\nabla h|^2}{h} - \frac{2n + 2}{n - 1} G(h) L(h) + \frac{2n}{n - 1} [G(h)]^2 + \frac{2}{n - 1} (1 + \frac{1}{\alpha})^2 \frac{|\nabla h|^4}{h^2} \]
\[ - \frac{4}{n - 1} (1 + \frac{1}{\alpha}) \frac{|\nabla h|^2}{h} L(h) + \frac{2}{n - 1} [L(h)]^2. \] (3.16)

Since \( G(h) \geq L(h) \), then
\[ \frac{2n}{n - 1} [G(h)]^2 - \frac{2n + 2}{n - 1} G(h) L(h) + \frac{2}{n - 1} [L(h)]^2 \]
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\[= \frac{2n}{n-1} [G(h)]^2 - \frac{2n}{n-1} G(h)L(h) + \frac{2}{n-1}[L(h)]^2 - \frac{2}{n-1}G(h)L(h)\]

\[= \frac{2n}{n-1} G(h)[G(h) - L(h)] - \frac{2}{n-1}L(h)[G(h) - L(h)]\]

\[= \frac{2}{n-1} [G(h) - L(h)][nG(h) - L(h)] \geq 0. \quad (3.17)\]

By (3.16) and (3.17), we obtain

\[2G(h)\Delta h + \frac{2n}{n-1}[G(h)]^2 + \frac{4}{n-1}G(h)\Delta h + \frac{2}{n-1}(\Delta h)^2\]

\[\geq \frac{2n+2}{n-1} G(h)(1 + \frac{1}{\alpha})|\nabla h|^2 + \frac{2n+2}{n-1} G(h)(1 + \frac{1}{\alpha})^2|\nabla h|^4 - \frac{4}{n-1}(1 + \frac{1}{\alpha})^2|\nabla h|^4 L(h). \quad (3.18)\]

According to (3.12), (3.15) and (3.18), we get

\[\nabla F \geq \nabla F \cdot X - \frac{4n+2}{\alpha} \frac{|\nabla h|^2}{h} G(h) + \frac{2n+2}{n-1} G(h)(1 + \frac{1}{\alpha}) \frac{|\nabla h|^2}{h} + \frac{4}{n-1}(1 + \frac{1}{\alpha})^2 \frac{|\nabla h|^4}{h^2}\]

\[- \frac{4}{n-1}(1 + \frac{1}{\alpha}) \frac{|\nabla h|^2}{h} L(h) - (2 + \frac{2n}{\alpha}) \frac{|\nabla h|^2}{h^2} - 2(n-2)\alpha |\nabla h|^2 + \frac{2 + 2(2-n)\alpha}{\alpha^2(2-n)} \frac{|\nabla h|^4}{h^2}\]

\[+ 2\alpha |\nabla h|^2 - 2\alpha(n-2)\nabla \lambda \nabla h + 2(n-2)\alpha \lambda_{\max} h |\nabla h|^2. \quad (3.19)\]

Since \(G(h) \geq L(h), \alpha > 0\) and \(\beta > 0\), then

\[- \frac{4\alpha + 2}{\alpha} G(h) + \frac{2n+2}{n-1} G(h)(1 + \frac{1}{\alpha}) - \frac{4}{n-1}(1 + \frac{1}{\alpha})L(h)] \frac{|\nabla h|^2}{h} + 2(n-2)\alpha \lambda_{\max} h |\nabla h|^2\]

\[= -2G(h) \frac{|\nabla h|^2}{h} + \frac{4}{n-1}(1 + \frac{1}{\alpha})[G(h) - L(h)] \frac{|\nabla h|^2}{h} + 2(n-2)\alpha \lambda_{\max} h |\nabla h|^2\]

\[= 2\alpha \beta h^{\frac{2}{(n-2)}} |\nabla h|^2 + \frac{4}{n-1}(1 + \frac{1}{\alpha})[G(h) - L(h)] \frac{|\nabla h|^2}{h} \geq 0. \quad (3.20)\]

By (3.20) and (3.19), we conclude that (3.8) is true. \[\square\]

**Lemma 3.3** Suppose that \(M\) is a compact generalized \((2-n)\)-quasi-Einstein manifold, \(\alpha > 0\) and \(\beta > 0\). If \(\nabla \lambda \nabla f \leq 0\) holds on \(M\), then there exists a constant \(\delta > 0\) such that if \(|\alpha - \frac{1}{n-2}| < \delta\) then

\[\Delta F \geq \nabla F \cdot X - \alpha(n-2)\nabla \lambda \nabla h \quad (3.21)\]

holds on \(D\).
Proof Since $M$ is a compact generalized $(2-n)$-quasi-Einstein manifold, then it is obvious that
\[
\lim_{n \to \frac{1}{n^2}} \left[ \frac{2}{n-1} \left(1 + \frac{1}{\alpha} \right)^2 - \left(2 + \frac{2}{\alpha} \right) + \frac{2 + 2(2-n)\alpha}{\alpha^2(2-n)} \right] \frac{|\nabla h|^4}{h^2} = 0
\]
and
\[
\lim_{n \to \frac{1}{n^2}} \left[ 2\lambda |\nabla h|^2 - 2\alpha(n-2)\lambda |\nabla h|^2 \right] = 0.
\]
Since $\nabla^2 f \leq 0$, then $\alpha(n-2)h \nabla^2 f \leq 0$. Therefore, there exists a constant $\delta > 0$ such that if $|\alpha - \frac{1}{n^2}| < \delta$ then
\[
\left[ \frac{2}{n-1} \left(1 + \frac{1}{\alpha} \right)^2 - \left(2 + \frac{2}{\alpha} \right) + \frac{2 + 2(2-n)\alpha}{\alpha^2(2-n)} \right] \frac{|\nabla h|^4}{h^2} + 2\lambda |\nabla h|^2 - 2\alpha(n-2)\lambda |\nabla h|^2 - \alpha(n-2)h \nabla^2 f \geq 0. \tag{3.22}
\]
By (3.22) and (3.8), we conclude that (3.21) is true. \qed

Lemma 3.4 Suppose that $M$ is a compact generalized $(2-n)$-quasi-Einstein manifold, $\alpha > 0$ and $\beta \geq 0$. If $\nabla^2 f \leq 0$ and $\nabla \lambda \neq 0$ holds on $M$, then there exists a constant $\delta > 0$ such that if $|\alpha - \frac{1}{n^2}| < \delta$ then
\[
|\nabla h|^2(x) \leq G(h(x_0)) - G(h(x)) \tag{3.23}
\]
holds for all $x \in M$, where $x_0$ is the maximum point of $F(h(x))$ on $M$ and
\[
G(h) = (n-2)\alpha \lambda_{\max} h^2 - \frac{(2-n)\alpha^2}{1+(2-n)\alpha} \beta h^{2+\frac{2}{n-2}}. \tag{3.24}
\]

Proof By Lemma 3.3, there exists a constant $\delta > 0$ such that if $|\alpha - \frac{1}{n^2}| < \delta$ then (3.21) holds. If $x_0 \in D$, then there exists a neighborhood $U$ of $x_0$ so that $U \subset D$. Moreover, $x_0$ is the maximum point of $F(h(x))$ on $U$. By Lemma 3.3, we conclude that $\Delta F \geq \nabla F \cdot X$ holds on $U$. Therefore, by the Hopf maximum principle in [6] we conclude that $F$ is constant on $U$. Since $\Delta F(x_0) \leq 0$, $\nabla F(x_0) = 0$, $\nabla \lambda \nabla f \leq 0$ and $\nabla \lambda \neq 0$, (3.21) tells us that $\nabla h(x_0) = 0$, which is a contradiction with $x_0 \in D$. Therefore, $x_0$ is not in $D$, which means that $\nabla h(x_0) = 0$. Thus $|\nabla h|^2(x) + G(h(x)) \leq G(h(x_0))$. \qed

4. Diameter estimate and main result
In this section, we consider the lower diameter estimate for compact generalized $(2-n)$-quasi-Einstein manifold $M$ using the gradient estimate obtained in Lemma 3.4. If $\beta \leq 0$ and $\lambda \geq 0$, by (2.1) and the Maximum principle in [6] we conclude that $M$ is an Einstein manifold. Under this consideration, we only discuss the diameter estimate for compact generalized $(2-n)$-quasi-Einstein manifold with $\beta > 0$. The main result of this paper is

Theorem 4.1 Suppose that $M$ is a compact generalized $(2-n)$-quasi-Einstein manifold satisfying (1.2). Let $\omega_f = \max_{x \in M} f(x) - \min_{x \in M} f(x)$ and
\[
d_1 = \frac{1}{\sqrt{\lambda_{\max} - \lambda_{\min} e^{\frac{\pi}{2\omega_f}}}} \left( \frac{\pi}{2} - \arcsin e^{\frac{\omega_f}{\pi}} \right), \quad d_2 = \frac{1}{\sqrt{\lambda_{\max} e^{\frac{\pi}{2\omega_f}}}} \left( \frac{\pi}{2} - \arcsin e^{\frac{\omega_f}{\pi}} \right).
\]
If \( \nabla \lambda \nabla f \leq 0 \) holds on \( M \), \( n \geq 3 \), \( \beta > 0 \), then the diameter of \( M \) satisfies

\[
diam M \geq \max \{d_1, d_2\}.
\]

**Proof** Suppose that \( \delta \) is the constant mentioned in Lemma 3.4. Let \( |\alpha - \frac{1}{n-2}| < \delta \). Then (3.23) holds. Assume that \( x_0 \) is the maximum point of \( F(h(x)) \) on \( M \). By Lemma 3.4, for all \( x \in M \),

\[
\mathcal{G}(h(x)) \leq |\nabla h|^2(x) + \mathcal{G}(h(x)) \leq \mathcal{G}(h(x_0)).
\]

Therefore, \( x_0 \) is the maximum point of \( \mathcal{G}(h(x)) \). According to (3.24), we get

\[
\mathcal{G}'(t) = -2[(n-2)\alpha \lambda_{\max} t - \alpha \beta t^{(1 + \frac{2}{n-2})}].
\]

Let \( x_1 \) and \( x_2 \) be the maximum and minimum points of \( h(x) \) on \( M \), respectively. Similar to the proof of Theorem 1.3 in [18], we conclude that \( x_0 = x_1 \) or \( x_0 = x_2 \). We only consider the case that \( x_0 = x_1 \). Choosing a minimizing geodesic \( \gamma \) jointing \( x_1 \) and \( x_2 \). Let \( h_1 = h(x_1) \), \( h_2 = h(x_2) \). Similar to the proof of Theorem 1.3 in [18], we have

\[
diam M \geq \int_{h_2}^{h_1} \frac{dh}{\sqrt{\mathcal{G}(h_1)} - \mathcal{G}(h(x))} = \int_{h_2}^{h_1} \frac{dh}{\sqrt{(n-2)\alpha \lambda_{\max} (h_1^2 - h_2^2) - \frac{(2-n)\alpha^2 \beta}{1+(2-n)\alpha} h_1^{(\frac{4}{n-2})} - h^{(\frac{4}{n-2})}}} = \int_{\frac{h_2}{h_1}}^{\frac{h_1}{h_2}} \frac{d\sigma}{\sqrt{(n-2)\alpha \lambda_{\max} (1 - \sigma^2) + \frac{(2-n)\alpha^2 \beta}{1+(2-n)\alpha} h_1^{\frac{2}{n-2}} - 1 + \sigma^{(\frac{4}{n-2})}}}.
\]

Since \( \beta > 0 \), by Lemma 2.2 we have \( \lambda_{\max} > 0 \). If \( \alpha > \frac{1}{n-2} \), then \( 1 + (2 - n)\alpha < 0 \). Therefore, by (4.1) we conclude that

\[
diam M \geq \frac{1}{\sqrt{(n-2)\alpha \lambda_{\max}}} \int_{\frac{h_2}{h_1}}^{\frac{h_1}{h_2}} \frac{d\sigma}{\sqrt{1 - \sigma^2}} = \frac{1}{\sqrt{(n-2)\alpha \lambda_{\max}}} \left[ \frac{\pi}{2} - \arcsin e^{-\sigma w} \right].
\]

Let \( \alpha \to \frac{1}{n-2} \) from the right side of \( \frac{1}{n-2} \). Then we have

\[
diam M \geq \frac{1}{\sqrt{\lambda_{\max}}} \left[ \frac{\pi}{2} - \arcsin e^{-\sigma w} \right].
\]

If \( 0 < \alpha < \frac{1}{n-2} \), we consider the function

\[
S(\sigma) = -[1 + (2 - n)\alpha](1 - \sigma^2) + \sigma^{(\frac{4}{n-2})} - 1
\]

on \( \left[ \frac{h_2}{h_1}, 1 \right] \). Since \( 0 < \alpha < \frac{1}{n-2} \), then \( n - 2)\alpha < 1 \), \( \sigma^{\frac{2}{(n-2)\alpha}} \geq 1 \), \( 2 + \frac{2}{(n-2)\alpha} < 0 \). Therefore, we have

\[
S'(\sigma) = \left[ 2 + \frac{2}{(n-2)\alpha} \right] \sigma^{\frac{2}{(n-2)\alpha}} - (n-2)\alpha |\sigma| < 0.
\]

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Thus, we conclude that $S(\sigma) > S(1) = 0$. Then

$$1 - \sigma^{(2 + \frac{2}{2-n})} < -[1 + (2 - n)\alpha](1 - \sigma^2).$$

(4.3)

Since $\beta > 0$ and $1 + (2 - n)\alpha > 0$, by (4.3) we get

$$\frac{(n-2)\alpha^2 \beta}{1 + (2 - n)\alpha} h_1^{\frac{2}{2-n}} [1 - \sigma^{(2 + \frac{2}{2-n})}] < -(n - 2)\alpha^2 \beta h_1^{\frac{2}{2-n}} (1 - \sigma^2).$$

(4.4)

Since $\alpha > 0$, then $x_2$ is a minimum point of $f(x)$. Let $\lambda_{\min} = \min_{x \in M} \lambda(x)$. According to (2.2) we have

$$\beta e^{\frac{2}{2-n}f(x_2)} \geq (n - 2)\lambda(x_2) \geq (n - 2)\lambda_{\min}.$$  

(4.5)

By (4.5), we obtain

$$\beta h_1^{\frac{2}{2-n}} = \beta e^{\frac{2}{2-n}w_f} e^{\frac{2}{2-n}f(x_2)} \geq (n - 2)\lambda_{\min} e^{\frac{2}{2-n}w_f}.$$  

(4.6)

According to (4.1), (4.4) and (4.5), we have

$$\text{diam} M \geq \frac{1}{\sqrt{(n - 2)\alpha \lambda_{\max} - (n - 2)^2 \alpha^2 \lambda_{\min} e^{\frac{2}{2-n}w_f}}}. \left(\pi - \arcsin e^{-\alpha w_f}\right).$$

Let $\alpha \rightarrow \frac{1}{n-2}$ from the left side of $\frac{1}{n-2}$. Then

$$\text{diam} M \geq \frac{1}{\sqrt{\lambda_{\max} - \lambda_{\min} e^{\frac{2}{2-n}w_f}}} \left(\pi - \arcsin e^{w_f}\right).$$

(4.7)

According to (4.2) and (4.7), we conclude that $\text{diam} M \geq \max\{d_1, d_2\}$. \hfill \Box

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**References**


