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Secondary constructions of (non)weakly regular plateaued functions over finite fields

Sihem MESNAGER[1,2,3], **Ferruh ÖZBUDAK**[4], **Ahmet SINAK**[2,5,*]
[1]Department of Mathematics, University of Paris VIII, 93526 Saint-Denis, France
[2]LAGA UMR 7539, CNRS, Sorbonne Paris Cité, University of Paris XIII, 93430 Villetaneuse, France
[3]Telecom Paris, 91120 Palaiseau, France
[4]Department of Mathematics, Institute of Applied Mathematics, Middle East Technical University, 06800 Ankara, Turkey
[5]Department of Mathematics and Computer Science, Necmettin Erbakan University, 42090 Konya, Turkey

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Abstract: Plateaued (vectorial) functions over finite fields have diverse applications in symmetric cryptography, coding theory, and sequence theory. Constructing these functions is an attractive research topic in the literature. We can distinguish two kinds of constructions of plateaued functions: secondary constructions and primary constructions. The first method uses already known functions to obtain new functions while the latter do not need to use previously constructed functions to obtain new functions. In this work, the first secondary constructions of (non)weakly regular plateaued (vectorial) functions are presented over the finite fields of odd characteristics. We also introduce some recursive constructions of (non)weakly regular plateaued p-ary functions by using already known such functions. We obtain nontrivial plateaued functions from the previously known trivial plateaued (partially bent) functions in the proposed construction methods.

Key words: p-ary functions, plateaued functions, (non)weakly regular plateaued functions, secondary construction

1. Introduction

Bent Boolean functions were defined by Rothaus [25] in the 1970s, as an extension, partially bent Boolean functions were defined by Carlet [3] in 1993, and later as an extension of these notions, the notion of plateaued Boolean functions has been proposed by Zheng and Zhang [27] in 1999. For more information on plateaued Boolean functions, the reader is referred to [1, 2, 13, 18]. The notion of plateaued Boolean functions has been generalized for arbitrary finite fields and then several researchers have studied these functions, in particular on their characterizations (see, [4, 7, 11, 15, 17, 19–21]). We refer the reader to the book [18] (Chapter 16) for a recent survey on plateaued functions in any characteristic. In the literature, although several secondary construction methods of bent and (vectorial) Boolean functions have been introduced (see, [6, 13, 24, 26] and the main references [1, 5, 18]), only a few secondary construction methods of (vectorial) bent functions have been obtained in [8–10, 23]. Designing plateaued functions via secondary construction methods is a relatively hard problem. In even characteristics, only a few secondary constructions of plateaued functions are known (see

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in [1, 2, 13, 14, 26]). More specifically, in [2], Carlet has proved that the direct sum preserves plateauedness, and the building of bent functions without extension of the number of variables can be directly applied to plateaued functions. Furthermore, Hodžić et al. [13] have recently proposed several secondary constructions of bent and plateaued Boolean functions using the concept of composite representation of Boolean functions. A very recent book [1] written by Carlet is an excellent reference for this research topic. In odd characteristics, the situation is more unclear because of the complex structure, and the gap between the design of bent functions and plateaued functions is much more significant. This paper's ultimate goal is to decrease this gap by starting proposing the first secondary constructions of plateaued functions over finite fields in odd characteristics. Given our goal, in this paper, we generalize the secondary constructions of (vectorial) bent functions proposed in [8–10, 23] for plateaued functions over the odd characteristic finite fields. Notably, the authors have recently introduced in [21] a nontrivial subclass of plateaued functions, (non)weakly regular plateaued functions. In this paper, we study for the first time the secondary and recursive constructions of these functions over the odd characteristic finite fields.

The paper is structured as follows. Section 2 gives the basic notations and backgrounds. In Section 3, we present the direct sum and semidirect sum secondary constructions of plateaued (vectorial) functions over the odd characteristic finite fields. In Section 4, we propose recursive construction methods of plateaued (vectorial) functions over the odd characteristic finite fields. We construct recursively new plateaued functions from already known ones. We conclude the paper in Section 5.

2. Preliminaries

For any set T, $\#T$ denotes the cardinality of T and $T^\star = T \setminus \{0\}$. The finite field with p^n elements is denoted by $\mathbb{F}_{p^n}$ for a positive integer n and an odd prime p. The extension field $\mathbb{F}_{p^n}$ can be viewed as a vector space $\mathbb{F}_p^n$ of n-dimension over $\mathbb{F}_p$. The absolute trace of $\beta \in \mathbb{F}_{p^n}$ over $\mathbb{F}_p$ is described as $\mathrm{Tr}_p^{p^n}(\beta) = \sum_{i=0}^{n-1} \beta^{p^i}$. Let f be a function from $\mathbb{F}_p^n$ to $\mathbb{F}_p$. A function $f : \mathbb{F}_p^n \to \mathbb{F}_p$ is uniquely defined by

$$f(x) = \sum_{\mathbf{u} \in \mathbb{F}_p^n} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} = \sum_{\mathbf{u} \in \mathbb{F}_p^n} a_{\mathbf{u}} x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_p^n$, $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{F}_p^n$ and $a_{\mathbf{u}} \in \mathbb{F}_p$. This is the algebraic normal form (ANF) of f. We use the notations $\widehat{f}$ and $Spec(f)$ to denote the Walsh–Hadamard transform (WHT) and Walsh spectrum of f (the value distribution of the WHT), respectively. The WHT of f is described as

$$\widehat{f}(\omega) = \sum_{x \in \mathbb{F}_p^n} \xi_p^{\,f(x) - \omega \cdot x},$$

where the dot "$\cdot$" is an inner product in $\mathbb{F}_p^n$ and ξ_p is a primitive p-th root of unity. The Walsh spectrum of f is defined as the multiset $Spec(f) = \{\widehat{f}(\omega) : \omega \in \mathbb{F}_p^n\}$. Furthermore, the Walsh support of f is defined as the set $\mathrm{Supp}(\widehat{f}) = \{\omega \in \mathbb{F}_{p^n} : \widehat{f}(\omega) \neq 0\}$. We call a function f is balanced over $\mathbb{F}_p$ if $\#\{x \in \mathbb{F}_{p^n} : f(x) = i\} = p^{n-1}$ for each $i \in \mathbb{F}_p$; otherwise, f is unbalanced. For a balanced function f, we have $\widehat{f}(0) = 0$.

A function $f : \mathbb{F}_{p^n} \to \mathbb{F}_p$ is said to be bent if all its WHT coefficients satisfy $|\widehat{f}(\omega)|^2 = p^n$, and f is s-plateaued if $|\widehat{f}(\omega)|^2 \in \{0, p^{n+s}\}$ for all $\omega \in \mathbb{F}_{p^n}$, where s is an integer with $0 \le s \le n$. We point out that a 0-plateaued function is a bent function. Meanwhile, as an extension of Boolean bent functions, the notion of partially bent Boolean functions was already defined by Carlet [3]. A function f is called partially bent function if $f(x+a)-f(x)$ is either balanced or constant for all $a \in \mathbb{F}_{p^n}$ (see also [10]). It is clear that f is bent iff $f(x+a)-f(x)$ is balanced for all $a \in \mathbb{F}_{p^n}^{\star}$. An element $a \in \mathbb{F}_{p^n}$ is called a linear structure of f if $f(x+a)-f(x)$ is constant. From the definition, partially bent functions have nonzero linear structures while bent functions do not have. It is well-known that the class of partially bent functions is a trivial subclass of the class of plateaued functions. We can say that partially bent functions are the trivial plateaued functions that have nonzero linear structures, on the other hand, plateaued functions without nonzero linear structures are nontrivial plateaued functions.

For an s-plateaued $f : \mathbb{F}_{p^n} \to \mathbb{F}_p$, we have $\#\mathrm{Supp}(\widehat{f}) = p^{n-s}$ from the Parseval identity. From [15], the WHT coefficients of an s-plateaued f satisfy

$$\widehat{f}(\omega) = \begin{cases} \pm p^{\frac{n+s}{2}} \xi_p^{f^\star(\omega)}, 0 & \text{if } n+s \text{ is odd and } p \equiv 1 \pmod 4 \text{ or } n+s \text{ is even,} \\ \pm i p^{\frac{n+s}{2}} \xi_p^{f^\star(\omega)}, 0 & \text{if } n+s \text{ is odd and } p \equiv 3 \pmod 4, \end{cases} \tag{2.1}$$

where $f^\star$ is a p-ary function over $\mathbb{F}_{p^n}$ with $f^\star(\omega) = 0$ for $\omega \notin \mathrm{Supp}(\widehat{f})$. Indeed, for every $\omega \in \mathrm{Supp}(\widehat{f})$, we can rewrite

$$\widehat{f}(\omega) = \begin{cases} \pm p^{\frac{n+s}{2}} \xi_p^{f^\star(\omega)}, & \text{if } p^n \equiv 1 \pmod 4, \\ \pm i p^{\frac{n+s}{2}} \xi_p^{f^\star(\omega)}, & \text{if } p^n \equiv 3 \pmod 4. \end{cases}$$

Motivated by this expression, Mesnager et al. [21] have described the concept of (non)weakly regular plateaued functions that is a nontrivial subclass of the plateaued functions in odd characteristics.

Definition 2.1 *[21] Let $f : \mathbb{F}_{p^n} \to \mathbb{F}_p$ be an s-plateaued function with $0 \le s \le n$. Then, f is said to be a weakly regular s-plateaued function if there exists a complex number u with $|u| = 1$ such that $\widehat{f}(\omega) \in \{0, u p^{\frac{n+s}{2}} \xi_p^{f^\star(\omega)}\}$ for all $\omega \in \mathbb{F}_{p^n}$, where $f^\star$ is a p-ary function over $\mathbb{F}_{p^n}$ with $f^\star(\omega) = 0$ for $\omega \notin \mathrm{Supp}(\widehat{f})$; otherwise, f is called nonweakly regular.*

It is safe to say that an s-plateaued f is weakly regular if $\widehat{f}(\omega) = u p^{(n+s)/2} \xi_p^{f^\star(\omega)}$ for all $\omega \in \mathrm{Supp}(\widehat{f})$, where $|u| = 1$ (in fact, $u \in \{\pm 1, \pm i\}$ is independent from ω) and $f^\star$ is a p-ary function over $\mathrm{Supp}(\widehat{f})$. Here, in particular, if the sign $u = 1$ for all ω, we say f is regular s-plateaued. If the sign $u \in \{\pm 1, \pm i\}$ depends on ω, then f is a nonweakly regular s-plateaued. By (2.1), regular s-plateaued functions can only exist when $n+s$ is even or $n+s$ is odd and $p \equiv 1 \pmod 4$.

For a vectorial function $F : \mathbb{F}_p^n \to \mathbb{F}_p^m$, for all $\lambda \in \mathbb{F}_{p^m}^\star$, its nonzero component functions $F_\lambda : \mathbb{F}_{p^n} \to \mathbb{F}_p$ are defined as $F_\lambda(x) = \mathrm{Tr}_p^{p^m}(\lambda F(x))$; equivalently, for all nonzero $\lambda \in \mathbb{F}_p^m$, $F_\lambda : \mathbb{F}_p^n \to \mathbb{F}_p$ are defined as $F_\lambda(x) = \lambda \cdot F(x)$ for every $x \in \mathbb{F}_p^n$, where the dot "$\cdot$" represents an inner product in $\mathbb{F}_p^m$. The extended WHT of vectorial F is defined as

$$\widehat{F}(\omega, \lambda) = \sum_{x \in \mathbb{F}_p^n} {\xi_p}^{\lambda \cdot F(x) - \omega \cdot x}$$

for all $\omega \in \mathbb{F}_p^n$ and nonzero $\lambda \in \mathbb{F}_p^m$. A function F is said to be vectorial bent if its extended WHT coefficients satisfy $|\widehat{F}(\omega, \lambda)|^2 = p^n$, and F is vectorial s-plateaued if $|\widehat{F}(\omega, \lambda)|^2 \in \{0, p^{n+s}\}$ for all $\omega \in \mathbb{F}_{p^n}$ and $\lambda \in \mathbb{F}_{p^m}^\star$, where s is an integer with $0 \le s \le n$. Note that a vectorial 0-plateaued function is a vectorial bent function. In other words, we say that a function F is vectorial bent if F_λ is bent for every $\lambda \in \mathbb{F}_{p^m}^\star$, and F is vectorial plateaued if F_λ is plateaued with possibly different amplitudes for every $\lambda \in \mathbb{F}_{p^m}^\star$, in particular, F is vectorial s-plateaued function if F_λ is s-plateaued for every $\lambda \in \mathbb{F}_{p^m}^\star$.

We can give a regularity property for vectorial bent and plateaued functions, which introduces a subclass of the class of these functions.

Definition 2.2 *A function $F : \mathbb{F}_p^n \to \mathbb{F}_p^m$ is vectorial (weakly) regular bent if F_λ is (weakly) regular bent for every $\lambda \in \mathbb{F}_{p^m}^\star$; otherwise, F is said to be a vectorial nonweakly regular bent function.*

Definition 2.3 *A function $F : \mathbb{F}_p^n \to \mathbb{F}_p^m$ is vectorial (weakly) regular plateaued if F_λ is (weakly) regular plateaued with possibly different amplitudes for every $\lambda \in \mathbb{F}_{p^m}^\star$; otherwise, F is said to be a vectorial nonweakly regular plateaued function. In particular, F is vectorial (weakly) regular s-plateaued if F_λ is (weakly) regular s-plateaued for every $\lambda \in \mathbb{F}_{p^m}^\star$.*

3. Secondary constructions of (non)weakly regular plateaued functions

This section introduces the first secondary constructions of plateaued functions over the odd characteristic finite fields: the so-called direct sum and semidirect sum of (vectorial) functions.

We first give the semidirect sum construction that was proposed for bent functions in [9] and vectorial bent functions in [8] over the odd characteristic finite fields. The semidirect sum of two (vectorial) functions is defined as follows.

Definition 3.1 *Let $f : \mathbb{F}_{p^n} \to \mathbb{F}_{p^k}$, $g : \mathbb{F}_{p^m} \to \mathbb{F}_{p^k}$ and $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ be vectorial functions. Then, the semidirect sum $h : \mathbb{F}_{p^n} \times \mathbb{F}_{p^m} \to \mathbb{F}_{p^k}$ of functions f and g is defined as*

$$h(x, y) = f(x) + g(y + F(x)). \tag{3.1}$$

We derive from [9, Theorem 1] the expression of the WHT of the semidirect sum of p-ary functions when $k = 1$.

Proposition 3.2 *Let $f : \mathbb{F}_{p^n} \to \mathbb{F}_p$, $g : \mathbb{F}_{p^m} \to \mathbb{F}_p$ and $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ be functions. Then for $(a, b) \in \mathbb{F}_{p^{n+m}}$, the WHT of the semidirect sum h of the form of (3.1) is given by*

$$\widehat{h}(a, b) = \widehat{F_{b,f}}(a)\widehat{g}(b), \tag{3.2}$$

where $F_{b,f}$ is the function from $\mathbb{F}_{p^n}$ to $\mathbb{F}_p$ defined as $F_{b,f}(x) = f(x) + F_b(x)$ for all $b \in \mathbb{F}_{p^m}$.

Remark 3.3 *One can easily observe that $(a, b) \in \mathrm{Supp}(\widehat{h})$ if and only if $a \in \mathrm{Supp}(\widehat{F_{b,f}})$ and $b \in \mathrm{Supp}(\widehat{g})$.*

The semidirect sum construction can be used to design plateaued functions. We derive from Proposition 3.2 the following characterizations.

Theorem 3.4 *Let $f : \mathbb{F}_{p^n} \to \mathbb{F}_p$ and $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ be two functions. Let s_1 and s_2 be two integers with $0 \le s_1 \le n$ and $0 \le s_2 \le m$. Let $g : \mathbb{F}_{p^m} \to \mathbb{F}_p$ be s_2-plateaued and $F_{b,f}$ be defined as in Proposition 3.2. Then, the semidirect sum h of the form of (3.1) is $(s_1 + s_2)$-plateaued iff $F_{b,f}$ is s_1-plateaued for all $b \in \mathrm{Supp}(\widehat{g})$.*

Proof Since g is s_2-plateaued, we have $|\widehat{g}(b)|^2 = p^{m+s_2}$ for all $b \in \mathrm{Supp}(\widehat{g})$. By Proposition 3.2, $F_{b,f}$ is s_1-plateaued for all $b \in \mathrm{Supp}(\widehat{g})$ iff $|\widehat{h}(a,b)|^2 = p^{n+m+s_1+s_2}$ for all $(a,b) \in \mathrm{Supp}(\widehat{h})$. The proof is complete. □

To be more precise, we can construct further (non)weakly regular plateaued functions over a larger size field from given ones over smaller size fields.

Corollary 3.5 *Let $f : \mathbb{F}_{p^n} \to \mathbb{F}_p$ and $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ be two functions. Let $g : \mathbb{F}_{p^m} \to \mathbb{F}_p$ be weakly regular plateaued and $F_{b,f}$ be plateaued defined as in Proposition 3.2 with the same amplitude for all $b \in \mathrm{Supp}(\widehat{g})$. Let h be the semidirect sum of the form of (3.1). Then,*

i) h is weakly regular plateaued if $F_{b,f}$ is weakly regular plateaued with the same sign u for all $b \in \mathrm{Supp}(\widehat{g})$.

ii) h is nonweakly regular plateaued if $F_{b,f}$ is weakly regular plateaued with the sign u_b that depends on b for $b \in \mathrm{Supp}(\widehat{g})$.

iii) h is nonweakly regular plateaued if $F_{b,f}$ is nonweakly regular plateaued for some $b \in \mathrm{Supp}(\widehat{g})$.

The semi-direct sum construction can be also used to design vectorial plateaued functions. From [8, Theorem 5], the WHT of the semidirect sum of two vectorial functions can be expressed as follows.

Proposition 3.6 *Let $f : \mathbb{F}_{p^n} \to \mathbb{F}_{p^k}$, $g : \mathbb{F}_{p^m} \to \mathbb{F}_{p^k}$ and $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ be vectorial functions. Then for $(a,b) \in \mathbb{F}_{p^{n+m}}$ and $\lambda \in \mathbb{F}_{p^k}^\star$, the WHT of the component function h_λ of the semidirect sum h of the form of (3.1) is given by*

$$\widehat{h_\lambda}(a,b) = \widehat{F_{b,\lambda}}(a)\widehat{g_\lambda}(b),$$

where $F_{b,\lambda}$ is the function from $\mathbb{F}_{p^n}$ to $\mathbb{F}_p$ defined as $F_{b,\lambda}(x) = f_\lambda(x) + F_b(x)$ for $b \in \mathbb{F}_{p^m}$.

We can derive from Proposition 3.6 the following characterizations of plateaued functions.

Theorem 3.7 *Let $f : \mathbb{F}_{p^n} \to \mathbb{F}_{p^k}$ and $g : \mathbb{F}_{p^m} \to \mathbb{F}_{p^k}$ be two vectorial plateaued functions, and $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ be a vectorial function. Let $F_{b,\lambda}$ be defined as in Proposition 3.6. Then, the semidirect sum h of the form of (3.1) is vectorial plateaued iff $F_{b,\lambda}$ is plateaued for all $b \in \mathrm{Supp}(\widehat{g_\lambda})$ and $\lambda \in \mathbb{F}_{p^k}^\star$.*

Proof From assumption, the component function g_λ is plateaued for every $\lambda \in \mathbb{F}_{p^k}^\star$. By Proposition 3.6, for every $\lambda \in \mathbb{F}_{p^k}^\star$, h_λ is plateaued iff $F_{b,\lambda}$ is plateaued for every $b \in \mathrm{Supp}(\widehat{g_\lambda})$. This completes the proof. □

Corollary 3.8 *Let $f : \mathbb{F}_{p^n} \to \mathbb{F}_{p^k}$ and $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ be two functions. Let $g : \mathbb{F}_{p^m} \to \mathbb{F}_{p^k}$ be vectorial s_2-plateaued and let $F_{b,\lambda}$ be given as in Proposition 3.6. Then, the semidirect sum h of the form of (3.1) is vectorial (s_1+s_2)-plateaued iff $F_{b,\lambda}$ is s_1-plateaued for every $b \in \mathrm{Supp}(\widehat{g_\lambda})$ and $\lambda \in \mathbb{F}_{p^k}^\star$.*

Proof For every $\lambda \in \mathbb{F}_{p^k}^\star$, since g_λ is s_2-plateaued, we have $|\widehat{g_\lambda}(b)|^2 = p^{m+s_2}$ for all $b \in \mathrm{Supp}(\widehat{g_\lambda})$. By Proposition 3.6, for every $\lambda \in \mathbb{F}_{p^k}^\star$, we have $|\widehat{F_{b,\lambda}}(a)|^2 = p^{n+s_1}$ for all $a \in \mathrm{Supp}(\widehat{F_{b,\lambda}})$ and $b \in \mathrm{Supp}(\widehat{g_\lambda})$ iff the component function $|\widehat{h_\lambda}(a,b)|^2 = p^{n+m+s_1+s_2}$ for all $(a,b) \in \mathrm{Supp}(\widehat{h_\lambda})$; equivalently, $F_{b,\lambda}$ is s_1-plateaued iff h is vectorial (s_1+s_2)-plateaued. This completes the proof. □

We now give the direct sum construction of plateaued (vectorial) functions, which can be seen as a special case of semidirect sum construction of these functions. To be more precise, when we assume that $F(x) = 0$ in the semidirect sum construction given in Definition 3.1, we can obtain the direct sum construction given in Definition 3.9.

Definition 3.9 *Let $f : \mathbb{F}_{p^n} \to \mathbb{F}_{p^k}$ and $g : \mathbb{F}_{p^m} \to \mathbb{F}_{p^k}$ be two vectorial functions. Then, the direct sum of f and g is the function h from $\mathbb{F}_{p^n} \times \mathbb{F}_{p^m}$ to $\mathbb{F}_{p^k}$ defined as $h(x, y) = f(x) + g(y)$ for $x \in \mathbb{F}_{p^n}$ and $y \in \mathbb{F}_{p^m}$.*

The direct sum construction is the well-known first secondary construction for Dillon's Boolean bent functions [12] and Rothaus [25]. Such a construction has been extended first by Tan et al. [23] for p-ary bent functions and then by Carlet [2] for Boolean plateaued functions.

We emphasize that the WHT of the direct sum h of p-ary functions f and g (when $k = 1$) can be easily expressed as, for $(a, b) \in \mathbb{F}_p^n \times \mathbb{F}_p^m$,

$$\widehat{h}(a, b) \quad = \widehat{f}(a)\widehat{g}(b). \tag{3.3}$$

In Proposition 3.2, when $F(x)$ is the zero function, namely, its component functions $F_b(x)$ are the zero functions for all $b \in \mathbb{F}_{p^m}$, we will have $F_{b,f}(x) = f(x)$ for all $b \in \mathbb{F}_{p^m}$ and $x \in \mathbb{F}_{p^n}$. To be more precise, for all $b \in \mathbb{F}_{p^m}$ the functions $F_{b,f}$ are the same function f. Hence, the WHT of the semidirect sum h in (3.2) is equal to the WHT of the direct sum h in (3.3).

From (3.3), we make a preliminary observation: $(a, b) \in \mathrm{Supp}(\widehat{h})$ if and only if $a \in \mathrm{Supp}(\widehat{f})$ and $b \in \mathrm{Supp}(\widehat{g})$, namely, $\mathrm{Supp}(\widehat{h}) = \mathrm{Supp}(\widehat{f}) \times \mathrm{Supp}(\widehat{g})$. Hence, one can derive directly from (3.3) the following corollary.

Corollary 3.10 *Let $f : \mathbb{F}_{p^n} \to \mathbb{F}_p$ be s_1-plateaued and $g : \mathbb{F}_{p^m} \to \mathbb{F}_p$ be s_2-plateaued, where $0 \le s_1 \le n$ and $0 \le s_2 \le m$. Then, the direct sum h of f and g is an $(s_1 + s_2)$-plateaued function over $\mathbb{F}_{p^{n+m}}$.*

As an example of this simplest secondary construction method, we give the following example.

Example 3.11 *Let $f : \mathbb{F}_3^4 \to \mathbb{F}_3$ and $g : \mathbb{F}_3^5 \to \mathbb{F}_3$ be two weakly regular 1-plateaued functions defined as $f(x_1, \ldots, x_4) = x_1^2 - x_2x_4^2 - x_3x_4^2 - x_2x_4 + x_3x_4 - x_2$ and $g(y_1, \ldots, y_5) = y_1^2 + y_1y_3 + y_1y_5^2 + y_2y_5^2 + y_1y_5 + y_2y_5 + y_4y_5 + y_2$. Then the direct sum $h : \mathbb{F}_3^4 \times \mathbb{F}_3^5 \to \mathbb{F}_3$ of f and g defined as $h(x_1, \ldots, x_9) = x_1^2 - x_2x_4^2 - x_3x_4^2 - x_2x_4 + x_3x_4 - x_2 + x_5^2 + x_5x_7 + x_5x_9^2 + x_6x_9^2 + x_5x_9 + x_6x_9 + x_8x_9 + x_6$ is the weakly regular 2-plateaued balanced function without nonzero linear structures over $\mathbb{F}_3^9$.*

One can use the direct sum to construct new (non)weakly regular plateaued functions over a larger field from two given ones over smaller fields.

Remark 3.12 *Let $f : \mathbb{F}_{p^n} \to \mathbb{F}_p$ be a nonweakly regular s_1-plateaued and $g : \mathbb{F}_{p^m} \to \mathbb{F}_p$ be a weakly regular s_2-plateaued, where $0 \le s_1 \le n$ and $0 \le s_2 \le m$. Then, the direct sum h of f and g is a nonweakly regular ($s_1 + s_2$)-plateaued function over $\mathbb{F}_{p^{n+m}}$.*

For $(a, b) \in \mathbb{F}_{p^n} \times \mathbb{F}_{p^m}$, the WHT of the nonzero component function of vectorial direct sum h of the vectorial functions f and g can be easily expressed as $\widehat{h_\lambda}(a, b) = \widehat{f_\lambda}(a)\widehat{g_\lambda}(b)$ for every $\lambda \in \mathbb{F}_{p^k}^\star$. This suggests the following direct sum construction of vectorial functions.

Remark 3.13 *Let* $f:\mathbb{F}_{p^n}\to\mathbb{F}_{p^k}$ *and* $g:\mathbb{F}_{p^m}\to\mathbb{F}_{p^k}$ *be two vectorial plateaued functions. Then, the direct sum* $h:\mathbb{F}_{p^n}\times\mathbb{F}_{p^m}\to\mathbb{F}_{p^k}$ *is a vectorial plateaued function. In other words, for every* $\lambda\in\mathbb{F}_{p^k}^{\star}$, *its component function* $h_\lambda(x,y)=f_\lambda(x)+g_\lambda(y)$ *is a plateaued function for* $x\in\mathbb{F}_{p^n}$ *and* $y\in\mathbb{F}_{p^m}$.

4. Recursive constructions of (non)weakly regular plateaued functions

In this section, we construct (non)weakly regular plateaued functions from given ones by using the recursive construction methods proposed in [9, 10] and a new method.

Leander and McGuire [16] introduced in 2009 the first construction method of Boolean bent functions from given near-bent functions. This method was then generalized in [10] to arbitrary characteristics by obtaining bent functions from given near-bent functions as follows.

Let $f_i:\mathbb{F}_{p^n}\to\mathbb{F}_p$ be a function for all $i\in\{0,\ldots,p-1\}$ such that for $0\le j\ne k\le p-1$, $\mathrm{Supp}(\widehat{f_j})\cap\mathrm{Supp}(\widehat{f_k})=\emptyset$. A function $h:\mathbb{F}_{p^n}\times\mathbb{F}_p\to\mathbb{F}_p$ is defined as

$$h(x,y)=(p-1)\sum_{i=0}^{p-1}\frac{y(y-1)\cdots(y-(p-1))}{y-i}f_i(x). \tag{4.1}$$

One can easily observe that $h(x,y)=f_y(x)$ for each $y\in\mathbb{F}_p$. The WHT of p-ary function h at $(a,b)\in\mathbb{F}_{p^n}\times\mathbb{F}_p$ was given in [10] by

$$\widehat{h}(a,b)=\sum_{y\in\mathbb{F}_p}\xi_p^{-by}\widehat{f_y}(a).$$

We now use the recursive construction presented above to produce the first construction of (non)weakly regular plateaued functions from some already known such functions with pairwise disjoint Walsh supports. Let $f_i:\mathbb{F}_{p^n}\to\mathbb{F}_p$ be s-plateaued functions with $1\le s\le n$ for all $i\in\{0,\ldots,p-1\}$ such that $\mathrm{Supp}(\widehat{f_j})\cap\mathrm{Supp}(\widehat{f_k})=\emptyset$ for $0\le j\ne k\le p-1$. Notice that $\mathrm{Supp}(\widehat{f_i})=\{a\in\mathbb{F}_{p^n}:\widehat{f_i}(a)\ne 0\}$ and $\#\mathrm{Supp}(\widehat{f_i})=p^{n-s}$ for all $i\in\{0,\ldots,p-1\}$. Hence, we have

$$\#\left(\bigcup_{i=0}^{p-1}\mathrm{Supp}(\widehat{f_i})\right)=\sum_{i=0}^{p-1}\#\mathrm{Supp}(\widehat{f_i})=p^{n+1-s},$$

and the set $\bigcup_{i=0}^{p-1}\mathrm{Supp}(\widehat{f_i})$ is the proper subset of $\mathbb{F}_{p^n}$ for an integer $s>1$. Then, the Walsh support of $h:\mathbb{F}_{p^n}\times\mathbb{F}_p\to\mathbb{F}_p$ is given by

$$\mathrm{Supp}(\widehat{h})=\left\{(a,b)\in\mathbb{F}_{p^n}\times\mathbb{F}_p:\ a\in\bigcup_{i=0}^{p-1}\mathrm{Supp}(\widehat{f_i})\ \text{and}\ b\in\mathbb{F}_p\right\}=\bigcup_{i=0}^{p-1}\mathrm{Supp}(\widehat{f_i})\times\mathbb{F}_p,$$

and $\#\mathrm{Supp}(\widehat{h})=p^{n+1-(s-1)}$. This implies that $(a,b)\in\mathrm{Supp}(\widehat{h})$ iff $a\in\mathrm{Supp}(\widehat{f_i})$ for exactly one $i\in\mathbb{F}_p$ and $b\in\mathbb{F}_p$. Then, for $(a,b)\in\mathrm{Supp}(\widehat{h})$, we have

$$\widehat{h}(a,b)=\xi_p^{-bi}\widehat{f_i}(a)=u_{i,a}p^{\frac{n+s}{2}}\xi_p^{h^\star(a,b)}, \tag{4.2}$$

where $|u_{i,a}| = 1$ (in fact, $u_{i,a}$ depends on f_i and a) and $h^\star(a,b) = f_i^\star(a) - bi$, where $a \in \mathrm{Supp}(\widehat{f_i})$ for each $i \in \mathbb{F}_p$. It is worth noting that the Walsh spectrum of h is given by

$$Spec(h) = \bigcup_{i=0}^{p-1} \bigcup_{b \in \mathbb{F}_p} \xi_p^{-bi} Spec(f_i).$$

The above observations prove the following theorem that constructs an $(s-1)$-plateaued function over $\mathbb{F}_{p^{n+1}}$ from some given s-plateaued functions over $\mathbb{F}_{p^n}$, where s is an integer with $1 \le s \le n$.

Theorem 4.1 *Let s be an integer with $1 \le s \le n$, and f_i be an s-plateaued for every $i \in \{0, \dots, p-1\}$ with pairwise disjoint Walsh supports. Then, h of the form of (4.1) is the $(s-1)$-plateaued function over $\mathbb{F}_{p^{n+1}}$.*

As an example of this construction method, we give the following construction of the ternary nontrivial plateaued function.

Example 4.2 *For $i = 0, 1, 2$, let $f_i : \mathbb{F}_3^3 \to \mathbb{F}_3$ be weakly regular 2-plateaued (in fact, partially bent) functions defined as $f_0(x) = 2x_1^2 + 2x_2, f_1(x) = 2x_1^2$, and $f_2(x) = 2x_1^2 + 2x_2 + x_3$, where $x = (x_1, x_2, x_3) \in \mathbb{F}_3^3$, such that $\mathrm{Supp}(\widehat{f_j}) \cap \mathrm{Supp}(\widehat{f_k}) = \emptyset$ for $0 \le j \ne k \le 2$. Then the function $h : \mathbb{F}_3^3 \times \mathbb{F}_3 \to \mathbb{F}_3$ defined as*

$$\begin{aligned} h(x_1, \dots, x_4) &= 2(x_4-1)(x_4-2)f_0(x) + 2x_4(x_4-2)f_1(x) + 2x_4(x_4-1)f_2(x) \\ &= 2x_2x_4^2 - x_3x_4^2 - x_2x_4 + x_3x_4 - x_1^2 - x_2 \end{aligned}$$

is the weakly regular 1-plateaued unbalanced function without nonzero linear structures over $\mathbb{F}_3^4$.

A non-weakly regular plateaued function can be also derived from some given weakly regular plateaued functions.

Corollary 4.3 *Let s be an integer with $1 \le s \le n$, and f_i be a (weakly) regular s-plateaued for every $i \in \{0, \dots, p-1\}$ with pairwise disjoint Walsh supports. Then, the $(s-1)$-plateaued h of the form of (4.1) is (weakly) regular if the sign u_i is constant for every i; otherwise, h is nonweakly regular over $\mathbb{F}_{p^{n+1}}$.*

Proof From (4.2), for all $(a,b) \in \mathrm{Supp}(\widehat{h})$, we have $\widehat{h}(a,b) = u_i p^{(n+s)/2} \xi_p^{h^\star(a,b)}$, where $|u_i| = 1$ (it may depend on f_i) and $h^\star(a,b) = f_i^\star(a) - bi$ is a p-ary function. This completes the proof. □

Remark 4.4 *In Corollary 4.3, if f_i is non-weakly regular for some i, then h is nonweakly regular.*

The following construction given in [9, 10] combines n variable p bent functions to construct an $(n+2)$ variable bent function. Let $f_i : \mathbb{F}_{p^n} \to \mathbb{F}_p$ be a function for all $i \in \{0, \dots, p-1\}$ and $h : \mathbb{F}_{p^n} \times \mathbb{F}_{p^2} \to \mathbb{F}_p$ be the function defined as

$$h(x, y_1, y_2) = f_{y_2}(x) + y_1 y_2 \tag{4.3}$$

where $x \in \mathbb{F}_{p^n}$ and $(y_1, y_2) \in \mathbb{F}_{p^2}$. For $(a,b) \in \mathbb{F}_{p^n} \times \mathbb{F}_{p^2}$, where $b = (i,j) \in \mathbb{F}_{p^2}$, the WHT of h is given by

$$\widehat{h}(a,i,j) = p\xi_p^{-ij} \widehat{f}_i(a). \tag{4.4}$$

Here, we can easily observe that $(a,i,j) \in \mathrm{Supp}(\widehat{h})$ iff $a \in \mathrm{Supp}(\widehat{f_i})$ for $i \in \mathbb{F}_p$. Below, we consider this construction for p-ary plateaued functions.

Theorem 4.5 *Let s be an integer with $0 \leq s \leq n$, and let h be the function of the form of (4.3). Then, h is s-plateaued over $\mathbb{F}_{p^{n+2}}$ iff f_i is s-plateaued over $\mathbb{F}_{p^n}$ for all $i \in \{0, \ldots, p-1\}$. In particular, h is (weakly) regular iff f_i is (weakly) regular with the same sign u for all $i \in \{0, \ldots, p-1\}$.*

Proof For all $(a,i,j) \in \mathrm{Supp}(\widehat{h})$, we have $|\widehat{h}(a,i,j)|^2 = p^2|\widehat{f_i}(a)|^2$ from (4.4). This suggests that h is $(n+2)$ variable s-plateaued iff f_i is n variable s-plateaued for all $i \in \{0, \ldots, p-1\}$. Similarly, the second case follows from (4.4). □

As seen in the following example, Theorem 4.5 can construct nontrivial plateaued functions from the known trivial plateaued (partially bent) functions.

Example 4.6 *Let $f_i : \mathbb{F}_3^3 \to \mathbb{F}_3$, for $i = 0, 1, 2$, be regular 1-plateaued (in fact, partially bent) functions defined as $f_0(x) = 2x_1^2 - x_1x_3 - x_2$, $f_1(x) = 2x_1^2 - x_1x_3 - x_2 - x_1$ and $f_2(x) = 2x_1^2 - x_1x_3 + x_2$, where $x = (x_1, x_2, x_3) \in \mathbb{F}_3^3$. Then, the function $h : \mathbb{F}_3^3 \times \mathbb{F}_3^2 \to \mathbb{F}_3$ defined as*

$$\begin{aligned} h(x_1, \ldots, x_5) &= 2(x_5-1)(x_5-2)f_0(x) + 2x_5(x_5-2)f_1(x) + 2x_5(x_5-1)f_2(x) + x_4x_5 \\ &= x_1x_5^2 + x_2x_5^2 - x_1^2 + x_1x_5 - x_1x_3 - x_2x_5 + x_4x_5 - x_2 \end{aligned}$$

is the regular 1-plateaued balanced function without nonzero linear structures over $\mathbb{F}_3^5$.

A nonweakly regular plateaued function can be derived from given weakly regular plateaued functions based on the above construction.

Corollary 4.7 *Let h be the function of the form of (4.3), and f_i be s-plateaued for all $i \in \{0, \ldots, p-1\}$. If f_i is weakly regular with the sign u_i that depends on f_i, then h is nonweakly regular plateaued. If f_i is nonweakly regular for some i, then h is nonweakly regular plateaued.*

Proof From (4.4), for all $(a,i,j) \in \mathrm{Supp}(\widehat{h})$, we have $\widehat{h}(a,i,j) = u_i p^{(n+2+s)/2} \xi_p^{h^\star(a,i,j)}$, where $|u_i| = 1$ (in fact, u_i depends on f_i) and $h^\star(a,i,j) = f_i^\star(a) - ij$ is a p-ary function; equivalently, h is nonweakly regular plateaued. The second one is obvious, thereby completing the proof. □

The above construction have recently generalized for vectorial functions in [8] as follows. Let $f_i : \mathbb{F}_{p^k}^n \to \mathbb{F}_{p^k}$ be a vectorial function for all $i \in \mathbb{F}_{p^k}$ and $h : \mathbb{F}_{p^k}^{n+2} \to \mathbb{F}_{p^k}$ be the function defined as

$$h(x, y_1, y_2) = f_{y_2}(x) + y_1 y_2 \tag{4.5}$$

where $x \in \mathbb{F}_{p^k}^n$ and $y_1, y_2 \in \mathbb{F}_{p^k}$. For $(a,i,j) \in \mathbb{F}_{p^k}^{n+2}$, where $a \in \mathbb{F}_{p^k}^n$ and $i, j \in \mathbb{F}_{p^k}$, the WHT of the component function h_λ is given by

$$\widehat{h_\lambda}(a,i,j) = p^k \xi_p^{-\mathrm{Tr}^k(ij\lambda^{-1})} (\widehat{\lambda \cdot f_{i\lambda^{-1}}})(a)$$

for every nonzero $\lambda \in \mathbb{F}_{p^k}$. We hence give the following construction of vectorial plateaued functions.

Theorem 4.8 *Let $f_i : \mathbb{F}_{p^k}^n \to \mathbb{F}_{p^k}$ be a vectorial plateaued function for all $i \in \mathbb{F}_{p^k}$ and let $h : \mathbb{F}_{p^k}^{n+2} \to \mathbb{F}_{p^k}$ be the function of the form of (4.5). Then, h is the vectorial plateaued function. In particular, if f_i is vectorial s-plateaued for all $i \in \mathbb{F}_{p^k}$, then h is the vectorial s-plateaued function.*

As an extension of the above construction, we introduce the following new construction method for bent and plateaued functions over $\mathbb{F}_p$. Given n variable plateaued functions, the following new construction produces $(n+4)$ variable one plateaued function. Let $f_i : \mathbb{F}_{p^n} \to \mathbb{F}_p$ be a function for all $i \in \{0,\dots,p-1\}$ and $h : \mathbb{F}_{p^n} \times \mathbb{F}_{p^4} \to \mathbb{F}_p$ be the function defined as

$$h(x,y) = f_{y_4}(x) + y_1y_2 + y_3y_4, \tag{4.6}$$

where $y = (y_1,y_2,y_3,y_4) \in \mathbb{F}_p^4$. The WHT of h can be stated as follows.

Theorem 4.9 *For $(a,b) \in \mathbb{F}_{p^n} \times \mathbb{F}_{p^4}$, where $b = (b_1,b_2,b_3,b_4)$, the WHT of h of the form of (4.6) can be expressed as $\widehat{h}(a,b) = p^2\xi_p^{-b_1b_2-b_3b_4}\widehat{f_{b_3}}(a)$.*

Proof For $(a,b) \in \mathbb{F}_{p^n} \times \mathbb{F}_{p^4}$, the WHT of h is given by

$$\begin{aligned}
\widehat{h}(a,b) &= \sum_{x,y\in\mathbb{F}_{p^n}} \xi_p^{h(x,y)-(a,b)\cdot(x,y)} \\
&= \sum_{x\in\mathbb{F}_{p^n}} \xi_p^{-a\cdot x}\left(\sum_{y_2\in\mathbb{F}_p}\xi_p^{-b_2y_2}\sum_{y_1\in\mathbb{F}_p}\xi_p^{y_1(y_2-b_1)}\right)\left(\sum_{y_4\in\mathbb{F}_p}\xi_p^{f_{y_4}(x)-b_4y_4}\sum_{y_3\in\mathbb{F}_p}\xi_p^{y_3(y_4-b_3)}\right) \\
&= \sum_{x\in\mathbb{F}_{p^n}} \xi_p^{f_{b_3}(x)-a\cdot x}\left(p\xi_p^{-b_1b_2}\right)\left(p\xi_p^{-b_3b_4}\right) = p^2\xi_p^{-b_1b_2-b_3b_4}\widehat{f_{b_3}}(a).
\end{aligned}$$

This completes the proof. □

Remark 4.10 *We have that $(a,b) \in \mathrm{Supp}(\widehat{h})$ iff $a \in \mathrm{Supp}(\widehat{f_{b_3}})$ and $b_3 \in \mathbb{F}_p$.*

We now present further possibilities of constructions of (non)weakly regular bent and plateaued functions based on the above construction.

Theorem 4.11 *Let h be the function of the form of (4.6). Then, h is s-plateaued over $\mathbb{F}_{p^{n+4}}$ iff f_i is s-plateaued for all $i \in \{0,\dots,p-1\}$. In particular, h is (weakly) regular iff f_i is (weakly) regular with the same sign u for all $i \in \{0,\dots,p-1\}$.*

Proof As in the proof of Theorem 4.5, this proof follows from Theorem 4.9. □

Corollary 4.12 *Let h be the function of the form of (4.6). Then, h is bent iff f_i is bent for all $i \in \{0,\dots,p-1\}$.*

As seen in the following example, Theorem 4.11 can construct nontrivial plateaued functions from the known trivial plateaued (partially bent) functions.

Example 4.13 *Let $f_i : \mathbb{F}_3^3 \to \mathbb{F}_3$, for $i = 0,1,2$, be regular 1-plateaued functions given in Example 4.6. Then, the function $h : \mathbb{F}_3^3 \times \mathbb{F}_3^4 \to \mathbb{F}_3$ defined as*

$$\begin{aligned}
h(x_1,\dots,x_7) &= 2(x_7-1)(x_7-2)f_0(x) + 2x_7(x_7-2)f_1(x) + 2x_7(x_7-1)f_2(x) + x_4x_5 + x_6x_7 \\
&= x_1x_7^2 + x_2x_7^2 + x_1x_7 - x_1^2 - x_1x_3 - x_2x_7 + x_4x_5 + x_6x_7 - x_2
\end{aligned}$$

is the regular 1-plateaued balanced function without nonzero linear structures over $\mathbb{F}_3^7$.

We can derive a nonweakly regular plateaued (resp., bent) function from given weakly regular plateaued (resp., bent) functions based on the above construction.

Corollary 4.14 *Let h be the function of the form of (4.6), and f_i be plateaued for all $i \in \{0, \ldots, p-1\}$. If f_i is weakly regular with the sign u_i that depends on i (or f_i is non-weakly regular for some i), then h is nonweakly regular plateaued.*

5. Conclusion

In this paper, inspired by the works of [8–10, 23], we generalized the secondary construction methods of bent functions for plateaued functions over the odd characteristic finite fields. We have provided for the first time the secondary constructions of (non)weakly regular plateaued (vectorial) functions. Moreover, the recursive constructions of (non)weakly regular plateaued (vectorial) functions from given ones have been introduced. We have employed the secondary and recursive constructions in this paper to provide the first constructions of (non)weakly regular plateaued (vectorial) functions. We say that this paper constructs new (non)weakly regular plateaued (vectorial) functions from already known ones over the odd characteristic finite fields. More importantly, the proposed construction methods construct nontrivial plateaued functions from the known trivial plateaued (partially bent) functions.

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