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3-Principalization over S_3-fields

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Abstract: Let $p \equiv 1 \pmod 9$ be a prime number and ζ_3 be a primitive cube root of unity. Then $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ is a pure metacyclic field with group $\mathrm{Gal}(k/\mathbb{Q}) \simeq S_3$. In the case that k possesses a 3-class group $C_{k,3}$ of type $(9,3)$, the capitulation of 3-ideal classes of k in its unramified cyclic cubic extensions is determined, and conclusions concerning the maximal unramified pro-3-extension $k_3^{(\infty)}$, that is the 3-class field tower of k, are drawn.

Key words: Maximal unramified pro-3-extension, capitulation, Galois action, pure metacyclic S_3-fields, pure cubic fields, finite 3-groups, descendant trees, presentations, relation rank, p-group generation algorithm

1. Introduction

For a prime $p \equiv 1 \pmod 9$, let $\Gamma = \mathbb{Q}(\sqrt[3]{p})$ be the pure cubic field with radicand p, and $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$, with a primitive third root of unity ζ_3, be the normal closure of Γ. Then k is a pure metacyclic field with automorphism group $\mathrm{Gal}(k/\mathbb{Q}) \simeq S_3$, the symmetric group of order 6, and, according to [3, Theorem 2.4(1), p. 258], the class number of k is divisible by 3. Most frequently, the 3-class group $C_{k,3} \simeq C_3$ is simply the cyclic group of order 3. There occur, however, interesting cases where $C_{k,3} \simeq C_9 \times C_3$ is nonelementary bicyclic with four maximal subgroups of index 3 and four second maximal subgroups of index 9. According to [1, Lemma 2.5, p. 4], the latter situation arises if and only if $C_{\Gamma,3} \simeq C_9$ and Γ is of principal factorization type α, in the sense of [3, Theorem 2.1, p. 254]. Since only little progress in determining the 3-class field tower $F_3^{(\infty)}$ of number fields F with $C_{F,3} \simeq C_9 \times C_3$ was achieved in the literature so far, we devote the present paper to the illumination of these uncharted waters by means of the S_3-fields $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ with $C_{k,3} \simeq C_9 \times C_3$.

With the aid of classical methods of algebraic number theory, we investigate the possibilities for the punctured capitulation type $\varkappa(k) = (\ker(T_{K_{1,3}/k}), \dots, \ker(T_{K_{3,3}/k}); \ker(T_{K_{4,3}/k}))$ consisting of the kernels of the transfer homomorphisms $T_{K_{i,3}/k} : C_{k,3} \to C_{K_{i,3},3}$ of 3-classes from k to its four unramified cyclic cubic extensions $K_{1,3}, \dots, K_{4,3}$, where $K_{4,3}$ is the distinguished extension corresponding to the product of the subgroups of index 9 in $C_{k,3}$, by Artin's reciprocity law.

Since Scholz and Taussky launched the capitulation problem [18], it is known that the type $\varkappa(k)$ can be used for an attempt to find the *second 3-class group* $G_2 = \mathrm{Gal}(k_3^{(2)}/k)$, i.e. the Galois group of the

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2010 *AMS Mathematics Subject Classification:* Primary 11R37, 11R29, 11R32, 11R20, 11R16, secondary 20D15, 20E22, 20F05, 20–04 second Hilbert 3-class field $k_3^{(2)}$, of k, which is a two-stage approximation of the entire 3-class tower group $G_\infty = \mathrm{Gal}(k_3^{(\infty)}/k)$. The identification of the latter is the point where innovative ideas involving the Galois action of $\mathrm{Gal}(k/\mathbb{Q}) \simeq S_3$ on G_∞ and sophisticated techniques estimating the relation rank $d_2(G_\infty)$ become mandatory. Before these theoretical means were available, fatal errors crept in when investigators tried to get the precise length ℓ_3 of the 3-class tower. The erroneous claim $\ell_3 = 2$ for $\mathbb{Q}(\sqrt{-9748})$ in [18, p. 41] was corrected by $\ell_3 = 3$ eighty years later in [5, Corollary 4.1.1, p. 775]. In the present paper, the type $\varkappa(k)$ is by far too insufficient in order to identify G_2. Here the Galois action is definitely required. Finally, the relation rank and the antitony principle for Artin patterns [16] will be employed to find G_∞ and ℓ_3 for $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$.

Let $k_3^{(1)}$ be the Hilbert 3-class field of k. When the 3-class group $C_{k,3}$ of k is of type $(9,3)$, the extension $k_3^{(1)}/k$ admits eight intermediate fields as illustrated in Figure 1.

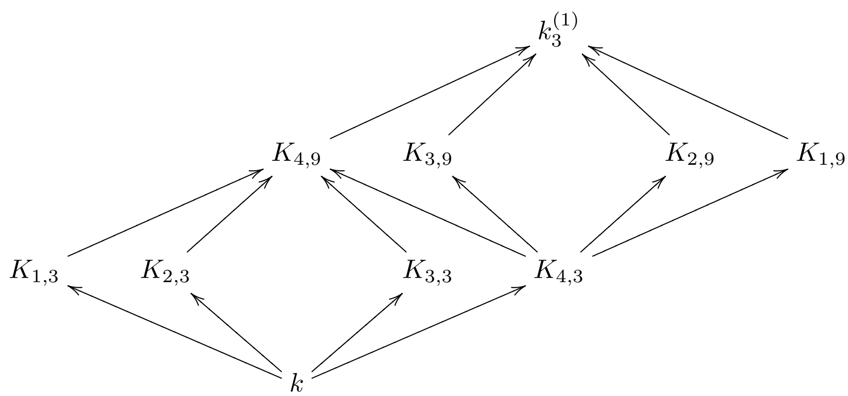

Figure 1. The unramified cubic and nonic subextensions of $k_3^{(1)}/k$.

The layout of this paper is the following. In Theorem 3.1 of Section 3, we construct the family $(K_{i,j})$ of all intermediate fields $k \subseteq K_{i,j} \subseteq k_3^{(1)}$, where $1 \le i \le 4$ and $j \in \{3, 9\}$. In Theorem 4.2 of Section 4, we investigate the capitulation of 3-ideal classes of k in the fields $K_{i,j}$. Our numerical results of subsection 5.1 has been computed with the aid of Magma [12]. For the 95 relevant cases $p < 20\,000$ in Table , the capitulation kernels $\ker(T_{K_{i,3}/k})$ of the class extension homomorphisms $T_{K_{i,3}/k} : C_{k,3} \to C_{K_{i,3},3}$ were computed and collected in the transfer kernel type $\varkappa$. As an application, we identify in subsection 5.2 the maximal unramified pro-3-extension $k_3^{(\infty)}$ of $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$. Finally, we provide computational evidence that the dominating proportion (at least 94%) of the fields k has a metabelian 3-class field tower $k_3^{(\infty)}$ with exactly two stages.

Notations:
Throughout this paper, we shall respect the usual notations as follows:

- The letter p designates a prime number congruent to 1 modulo 3;
- $\Gamma = \mathbb{Q}(\sqrt[3]{d})$: a pure cubic field, where $d \ge 2$ is a cube-free integer;
- $k_0 = \mathbb{Q}(\zeta_3)$: the cyclotomic field, where $\zeta_3 = e^{2i\pi/3}$;

- $k=\Gamma(\zeta_3)$: the normal closure of Γ;
- Γ' and Γ'': the two conjugate cubic fields of Γ, contained in k;
- $u=[E_k:E_0]$: the index of the subgroup E_0 generated by the units of intermediate fields of the extension $k/\mathbb{Q}$ in the group of units E_k of k;
- $\langle\tau\rangle=\mathrm{Gal}\,(k/\Gamma)$, $\tau^2=id$, $\tau(\zeta_3)=\zeta_3^2$ and $\tau(\sqrt[3]{d})=\sqrt[3]{d}$;
- $\langle\sigma\rangle=\mathrm{Gal}\,(k/k_0)$, $\sigma^3=id$, $\sigma(\zeta_3)=\zeta_3$ and $\sigma(\sqrt[3]{d})=\zeta_3\sqrt[3]{d}$;
- For an arbitrary algebraic number field F:
  - $C_{F,3}$: the 3-class group of F;
  - $\mathcal{O}_F$: the ring of integers of F;
  - $F_3^{(1)}$: the Hilbert 3-class field of F;
  - $[\mathcal{I}]$: the class of a fractional ideal $\mathcal{I}$ in the class group of F.

2. Preliminaries

In [11], Ismaili established that the 3-class group $C_{k,3}$ of $k=\mathbb{Q}(\sqrt[3]{d},\zeta_3)$ is of type $(3,3)$ if and only if 3 divides exactly the class number of Γ and $u=3$, where u is the units index defined in the notations, and he determined all the integers d which satisfy this property by distinguishing three types of fields k. Here, we present and discuss a detailed study of the case where the 3-class group $C_{k,3}$ is of type $(9,3)$. Let us start with Theorem 2.1 below which describe the structure of $C_{k,3}$.

Theorem 2.1 *Let Γ be a pure cubic field, k be its normal closure, $C_{k,3}$ (resp. $C_{\Gamma,3}$) the 3-class group of k (resp. Γ), and u be the units index defined in the notations, then:*

$$C_{k,3}\simeq\mathbb{Z}/9\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}\quad\Longleftrightarrow\quad[C_{\Gamma,3}\simeq\mathbb{Z}/9\mathbb{Z}\quad \textit{and}\quad u=1].$$

Proof See [1, Lemma 2.5, p. 4]. □

Therefore, Theorem 2.2 bellow classifies all the integers d such that the 3-class group $C_{k,3}$ is of type $(9,3)$.

Theorem 2.2 *Let $\Gamma=\mathbb{Q}(\sqrt[3]{d})$ be a pure cubic field, where $d\geq 2$ is a cube free integer, $k=\mathbb{Q}(\sqrt[3]{d},\zeta_3)$ its normal closure, and u the units index defined in the notations.*

1) *If the field k has a 3-class group of type $(9,3)$, then $d=p^e$, where p is a prime congruent to $1\,(\mathrm{mod}\,9)$ and $e=1$ or 2.*
2) *Conversely, if p is a prime congruent to $1\,(\mathrm{mod}\,9)$, and if 9 divides exactly the class number of $\Gamma=\mathbb{Q}(\sqrt[3]{p})$ and $u=1$, then the 3-class group of k is of type $(9,3)$.*

Proof See [1, Theorem 1.1, p. 2]. □

Therefore, let k be the special field $\mathbb{Q}(\sqrt[3]{p},\zeta_3)$, with $p \equiv 1 \pmod 9$. Calegari and Emerton [6, Lemma 5.11] proved that $\operatorname{rank} C_{k,3} = 2$ if 9 divides the class number of $\Gamma = \mathbb{Q}(\sqrt[3]{p})$. The converse of the Calegari–Emerton result is shown by Frank Gerth III in [9, Theorem 1, p. 471]. In the following, we assume that 9 divides exactly the class number of $\Gamma = \mathbb{Q}(\sqrt[3]{p})$, where $p \equiv 1 \pmod 9$. Under this assumption, $C_{k,3}$ is of type $(9,3)$ if and only if $u = 1$. Theorems 2.3 and 2.4 bellow give the generators of $C_{k,3}$.

Theorem 2.3 *Let $k = \mathbb{Q}(\sqrt[3]{p},\zeta_3)$, where p is a prime such that $p \equiv 1 \pmod 9$. Let $C_{k,3}^{+} = \{\mathcal{C} \in C_{k,3} \mid \mathcal{C}^{\tau} = \mathcal{C}\}$, $C_{k,3}^{-} = \{\mathcal{C} \in C_{k,3} \mid \mathcal{C}^{\tau} = \mathcal{C}^{-1}\}$, and $C_{k,3}^{1-\sigma} = \{\mathcal{A}^{1-\sigma} \mid \mathcal{A} \in C_{k,3}\}$. Assume that $C_{k,3}$ is of type $(9,3)$. Then:*

1. *$C_{k,3} = \langle \mathcal{A}, \mathcal{B}\rangle$, $C_{k,3}^{+} = \langle \mathcal{A}\rangle$, $C_{k,3}^{-} = \langle \mathcal{B}\rangle$, where $\mathcal{A} \in C_{k,3}$ such that $\mathcal{A}^9 = 1$, $\mathcal{A}^3 \neq 1$, and $\mathcal{B} \in C_{k,3}$ such that $\mathcal{B}^3 = 1$, $\mathcal{B} \neq 1$.*
2. *The ambiguous class group $C_{k,3}^{(\sigma)}$ of $k|k_0$ is a subgroup of $C_{k,3}^{+}$ of order 3, and $C_{k,3}^{(\sigma)} = \langle \mathcal{A}^3\rangle = \langle \mathcal{B}^{1-\sigma}\rangle$ with $\mathcal{A} \notin C_{k,3}^{(\sigma)}$.*
3. *$C_{k,3}^{-} = \langle (\mathcal{A}^2)^{\sigma-1}\rangle$.*
4. *The principal genus $C_{k,3}^{1-\sigma}$ of $C_{k,3}$ is an elementary bicyclic 3-group of type $(3,3)$, and $C_{k,3}^{1-\sigma} = C_{k,3}^{(\sigma)} \times C_{k,3}^{-} = \langle \mathcal{A}^3, \mathcal{B}\rangle$.*

Proof See [2, Proposition 3.4, pp. 9-10]. □

Theorem 2.4 *Let $k = \mathbb{Q}(\sqrt[3]{p},\zeta_3)$, where p is a prime such that $p \equiv 1 \pmod 9$. The prime 3 decomposes in* k *under the form $3\mathcal{O}_k = \mathcal{P}^2\mathcal{Q}^2\mathcal{R}^2$, where $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are prime ideals of k. Put $h = \frac{h_k}{27}$, where h_k is the class number of k. Assume 9 divides exactly the class number of $\mathbb{Q}(\sqrt[3]{p})$ and $u = 1$. If 3 is not cubic residue modulo p, then:*

1. *$[\mathcal{R}^h]$ generates $C_{k,3}^{+}$;*
2. *$C_{k,3}$ is generated by $[\mathcal{R}^h]$ and $[\mathcal{R}^h][\mathcal{P}^h]^2$, i.e. $C_{k,3} = \langle [\mathcal{R}^h]\rangle \times \langle [\mathcal{R}^h][\mathcal{P}^h]^2\rangle$.*

Proof See [2, Theorem 3.5, pp. 10-11]. □

3. Unramified cubic and nonic subextensions of $k_3^{(1)}/k$

Let $k = \mathbb{Q}(\sqrt[3]{p},\zeta_3)$, where p is a prime such that $p \equiv 1 \pmod 9$, $k_3^{(1)}$ the Hilbert 3-class field of $k_3^{(0)} = k$, $k_3^{(2)}$ the Hilbert 3-class field of $k_3^{(1)}$, and $G = \mathrm{Gal}(k_3^{(2)}/k)$. Let $C_{k,3}$ be the 3-ideal class group of k, then by class field theory, $\mathrm{Gal}(k_3^{(1)}/k) \simeq C_{k,3}$.

Assume that $C_{k,3}$ is of type $(9,3)$. In the sequel, we adopt the conventions of [16, § 4.2, pp. 76–78] concerning the normal lattice of metabelian 3-groups $G = \langle x, y\rangle$, with two generators satisfying $x^9 \in G'$ and $y^3 \in G'$, where G/G' is of type $(9,3)$. If we denote by $(H_{i,j})_i$ the family of all normal intermediate groups

$G' \subseteq H_{i,j} \subseteq G$, with $1 \le i \le 4$, $j \in \{3, 9\}$, the 3-group G has four second maximal normal subgroups of index 9 as follows:

$$H_{1,9} = \langle y, G' \rangle,\ H_{2,9} = \langle x^3 y, G' \rangle,\ H_{3,9} = \langle x^3 y^{-1}, G' \rangle,\ H_{4,9} = \langle x^3, G' \rangle,$$

and four maximal normal subgroups of index 3 as follows:

$$H_{1,3} = \langle x, G' \rangle,\ H_{2,3} = \langle xy, G' \rangle,\ H_{3,3} = \langle xy^{-1}, G' \rangle,\ H_{4,3} = \langle x^3, y, G' \rangle.$$

It should be noted that $H_{4,3} = \prod H_{i,9}$, the quotient group $H_{4,3}/G' = \langle x^3, y \rangle$ is bicyclic of type $(3, 3)$, and $H_{4,9} = \bigcap H_{i,3} = G^3 G'$ coincides with the Frattini subgroup $\Phi(G)$ of G. However, the group $H_{i,9}$ is only contained in $H_{4,3}$, for $1 \le i \le 3$. Figure 2 above illustrate these intermediate groups.

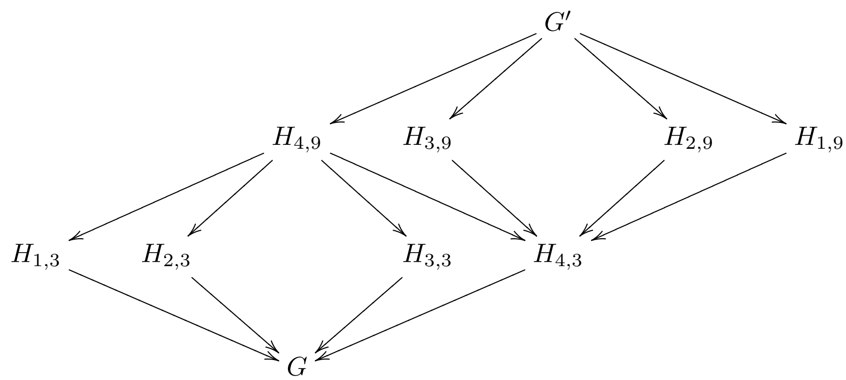

Figure 2. The group G with G/G' of type $(9, 3)$.

In the following Theorem 3.1, we determine via the Galois correspondence of $k_3^{(1)}/k$ the family $(K_{i,j})$ of all fields $k \subseteq K_{i,j} \subseteq k_3^{(1)}$, where $1 \le i \le 4$, $j \in \{3, 9\}$, satisfying $H_{i,j} = \mathrm{Gal}(k_3^{(2)}/K_{i,j})$, $H_{i,j}/G' \simeq \mathcal{N}_{K_{i,j}/k}(C_{K_{i,j},3})$, and $H_{i,j}/H'_{i,j} \simeq \mathrm{Gal}((K_{i,j})_3^{(1)}/K_{i,j}) \simeq C_{K_{i,j},3}$.

Theorem 3.1 *Let $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$, where p is a prime such that $p \equiv 1 \pmod 9$, $C_{k,3}^+ = \{\mathcal{C} \in C_{k,3} \mid \mathcal{C}^\tau = \mathcal{C}\}$, $C_{k,3}^- = \{\mathcal{C} \in C_{k,3} \mid \mathcal{C}^\tau = \mathcal{C}^{-1}\}$, and $C_{k,3}^{(\sigma)}$ be the ambiguous ideal class group of $k|k_0$.*
Assume that $C_{k,3}$ is of type $(9, 3)$. Let $C_{k,3}^+ = \langle \mathcal{A} \rangle$ and $C_{k,3}^- = \langle \mathcal{B} \rangle$, where $\mathcal{A} \in C_{k,3}$ such that $\mathcal{A}^9 = 1$, $\mathcal{A}^3 \neq 1$, and $\mathcal{B} \in C_{k,3}$ such that $\mathcal{B}^3 = 1$, $\mathcal{B} \neq 1$. Then, the extension $k_3^{(1)}/k$ admits eight intermediate extensions as follows:

1) Four unramified cyclic extensions of degree 3 denoted $K_{i,3}$, $1 \le i \le 4$, given by:

- *The field $K_{1,3}$ corresponds by class field theory to $C_{k,3}^+ = \langle \mathcal{A} \rangle$,*
- *The field $K_{2,3}$ corresponds to $\langle \mathcal{A}\mathcal{B} \rangle = \langle \mathcal{A}^\sigma \rangle$,*
- *The field $K_{3,3}$ corresponds to $\langle \mathcal{A}\mathcal{B}^2 \rangle = \langle \mathcal{A}^{\sigma^2} \rangle$,*

– *The field* $K_{4,3}$ *corresponds to the principal genus* $C_{k,3}^{1-\sigma} = \langle \mathcal{A}^3, \mathcal{B} \rangle$ *.*

Furthermore, $K_{4,3} = k(\sqrt[3]{\pi_1\pi_2^2}) = (k/k_0)^* = k\Gamma^* = k(\Gamma^\sigma)^* = k\left(\Gamma^{\sigma^2}\right)^*$*, where* $(k/k_0)^*$ *is the relative genus field of* k/k_0*,* F^* *for a number field* F *is the absolute genus field of* F*, and* π_1, π_2 *are two primes of* k_0 *such that* $p = \pi_1\pi_2$*.*

2) Three unramified cyclic extensions of degree 9 *denoted* $K_{i,9}$, $1 \leq i \leq 3$*, given by:*

– *The field* $K_{2,9}$ *corresponds by class field theory to the subgroup* $\langle \mathcal{A}^3\mathcal{B} \rangle$*,*

– *The field* $K_{3,9}$ *corresponds to the subgroup* $\langle \mathcal{A}^3\mathcal{B}^2 \rangle$*,*

– *The field* $K_{1,9}$ *corresponds to the subgroup* $C_{k,3}^{-} = \langle \mathcal{B} \rangle$*.*

Furthermore, $K_{1,9} = k \cdot {\Gamma_3}^{(1)} = k \cdot (\Gamma^\sigma)_3^{(1)} = k \cdot \left(\Gamma^{\sigma^2}\right)_3^{(1)}$*, where* ${F_3}^{(1)}$ *for a number field* F *is the Hilbert* 3*-class field of* F*.*

3) One bicyclic bicubic extension of degree 9 *denoted* $K_{4,9}$*, and given by* $K_{4,9} = K_{i,3} \cdot K_{j,3}$, $i \neq j$*, which corresponds by class field theory to the ambiguous ideal class group* $C_{k,3}^{(\sigma)} = \langle \mathcal{A}^3 \rangle$ *of the extension* k/k_0*.*

Proof We will start our proof by assuming that the 3-ideal class group $C_{k,3}$ is of type $(9,3)$. Then $C_{k,3}^{+} = \langle \mathcal{A} \rangle$ and $C_{k,3}^{-} = \langle \mathcal{B} \rangle$, where $\mathcal{A} \in C_{k,3}$ such that $\mathcal{A}^9 = 1$, $\mathcal{A}^3 \neq 1$, and $\mathcal{B} \in C_{k,3}$ such that $\mathcal{B}^3 = 1$, $\mathcal{B} \neq 1$. According to the class field theory, the results of Theorem 3.1 follow immediately from the fact that the 3-ideal class group $C_{k,3} = \langle \mathcal{A}, \mathcal{B} \rangle$ admits

- Four cubic subgroups $H_{i,3}$ of order 9, where $1 \leq i \leq 4$, ordered as follows:
  - three cyclic subgroups $H_{i,3}$ of order 9, for $1 \leq i \leq 3$, given by:
    - $H_{1,3} = C_{k,3}^{+} = \langle \mathcal{A} \rangle = \{\mathcal{C} \in C_{k,3} \mid \mathcal{C}^\tau = \mathcal{C}\}$,
    - $H_{2,3} = \langle \mathcal{A}\mathcal{B} \rangle = \langle \mathcal{A}^\sigma \rangle = \{\mathcal{C} \in C_{k,3} \mid \mathcal{C}^{\tau\sigma} = \mathcal{C}\}$,
    - $H_{3,3} = \langle \mathcal{A}\mathcal{B}^{-1} \rangle = \langle \mathcal{A}\mathcal{B}^2 \rangle = \langle \mathcal{A}^{\sigma^2} \rangle = \{\mathcal{C} \in C_{k,3} \mid \mathcal{C}^{\tau\sigma^2} = \mathcal{C}\}$.

    Then, we have:

$$C_{k,3}/H_{1,3} = C_{k,3}/C_{k,3}^{+} = C_{k,3}/\langle \mathcal{A} \rangle \simeq \mathbb{Z}/3\mathbb{Z},$$

$$C_{k,3}/H_{2,3} = C_{k,3}/\langle \mathcal{A}^\sigma \rangle \simeq \mathbb{Z}/3\mathbb{Z},$$

$$C_{k,3}/H_{3,3} = C_{k,3}/\langle \mathcal{A}^{\sigma^2} \rangle \simeq \mathbb{Z}/3\mathbb{Z}.$$

    Clearly we see that $\mathrm{Gal}(K_{1,3}/k) \simeq \mathrm{Gal}(K_{2,3}/k) \simeq \mathrm{Gal}(K_{3,3}/k) \simeq \mathbb{Z}/3\mathbb{Z}$, which means that $K_{1,3}$, $K_{2,3}$, $K_{3,3}$ are unramified cyclic extensions of degree 3 over k corresponding respectively to the subgroups $H_{1,3} = \langle \mathcal{A} \rangle$, $H_{2,3} = \langle \mathcal{A}^\sigma \rangle$, $H_{3,3} = \langle \mathcal{A}^{\sigma^2} \rangle$ of $C_{k,3}$.
  - the fourth subgroup $H_{4,3}$ of order 9 is exactly the principal genus $C_{k,3}^{1-\sigma}$, given by

$$H_{4,3} = \prod_{i=1}^{4} H_{i,9} = C_{k,3}^{1-\sigma} = C_{k,3}^{(\sigma)} \times C_{k,3}^{-} = \langle \mathcal{A}^3, \mathcal{B} \rangle.$$

We see that $C_{k,3}/H_{4,3} = C_{k,3}/C_{k,3}^{1-\sigma} \simeq \mathbb{Z}/3\mathbb{Z}$. According to genus theory

$$C_{k,3}/C_{k,3}^{1-\sigma} \simeq \mathrm{Gal}\left((k/k_0)^*/k\right),$$

which is exactly the genus group, for more details see [7, §2, p. 85]. Then, $\mathrm{Gal}(K_{4,3}/k) \simeq \mathrm{Gal}\left((k/k_0)^*/k\right) \simeq \mathbb{Z}/3\mathbb{Z}$, which means that $K_{4,3} = (k/k_0)^*$ is an unramified cyclic extension of degree 3 over k corresponds, according to Theorem 2.3, to the subgroup $H_{4,3} = \langle \mathcal{A}^3, \mathcal{B}\rangle$ of $C_{k,3}$.

Furthermore, since the discriminant of $\mathcal{O}_\Gamma$ is divisible by a single prime number p such that $p \equiv 1 \pmod 3$, then according to [11, Corollary 2.1, p. 21] we get

$$\begin{aligned}
\Gamma^* &= M(p).\Gamma,\\
(\Gamma^\sigma)^* &= M(p).\Gamma^\sigma,\\
\left(\Gamma^{\sigma^2}\right)^* &= M(p).\Gamma^{\sigma^2},
\end{aligned}$$

where Γ^* (respectively $(\Gamma^\sigma)^*$, $\left(\Gamma^{\sigma^2}\right)^*$) is the absolute genus field of Γ (respectively Γ^σ, Γ^{σ^2}), and $M(p)$ is the unique cubic subfield $\mathbb{Q}(\zeta_p)$ of degree 3. By switching to the composition we obtain

$$k.\Gamma^* = k.(\Gamma^\sigma)^* = k.\left(\Gamma^{\sigma^2}\right)^* = k.M(p).$$

The fact that $p \equiv 1 \pmod 3$ imply according to [10, Chapter 9, Section 1, Proposition 9.1.4, p. 110] that $p = \pi_1\pi_2$ with $\pi_1^\tau = \pi_2$ and $\pi_1 \equiv \pi_2 \equiv 1 \pmod{3\mathcal{O}_{k_0}}$, then by [8, § 3, Lemma 3.2, p. 56], we see that $(k/k_0)^* = k(\sqrt[3]{\pi_1\pi_2^2})$. We conclude that $K_{4,3} = (k/k_0)^* = k\Gamma^* = k(\sqrt[3]{\pi_1\pi_2^2})$.

- Four cyclic cubic subgroups $H_{i,9}$ of order 3, where $1 \le i \le 4$, ordered as follows:
  - $H_{1,9} = C_{k,3}^- = \langle\mathcal{B}\rangle = \{\mathcal{C} \in C_{k,3} \mid \mathcal{C}^\tau = \mathcal{C}^{-1}\}$,
  - $H_{2,9} = \langle\mathcal{A}^3\mathcal{B}\rangle$,
  - $H_{3,9} = \langle\mathcal{A}^3\mathcal{B}^{-1}\rangle = \langle\mathcal{A}^3\mathcal{B}^2\rangle$,
  - $H_{4,9} = C_{k,3}^{(\sigma)} = \langle\mathcal{A}^3\rangle$ is the ambiguous ideal class group of k/k_0.

Therefore, $H_{4,9} = \bigcap_{i=1}^4 H_{i,3}$, and for each $1 \le i \le 3$, $H_{i,9}$ is contained only in $H_{4,3}$.

On one hand, we get

$$C_{k,3}/H_{4,9} = C_{k,3}/C_{k,3}^{(\sigma)} = C_{k,3}/\langle\mathcal{A}^3\rangle \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z},$$

we see that $\mathrm{Gal}(K_{4,9}/k) \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, which signifies that $K_{4,9}$ is an unramified bicyclic bicubic extension of k corresponding to the subgroup $H_{4,9} = C_{k,3}^{(\sigma)}$ of $C_{k,3}$.
With the same reasoning, we obtain

$$C_{k,3}/H_{1,9} = C_{k,3}/C_{k,3}^- = C_{k,3}/\langle\mathcal{B}\rangle \simeq \mathbb{Z}/9\mathbb{Z},$$

$$C_{k,3}/H_{2,9} = C_{k,3}/\langle \mathcal{A}^3\mathcal{B}\rangle \simeq \mathbb{Z}/9\mathbb{Z},$$

$$C_{k,3}/H_{3,9} = C_{k,3}/\langle \mathcal{A}^3\mathcal{B}^2\rangle \simeq \mathbb{Z}/9\mathbb{Z}.$$

Clearly we see that $\mathrm{Gal}\,(K_{1,9}/k) \simeq \mathrm{Gal}\,(K_{2,9}/k) \simeq \mathrm{Gal}\,(K_{3,9}/k) \simeq \mathbb{Z}/9\mathbb{Z}$, which means that $K_{1,9}$, $K_{2,9}$, $K_{3,9}$ are unramified cyclic extensions of degree 9 over k corresponding respectively to the subgroups $H_{1,9} = \langle \mathcal{B}\rangle$, $H_{2,9} = \langle \mathcal{A}^3\mathcal{B}\rangle$, $H_{3,9} = \langle \mathcal{A}^3\mathcal{B}^2\rangle$ of $C_{k,3}$.

On the other hand, according to [8, § 2, Lemma 2.1, p. 53] we have

$$C_{k,3}/H_{1,9} = C_{k,3}/C_{k,3}^{-} \simeq C_{k,3}^{+},$$

by [8, § 2, Lemma 2.2, p. 53] we have

$$C_{k,3}^{+} \simeq C_{\Gamma,3},$$

and according to class field theory

$$C_{\Gamma,3} \simeq \mathrm{Gal}\left({\Gamma_3}^{(1)}|\Gamma\right),$$

then we obtain

$$C_{k,3}/H_{1,9} \simeq \mathrm{Gal}\left({\Gamma_3}^{(1)}|\Gamma\right) \simeq \mathrm{Gal}\left(k{\Gamma_3}^{(1)}|k\right),$$

so according to class field theory, $k{\Gamma_3}^{(1)}$ is an unramified cyclic extension of degree 9 of k corresponding to the subgroup $H_{1,9} = C_{k,3}^{-}$ of $C_{k,3}$. Thus, $K_{1,9} = k{\Gamma_3}^{(1)}$.

Next, we show that $k{\Gamma_3}^{(1)} = k\,(\Gamma^{\sigma})_3^{(1)} = k\left(\Gamma^{\sigma^2}\right)_3^{(1)}$. Suppose that ${\Gamma_3}^{(1)} \neq (\Gamma^{\sigma})_3^{(1)}$, then ${\Gamma_3}^{(1)} \cdot (\Gamma^{\sigma})_3^{(1)}$ is an unramified extension of ${\Gamma_3}^{(1)}$ different to ${\Gamma_3}^{(1)}$. This contradicts the fact that the tower of class fields of Γ stops at the first stage, since the 3-ideal class group $C_{\Gamma,3}$ of Γ is cyclic.

□

4. Capitulation of 3-ideal classes of $\mathbb{Q}(\sqrt[3]{p},\zeta_3)$

Let $k = \mathbb{Q}(\sqrt[3]{p},\zeta_3)$, where p is a prime such that $p \equiv 1\,(\mathrm{mod}\,9)$, and $C_{k,3}$ the 3-class group of k. Suppose that $C_{k,3}$ is of type $(9,3)$. Let $(K_{i,j})$ be the family of all intermediate subfields of $k \subseteq K_{i,j} \subseteq k_3^{(1)}$, where $1 \leq i \leq 4$ and $j \in \{3,9\}$. We denote by $\varkappa_{i,j} = \ker\left(T_{K_{i,j}/k}\right)$ the kernel of the homomorphism $T_{K_{i,j}/k} : C_{k,3} \longrightarrow C_{K_{i,j},3}$ induced by extension of ideals of k to $K_{i,j}$. We denote by κ the quartet of Taussky's conditions [19]: B if $\varkappa_{i,j} \cap N_{K_{i,j}/k}(C_{K_{i,j},3}) = 1$, A otherwise.

Definition 4.1 *Let $\mathcal{A}_{i,j}$ be a generator of the subgroup $H_{i,j}$ of $C_{k,3}$, with $1 \leq i \leq 4$, $j \in \{3,9\}$ corresponding to the field $K_{i,j}$. Let $l_i \in \{0,1,2,3,4\}$ with $1 \leq i \leq 4$.*
We will say that the capitulation is of punctured type $\varkappa = (l_1, l_2, l_3; l_4)$ to express the fact that when $l_i = n$ for some $n \in \{1,2,3,4\}$, then only the class $\mathcal{A}_{n,9}$ and its powers capitulate in $K_{i,3}$. If all classes of order 3 capitulate in $K_{i,3}$, then we put $l_i = 0$.

The main result of this paper is as follows:

Theorem 4.2 *Let $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$, where p is a prime number such that $p \equiv 1 \,(\mathrm{mod}\, 9)$, and $C_{k,3}$ is its 3-class group. Assume that $C_{k,3}$ is of type $(9,3)$. Then:*

1. *(a) $K_{1,3}^{\sigma} = K_{2,3}$, $K_{2,3}^{\sigma} = K_{3,3}$, and $K_{3,3}^{\sigma} = K_{1,3}$ (σ permutes $K_{1,3}$, $K_{2,3}$ and $K_{3,3}$).*

   (b) $K_{1,3}^{\tau} = K_{1,3}$, $K_{2,3}^{\tau} = K_{3,3}$, and $K_{3,3}^{\tau} = K_{2,3}$,

   (c) $K_{4,9}^{\tau} = K_{4,9}$, $K_{2,9}^{\tau} = K_{3,9}$, and $K_{3,9}^{\tau} = K_{2,9}$,

   for all extensions of the automorphisms σ and τ.

2. *The three classes $\mathcal{A}$, $\mathcal{A}^{\sigma}$ and $\mathcal{A}^{\sigma^2}$ do not capitulate in $K_{i,3}$, for $1 \le i \le 4$.*

3. *Exactly the class $\mathcal{A}^3$ and its powers capitulate in $K_{4,3}$, i.e. $\ker\left(T_{K_{4,3}/k}\right) = \langle \mathcal{A}^3 \rangle$.*

4. *The capitulation kernels of the fields $K_{2,3}$ and $K_{3,3}$ have the same order.*

5. *The three classes $\mathcal{A}$, $\mathcal{A}^{\sigma}$ and $\mathcal{A}^{\sigma^2}$ capitulate in $K_{1,9}$.*

6. *The capitulation kernels of the fields $K_{2,9}$ and $K_{3,9}$ have the same order.*

7. *Possible types of capitulation in $K_{i,3}$, $1 \le i \le 4$, are $\varkappa = (4,4,4;4)$, $(1,2,3;4)$ and $(0,0,0;4)$. Possible Taussky types in $K_{i,3}$, $1 \le i \le 4$, are $\kappa = (\mathrm{AAA};\mathrm{A})$ or $(\mathrm{BBB};\mathrm{A})$.*

Proof Let $C_{k,3}^{(\sigma)}$ be the 3-ambigous class group of k/k_0 and $C_{k,3}^{1-\sigma} = \{\mathcal{A}^{1-\sigma} \,|\, \mathcal{A} \in C_{k,3}\}$ the principal genus of $C_{k,3}$.

By Theorem 2.3 we have $C_{k,3}^{(\sigma)} = \langle \mathcal{A}^3 \rangle = \langle \mathcal{B}^{1-\sigma} \rangle$, and $C_{k,3}^{1-\sigma} = C_{k,3}^{-} \times C_{k,3}^{(\sigma)} = \langle \mathcal{B}, \mathcal{A}^3 \rangle$ is a 3-group of $C_{k,3}$ of type $(3,3)$, where $\mathcal{B} \in C_{k,3}$ such that $C_{k,3}^{-} = \langle \mathcal{B} \rangle = \langle (\mathcal{A}^2)^{\sigma - 1} \rangle$.

1. We will agree that for all i, $1 \le i \le 4$, $j = 3$ or 9, and for all $\omega \in \mathrm{Gal}\,(k|\mathbb{Q})$, $H_{i,j}^{\omega} = \{\mathcal{C}^{\omega}/\mathcal{C} \in H_{i,j}\}$.

   (a) According to Theorem 3.1, $H_{1,3} = C_{k,3}^{+} = \langle \mathcal{A} \rangle$, $H_{2,3} = \{\mathcal{C} \in C_{k,3} | \mathcal{C}^{\tau\sigma} = \mathcal{C}\} = \langle \mathcal{A}^{\sigma} \rangle$, and $H_{3,3} = \{\mathcal{C} \in C_{k,3} | \mathcal{C}^{\tau\sigma^2} = \mathcal{C}\} = \langle \mathcal{A}^{\sigma^2} \rangle$. Then, $H_{1,3}^{\sigma} = H_{2,3}$, $H_{2,3}^{\sigma} = H_{3,3}$ and $H_{3,3}^{\sigma} = H_{1,3}$ (σ permutes $H_{1,3}$, $H_{2,3}$ and $H_{3,3}$).

   (b) As $H_{1,3} = C_{k,3}^{+} = \{\mathcal{C} \in C_{k,3} \,|\, \mathcal{C}^{\tau} = \mathcal{C}\}$, then $H_{1,3}^{\tau} = H_{1,3}$. We have $H_{2,3}^{\tau} = \langle (\mathcal{A}^{\sigma})^{\tau} \rangle = \langle \mathcal{A}^{\tau\sigma} \rangle$, and since $\mathcal{A}^{\tau\sigma} = \mathcal{A}^{\sigma^2\tau} = (\mathcal{A}^{\tau})^{\sigma^2} = \mathcal{A}^{\sigma^2} \in H_{3,3}$, then $H_{2,3}^{\tau} = H_{3,3}$. $H_{3,3}^{\tau} = \langle (\mathcal{A}^{\sigma^2})^{\tau} \rangle = \langle \mathcal{A}^{\tau\sigma^2} \rangle = \langle \mathcal{A}^{\sigma\tau} \rangle = \langle (\mathcal{A}^{\tau})^{\sigma} \rangle = \langle \mathcal{A}^{\sigma} \rangle$, then $H_{3,3}^{\tau} = H_{2,3}$.

   (c) We proceed as in (b). $H_{1,9}^{\tau} = H_{1,9}$ because $H_{1,9} = C_{k,3}^{-} = \{\mathcal{C} \in C_{k,3} \,|\, \mathcal{C}^{\tau} = \mathcal{C}^{-1}\}$. We have $H_{2,9} = \langle \mathcal{A}^3\mathcal{B} \rangle$, and $H_{3,9} = \langle \mathcal{A}^3\mathcal{B}^2 \rangle$, and since $\mathcal{A}^{\tau} = \mathcal{A}$ and $\mathcal{B}^{\tau} = \mathcal{B}^{-1} = \mathcal{B}^2$, then $H_{2,9}^{\tau} = H_{3,9}$ and $H_{3,9}^{\tau} = H_{2,9}$.

   The relations between the fields $K_{i,j}$ in (1) are nothing else than the translations of the corresponding relations for the subgroups $H_{i,j}$ via class field theory.

2. For each $1 \leq i \leq 4$, $K_{i,3}$ is an unramified cyclic extension of degree 3 over k. It is clear that for each class $\mathcal{C} \in C_{k,3}$ we have $\mathcal{C}^3 = (N_{K_{i,3}/k} \circ T_{K_{i,3}/k})(\mathcal{C})$. If the class $\mathcal{C}$ capitulates in $K_{i,3}$, then $T_{K_{i,3}/k}(\mathcal{C}) = 1$ and $\mathcal{C}^3 = 1$. We conclude that the ideal classes which capitulate in $K_{i,3}$ are of order 3. Since the classes $\mathcal{A}$, $\mathcal{A}^\sigma$, and $\mathcal{A}^{\sigma^2}$ are of order 9, then these classes cannot capitulate in $K_{i,3}$.

3. By Theorem 3.1 we have $K_{4,3} = (k/k_0)^*$ is the relative genus field of k/k_0, and by Theorem 2.3 we have $\langle \mathcal{A}^3 \rangle = \langle \mathcal{A}^{3\sigma} \rangle = \langle \mathcal{A}^{3\sigma^2} \rangle = C_{k,3}^{(\sigma)}$. We conclude according to Tannaka–Terada theorem [20], that all ambiguous ideal classes of $C_{k,3}$ capitulate in the relative genus field $(k/k_0)^*$. Thus, the class $\mathcal{A}^3$ and its powers capitulate in $K_{4,3}$. We shall prove that the unique classes which capitulate in $K_{4,3}$ are only the ambiguous ideal classes. We have $C_{k,3}^{-} = \langle \mathcal{B} \rangle = \langle (\mathcal{A}^2)^{\sigma-1} \rangle$ and $C_{k,3}^{(\sigma)} = \langle \mathcal{A}^3 \rangle = \langle \mathcal{B}^{1-\sigma} \rangle$. On one hand we have $\mathcal{A}^{1+2\sigma} = \mathcal{A}^{\sigma(1-\sigma)} = ((\mathcal{A}^{-1})^{\sigma-1})^\sigma = ((\mathcal{A}^2)^{\sigma-1})^\sigma = \mathcal{B}^\sigma$
because $(\mathcal{A}^3)^{\sigma-1} = 1$. One the other hand, we have $\mathcal{B}^{1-\sigma}\mathcal{B}^2 = \mathcal{B}^{3-\sigma} = \mathcal{B}^{-\sigma}$, then $(\mathcal{B}^{1-\sigma}\mathcal{B}^2)^{-1} = \mathcal{B}^\sigma$. So we get $\mathcal{B}^\sigma \in \langle \mathcal{B}^{1-\sigma}\mathcal{B}^2 \rangle = \langle \mathcal{A}^3\mathcal{B}^2 \rangle$, because $\langle \mathcal{A}^3 \rangle = \langle \mathcal{B}^{1-\sigma} \rangle$. Since $C_{k,3} = \langle \mathcal{A}, \mathcal{B} \rangle$ is of type $(9,3)$, then a class $\chi \in C_{k,3}$ of order 3 capitulates in the cubic cyclic unramified extension $K_{4,3}/k$, if and only if $\mathcal{B}$ capitulates in the extension $K_{4,3}/k$, because a class χ of order 3 is in one of the subgroups $\langle \mathcal{B} \rangle$, $\langle \mathcal{A}^3\mathcal{B} \rangle$, $\langle \mathcal{A}^3\mathcal{B}^2 \rangle$ and that $\mathcal{B}^\sigma \in \langle \mathcal{A}^3\mathcal{B}^2 \rangle$.
If $\mathcal{B}$ capitulates in the extension $K_{4,3}/k$, then $\mathcal{B}^\sigma$ capitulates also in $K_{4,3}/k$. Since $\mathcal{A}^{1+2\sigma} = \mathcal{B}^\sigma$, then $\mathcal{A}^{1+2\sigma}$ capitulates also in $K_{4,3}/k$, so $T_{K_{4,3}/k}\left(\mathcal{A}^{1+2\sigma}\right) = 1$. Then

$$\left(T_{K_{4,3}/k}\left(\mathcal{A}^\sigma\right)\right)^2 = T_{K_{4,3}/k}\left(\mathcal{A}^{-1}\right) = T_{K_{4,3}/k}\left(\mathcal{A}^2\right)$$

because $T_{K_{4,3}/k}\left(\mathcal{A}^3\right) = 1$, so $\left(T_{K_{4,3}/k}\left(\mathcal{A}^\sigma\right)\right)^2 = \left(T_{K_{4,3}/k}\left(\mathcal{A}\right)\right)^2$ and then
$T_{K_{4,3}/k}\left(\mathcal{A}^\sigma\right) = T_{K_{4,3}/k}\left(\mathcal{A}\right)$, so we get $(\Gamma_3')^{(1)} = \Gamma_3^{(1)}$, where $\Gamma_3^{(1)}$ (resp. $(\Gamma_3')^{(1)} = (\Gamma_3^\sigma)^{(1)}$) is the Hilbert 3-class field of Γ (resp. Γ'), which is a contradiction.
Thus $\mathcal{B}$ does not capitulate in $K_{4,3}/k$, and then only $\mathcal{A}^3$ and its powers capitulate in $K_{4,3}/k$.

4. The capitulation kernels of the fields $K_{2,3}$ and $K_{3,3}$ have the same order, because $K_{2,3}$ and $K_{3,3}$ are isomorphic by (1)(b).

5. Let $\mathcal{I}$ be an ideal of Γ such that $[\mathcal{I}]$ generates $C_{\Gamma,3}$. Then $[\mathcal{I}^\sigma]$ generates $C_{\Gamma',3}$, and $[\mathcal{I}^{\sigma^2}]$ generates $C_{\Gamma'',3}$. Let $\mathcal{A} = [T_{k/\Gamma}(\mathcal{I})]$, so $\mathcal{A}^\sigma = [T_{k/\Gamma}(\mathcal{I})^\sigma] = [T_{k/\Gamma'}(\mathcal{I}^\sigma)]$ and $\mathcal{A}^{\sigma^2} = [T_{k/\Gamma}(\mathcal{I})^{\sigma^2}] = [T_{k/\Gamma''}(\mathcal{I}^{\sigma^2})]$. The ideal $\mathcal{I}$ (resp. $\mathcal{I}^\sigma$ and $\mathcal{I}^{\sigma^2}$) becomes principal in $\Gamma_3^{(1)}$ (resp. $(\Gamma_3')^{(1)}$ and $(\Gamma_3'')^{(1)}$) because $\Gamma_3^{(1)}$ (resp. $(\Gamma_3')^{(1)}$ and $(\Gamma_3'')^{(1)}$) is the Hilbert 3-class field of Γ (resp. Γ' and Γ''). Then $\mathcal{I}$ (resp. $\mathcal{I}^\sigma$ and $\mathcal{I}^{\sigma^2}$) becomes principal in $k.\Gamma_3^{(1)}$ (resp. $k.(\Gamma_3')^{(1)}$ and $k.(\Gamma_3'')^{(1)}$). So, the class $\mathcal{A}$ (resp. $\mathcal{A}^\sigma$ and $\mathcal{A}^{\sigma^2}$) capitulates in $k.\Gamma_3^{(1)}$ (resp. $k.(\Gamma_3')^{(1)}$ and $k.(\Gamma_3'')^{(1)}$). Thus, $\mathcal{A}$, $\mathcal{A}^\sigma$ and $\mathcal{A}^{\sigma^2}$ capitulate in $K_{1,9}$ because $k.\Gamma_3^{(1)} = k.(\Gamma_3')^{(1)} = k.(\Gamma_3'')^{(1)} = K_{1,9}$.

6. According to (1)(c), the fields $K_{2,9}$ and $K_{3,9}$ are isomorphic. Then, the capitulation kernels of $K_{2,9}$ and $K_{3,9}$ have the same order.

7. Let $\mathcal{C}$ be an ideal class of $C_{k,3}$ of order 3.

    (i) Assume that all classes $\mathcal{C}$ of order 3 capitulate in $K_{j,3}/k$ for each $j \in \{1,2,3\}$. Since exactly the class $\mathcal{A}^3$ and its powers capitulate in $K_{4,3}$, we deduce that the type of capitulation is $(0,0,0;4)$ and the Taussky type is (AAA;A).

    (ii) Now, assume that exactly the class $\mathcal{C}$ capitulates in the extension $K_{1,3}/k$. According to assertion (1)(a) we have $K_{1,3}^{\sigma} = K_{2,3}$, $K_{2,3}^{\sigma} = K_{3,3}$, and $K_{3,3}^{\sigma} = K_{1,3}$. Then exactly one class of $C_{k,3}$ of order 3 and its powers capitulate in the extensions $K_{2,3}/k$ and $K_{3,3}/k$. Here there are two cases:

    case 1: Assume that exactly the class $\mathcal{A}^3$ capitulates in the extension $K_{1,3}/k$. By assertion (1)(a) we have $K_{1,3}^{\sigma} = K_{2,3}$, $K_{2,3}^{\sigma} = K_{3,3}$, then exactly the class $(\mathcal{A}^3)^{\sigma}$ capitulates in the extension $K_{2,3}/k$ and exactly the class $(\mathcal{A}^3)^{\sigma^2}$ capitulates in the extension $K_{3,3}/k$. Since exactly the class $\mathcal{A}^3$ and its powers capitulate in $K_{4,3}$, we conclude that the possible type of capitulation is $(4,4,4;4)$ and the Taussky type is (AAA;A).

    case 2: According to Theorem 2.3, $C_{k,3}^{-} = \langle \mathcal{B} \rangle$ and $C_{k,3}^{(\sigma)} = \langle \mathcal{A}^3 \rangle = \langle \mathcal{B}^{1-\sigma} \rangle$. Since $\mathcal{B}^{1-\sigma}\mathcal{B}^2 = \mathcal{B}^{3-\sigma} = \mathcal{B}^{-\sigma}$, then $(\mathcal{B}^{1-\sigma}\mathcal{B}^2)^{-1} = \mathcal{B}^{\sigma}$. It follows that $\mathcal{B}^{\sigma} \in \langle \mathcal{B}^{1-\sigma}\mathcal{B}^2 \rangle = \langle \mathcal{A}^3\mathcal{B}^2 \rangle$ because $\langle \mathcal{A}^3 \rangle = \langle \mathcal{B}^{1-\sigma} \rangle$. Now, assume that exactly the class $\mathcal{B}$ capitulates in $K_{1,3}/k$. Since $K_{1,3}^{\sigma} = K_{2,3}$ and $K_{2,3}^{\sigma} = K_{3,3}$ by assertion (1)(a), we deduce that exactly the class $\mathcal{B}^{\sigma}$ capitulates in $K_{2,3}/k$ and exactly the class $\mathcal{B}^{\sigma^2}$ capitulates in $K_{3,3}/k$. Then, exactly the class $\mathcal{A}^3\mathcal{B}^2$ capitulates in $K_{2,3}/k$ and exactly the class $\mathcal{A}^3\mathcal{B}$ capitulates in $K_{3,3}/k$. As exactly the class $\mathcal{A}^3$ and its powers capitulate in $K_{4,3}$, the possible type of capitulation is $(1,2,3;4)$ and the Taussky type is (BBB;A).

$\square$

5. Computational results and applications

5.1. Computational results

Let $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ be the normal closure of the pure cubic field $\Gamma = \mathbb{Q}(\sqrt[3]{p})$ with prime radicand $p \equiv 1 \pmod 9$ of Dedekind's second species. Then k is a pure metacyclic field with absolute group $\mathrm{Gal}(k/\mathbb{Q}) \simeq S_3$ the symmetric group of order six. Assume that k possesses a 3-class group $C_{k,3} \simeq C_9 \times C_3$. According to Theorem 2.1, the 3-class group of Γ is $C_{\Gamma,3} \simeq C_9$, and Γ is of principal factorization type α, in the sense of [3, Theorem. 2.1, p. 254].

In Theorem 4.2, we investigated the principalization of k in its four unramified cyclic cubic extensions $K_{1,3}, \dots, K_{4,3}$, and we found three possibilities for the kernels $\ker(T_{K_{i,3}/k})$ of the transfer homomorphisms $T_{K_{i,3}/k} : C_{k,3} \to C_{K_{i,3},3}$, $\mathfrak{a}\mathcal{P}_k \mapsto (\mathfrak{a}\mathcal{O}_{K_{i,3}})\mathcal{P}_{K_{i,3}}$.

Table has been computed with the aid of the computational algebra system MAGMA [12]. * For each of the 95 pure metacyclic fields $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ with prime radicands $p \equiv 1 \pmod 9$ in the range $0 < p < 20\,000$ and 3-class group of type $(9,3)$, the capitulation kernels $\ker(T_{K_{i,3}/k})$ of the class extension homomorphisms $T_{K_{i,3}/k} : C_{k,3} \to C_{K_{i,3},3}$ were computed and collected in the transfer kernel type $\varkappa$. An

*http://magma.maths.usyd.edu.au

asterisk indicates the second variant of harmonically balanced capitulation $\varkappa = (123;4)$ with abelian type invariants $\alpha = [(27,3)^3;(9,9,3)]$.

Table . Capitulation types $\varkappa$ of $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ in the range $p < 20\,000$.

No.	p	$\varkappa$	No.	p	$\varkappa$	No.	p	$\varkappa$	No.	p	$\varkappa$
1	199	4444	25	4951	4444	49	9829	4444	73	14293	4444
2	271	4444	26	5059	4444	50	10243	4444	74	14419	4444
3	487	4444	27	5077	1234∗	51	10459	0004	75	14563	4444
4	523	4444	28	5347	4444	52	10531	1234∗	76	14779	4444
5	1297	1234∗	29	5437	1234	53	10657	4444	77	14923	4444
6	1621	4444	30	5527	1234∗	54	10837	0004	78	15121	0004
7	1693	4444	31	5851	4444	55	10909	4444	79	15319	4444
8	1747	1234	32	6067	4444	56	11251	4444	80	15427	4444
9	1999	4444	33	6247	4444	57	11287	4444	81	16381	1234∗
10	2017	4444	34	6481	1234	58	11467	4444	82	16417	4444
11	2143	1234∗	35	6949	4444	59	11503	4444	83	16633	4444
12	2377	4444	36	7219	0004	60	11593	4444	84	16993	1234∗
13	2467	1234	37	7507	1234∗	61	11701	4444	85	17137	1234
14	2593	1234	38	7687	4444	62	11719	4444	86	17209	1234∗
15	2917	4444	39	8011	1234	63	12097	4444	87	17497	1234∗
16	3511	4444	40	8209	4444	64	12511	1234	88	17569	4444
17	3673	4444	41	8677	1234	65	12637	4444	89	18379	0004
18	3727	4444	42	8821	4444	66	12853	4444	90	18451	1234
19	3907	4444	43	9001	4444	67	12907	0004	91	18523	4444
20	4159	4444	44	9109	4444	68	13159	4444	92	18541	4444
21	4519	1234∗	45	9343	1234	69	13177	4444	93	19387	1234
22	4591	4444	46	9613	1234∗	70	13339	4444	94	19441	1234∗
23	4789	1234	47	9631	4444	71	13411	4444	95	19927	1234∗
24	4933	4444	48	9721	4444	72	13807	1234			

The following distribution of capitulation types $\varkappa$ arises in Table :

1. 61 (64%) with $\varkappa = (444;4)$ (distinguished capitulation),
2. 14 (15%) with $\varkappa = (123;4)$, $\alpha = [(27,3),(27,3),(27,3);(9,3,3)]$ ($1st$ variant),
3. 14 (15%) with $\varkappa = (123;4*)$, $\alpha = [(27,3),(27,3),(27,3);(9,9,3)]$ ($2nd$ variant),
4. 6 (6%) with $\varkappa = (000;4)$ (total capitulation).

Our numerical results confirm the occurrence of precisely three situations for the punctured capitulation type $\varkappa(k) = (\ker(T_{K_{1,3}/k}),\ldots,\ker(T_{K_{3,3}/k});\ker(T_{K_{4,3}/k}))$, which we want to dub with succinct names in Definition 5.1. Recall that $H_{4,9} = \cap_{j=1}^{4} H_{j,3}$ in Figure 2 is the distinguished subgroup of $C_{k,3}$ which is generated by third powers of 3-ideal classes, i.e. the Frattini subgroup. We also mention the corresponding types in [13, Tables 1 and 2].

Definition 5.1 *The punctured capitulation type, with puncture at the fourth component, for the subfield* $K_{4,3}$ *associated with the subgroup* $H_{4,3} = \prod_{j=1}^{4} H_{j,9}$ *in Figure 2, is called*

1. *distinguished, if* $\varkappa(k) = (H_{4,9}, H_{4,9}, H_{4,9}; H_{4,9})$ *, briefly* $(444;4)$ *or type* A.20 *,*

2. *harmonically balanced, if* $\varkappa(k) = (H_{1,9}, H_{2,9}, H_{3,9}; H_{4,9})$ *, briefly* $(123;4)$ *or type* E.12 *,*

3. *total, if* $\varkappa(k) = (H_{4,3}, H_{4,3}, H_{4,3}; H_{4,9})$ *, briefly* $(000;4)$ *or type* b.15 *.*

For the actual numerical determination of the (punctured) capitulation type $\varkappa$, we introduce the concept of Artin pattern of k.

Definition 5.2 *Let* $\alpha(k) = [\mathrm{ATI}(C_{K_{i,3},3})]_{1\le i\le 4}$ *be the family of abelian type invariants (ATI) (i.e.* 3*-primary type invariants) of the* 3*-class groups* $C_{K_{i,3},3}$ *of the four unramified cyclic cubic extensions of* k*. Then the pair* $\mathrm{AP}(k) = (\varkappa(k), \alpha(k))$ *is called the Artin pattern of* k*.*

It turns out that there is a bijective correspondence between $\varkappa$ and α for the distinguished and total capitulation, whereas there are two variants of harmonically balanced capitulation.

Anyway, it is never required to perform the difficult computation of the capitulation type $\varkappa$. It is sufficient to determine the abelian type invariants α, which is computationally easier. Remark 5.3 is a consequence of our results in Table , for which both, $\varkappa$ and α, were computed but only $\varkappa$ is listed, for brevity.

Remark 5.3 *For a pure metacyclic field* $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ *with prime radicand* $p \equiv 1 \pmod 9$*, bounded by* $p \le 20\,000$*, and* 3*-class group* $C_{k,3} \simeq C_9 \times C_3$*, the following statements determine* $\varkappa(k)$ *by means of* $\alpha(k)$*:*

1. $\varkappa(k) = (444;4) \iff \alpha(k) = [(9,3)^3;(9,3)]$ *(briefly for* $[(9,3),(9,3),(9,3);(9,3)]$ *).*

2. $\varkappa(k) = (000;4) \iff \alpha(k) = [(9,3,3)^3;(3,3,3,3)]$ *.*

3. $\varkappa(k) = (123;4) \iff \alpha(k) = \begin{cases} \textit{either } [(27,3)^3;(9,3,3)] & \textit{(1st variant)} \\ \textit{or } [(27,3)^3;(9,9,3)] & \textit{(2nd variant, with an asterisk in Table).} \end{cases}$

Conjecture 5.4 *Based on the computations for Table , we conjecture the truth of the statements in Remark 5.3 for any prime* $p \equiv 1 \pmod 9$*, not necessarily bounded from above by* $20\,000$*. (In fact, item 1. will be proved rigorously in Theorem 5.9.)*

5.2. Applications

Now, we are in the position to employ the strategy of pattern recognition via Artin transfers † [17] in order to determine the 3-class field tower $k_3^{(\infty)}$ of k by means of $\mathrm{AP}(k) = (\varkappa(k), \alpha(k))$.

5.2.1. Relation rank and Galois action

Constraints arise from two issues, bounds for the relation rank of the tower group $G = \mathrm{Gal}(k_3^{(\infty)}/k)$, and the Galois action of $\mathrm{Gal}(k/\mathbb{Q})$ on $C_{k,3} \simeq G/G'$. By $\langle o, i\rangle$ we denote groups in the SmallGroups database of Magma [12]. In the subsequent figures, the order o is given on a scale, and we abbreviate the identifiers by $\langle i\rangle$.

† http://www.algebra.at/DCM@ICMA2020Casablanca.pdf

Theorem 5.5 *For any pure metacyclic field* $k=\mathbb{Q}(\sqrt[3]{d},\zeta_3)$ *with cube free radicand* $d\geq 2$ *and* 3*-class rank* $\varrho=2$*, the Galois group* $G=\mathrm{Gal}(k_3^{(\infty)}/k)$ *of the* 3*-class field tower must satisfy the following conditions.*

1. *The relation rank* d_2 *of* G *must be bounded by* $2\leq d_2\leq 5$*.*
2. *The automorphism group* $\mathrm{Aut}(Q)$ *of the Frattini quotient* $Q=G/\Phi(G)$ *must contain a subgroup isomorphic to* $S_3=\langle 6,1\rangle$*. (This is true for any* S_3*-field* k*, not necessarily pure metacyclic.)*

Proof According to the Burnside basis theorem, the generator rank d_1 of G coincides with the generator rank of the Frattini quotient $Q=G/\Phi(G)=G/(G'\cdot G^3)$, resp. the derived quotient $G/G'\simeq C_{k,3}$, that is the 3-class rank ϱ of k.

1. According to the Shafarevich Theorem [15, Theorem 5.1, p. 28], the relation rank d_2 of G is bounded by $d_1\leq d_2\leq d_1+r+\vartheta$, where the torsion free unit rank $r=r_1+r_2-1$ of the totally complex field k with signature $(r_1,r_2)=(0,3)$ is $r=2$, and $\vartheta=1$, since k contains the primitive third roots of unity. Together with the generator rank $d_1=\varrho=2$ this gives the bounds $2\leq d_2\leq 2+2+1=5$. (For other complex, resp. totally real, S_3-fields k, we have $\vartheta=0$ and the upper bound changes to 4, resp. 7.)
2. The absolute Galois group $\mathrm{Gal}(k/\mathbb{Q})\simeq S_3$ of k acts on the 3-class group $C_{k,3}\simeq G/G'$ and thus also on the Frattini quotient $Q=G/\Phi(G)=G/(G'\cdot G^3)$, whence $\mathrm{Aut}(Q)$ contains a subgroup isomorphic to $S_3=\langle 6,1\rangle$.

□

By the same proof as for item 2. of Theorem 5.5, with $G/G'\simeq C_{k,3}$ replaced by

$$G_n/G_n'\simeq \mathrm{Gal}(k_3^{(n)}/k)/\mathrm{Gal}(k_3^{(n)}/k_3^{(1)})\simeq \mathrm{Gal}(k_3^{(1)}/k)\simeq C_{k,3}$$

we obtain the same requirement for the Galois action on G_n (but *not* for the relation rank of G_n!):

Corollary 5.6 *Let* n *be a positive integer, and denote by* $G_n=\mathrm{Gal}(k_3^{(n)}/k)$ *the Galois group of the* n*-th Hilbert* 3*-class field* $k_3^{(n)}$ *of* k*. The automorphism group* $\mathrm{Aut}(Q)$ *of the Frattini quotient* $Q=G_n/\Phi(G_n)$ *must contain a subgroup isomorphic to* $S_3=\langle 6,1\rangle$*.*

Furthermore, it will also be required to exploit data concerning the *second layer* of unramified abelian (three cyclic nonic and a single bicyclic bicubic) extensions.

Definition 5.7 *Let* $\varkappa_2(k)=(\ker(T_{K_{1,9}/k}),\ldots,\ker(T_{K_{3,9}/k});\ker(T_{K_{4,9}/k}))$ *be the punctured capitulation type and* $\alpha_2(k)=[\mathrm{ATI}(C_{K_{i,9},3})]_{1\leq i\leq 4}$ *be the family of abelian type invariants of the* 3*-class groups* $C_{K_{i,9},3}$ *of the four unramified abelian nonic extensions of* k*, and put* $\mathrm{AP}_2(k)=(\varkappa_2(k),\alpha_2(k))$*.*

According to item *5.* of Theorem 4.2, we know that $\ker(T_{K_{1,9}/k})=C_{k,3}$.

By a 3-group of type $(9,3)$ we understand a finite group G with derived quotient $G/G'\simeq C_9\times C_3$. Such groups of second maximal class, that is of coclass $\mathrm{cc}(G)=2$, were called *CF-groups* by Ascione et al. [4, § 7, pp. 272-274]. Ascione denoted those of nilpotency class $\mathrm{cl}(G)=3$ by capital letters A, …, H as in Figure 3. However, most of our 3-class tower groups $G=\mathrm{Gal}(k_3^{(\infty)}/k)$ arise as descendants of step size $s=2$ of the group $\langle 81,3\rangle$ with remarkable metabelian bifurcation to coclass $\mathrm{cc}=3$, in the sense of [14].

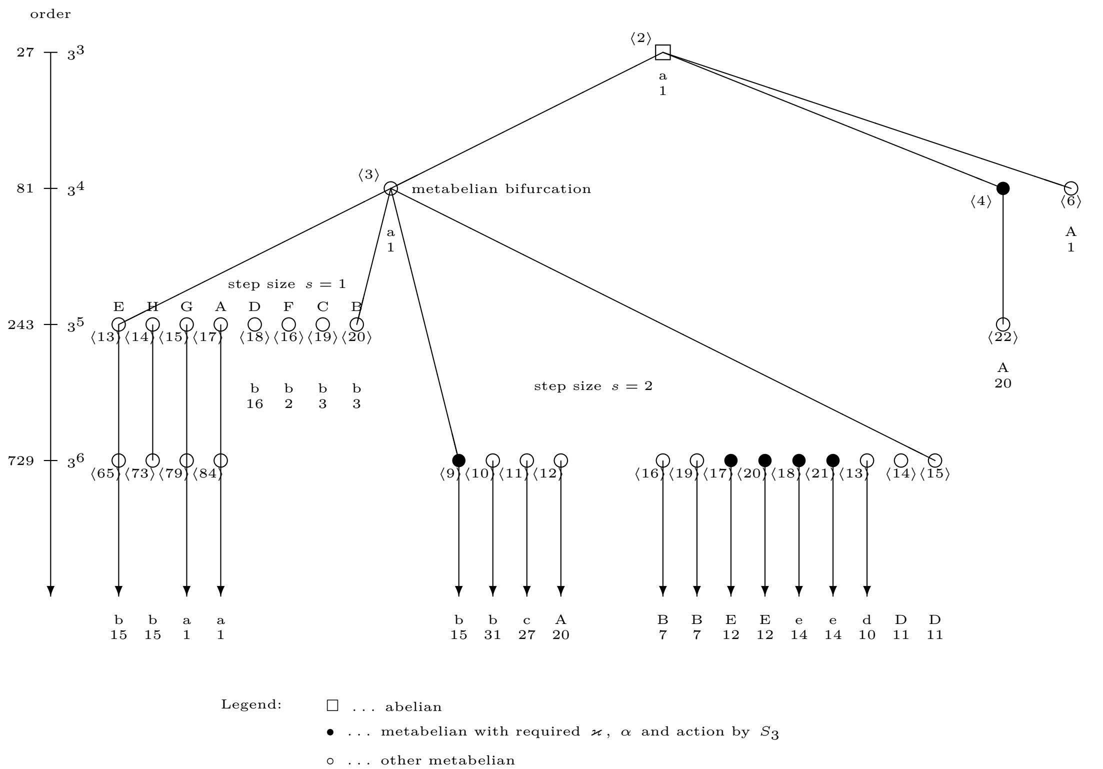

Figure 3. Finite 3-groups G with commutator quotient $G/G' \simeq C_9 \times C_3$.

5.2.2. Distinguished capitulation

Proposition 5.8 *A power commutator presentation of the finite metabelian 3-group $\langle 81, 4\rangle$ with class 2 and coclass 2 in terms of the commutator $s_2 = [y, x]$ is given by*

$$\langle x, y, s_2 \mid x^9 = 1,\ y^3 = s_2\rangle. \tag{5.1}$$

Proof The presentation of $\langle 81, 4\rangle$ is part of the SmallGroups database, implemented in Magma [12]. □

Theorem 5.9 *Let $k = \mathbb{Q}(\zeta_3, \sqrt[3]{p})$ be a pure metacyclic S_3-field with prime radicand $p \equiv 1 \,(\mathrm{mod}\, 9)$, 3-class group $C_{k,3}$ of type $(9, 3)$ and distinguished capitulation $\varkappa(k) = (444; 4)$. Then*

1. *The Galois group G_2 of the second Hilbert 3-class field $k_3^{(2)}$ of k is unambiguously determined as $\mathrm{Gal}(k_3^{(2)}/k) \simeq \langle 81, 4\rangle$ (see Figure 3) with $\varkappa_2(k) = ((C_{k,3})^3; C_{k,3})$ and $\alpha_2(k) = [(9)^3; (3, 3)]$.*

2. *The abelian type invariants of the 3-class groups $C_{K_{i,3},3}$ of the four unramified cyclic cubic extensions $K_{i,3}$, $1 \le i \le 4$, of k are given by $\alpha(k) = [(9, 3)^3; (9, 3)]$.*

3. *The 3-class field tower of k must stop at the second stage, that is, $k_3^{(2)} = k_3^{(\infty)}$ is the maximal unramified pro-3-extension of k.*

Proof In [16, §§ 5.1–5.4, pp. 78–87] we proved the theorem on the antitony $\varkappa(P) \ge \varkappa(D)$ and $\alpha(P) \le \alpha(D)$ of the components of the Artin pattern $(\varkappa, \alpha)$ with respect to (parent, descendant)-pairs (P, D), where P is a quotient of D. If we search the descendant tree in Figure 3 for groups G with punctured transfer kernel type $\varkappa = (444; 4)$ (A.20), we must therefore also consider possible parents with $\varkappa = (044; 4)$ (b.31) or $\varkappa = (044; 0)$ (c.27), since the kernel 0 may shrink to 4 for a descendant. We conduct the search by ascending order of the groups in the tree with abelian root $\langle 27, 2\rangle \simeq C_9 \times C_3$. The smallest order with a candidate is 81 and yields the unique hit $\langle 81, 4\rangle$, which shares the kernels $\varkappa$ and the abelian type invariants $\alpha = [(9, 3)^3; (9, 3)]$ with its terminal child $\langle 243, 22\rangle$ (a leaf of the tree). There are no further hits with coclass $\mathrm{cc} = 2$ among the eight descendants with step size $s = 1$ of the ramification vertex $\langle 81, 3\rangle$, i.e. all the CF-groups A, …, H of Ascione are discouraged as candidates. At this point, we have to enter a realm which did not occur in the literature yet: We follow the metabelian bifurcation to coclass $\mathrm{cc} = 3$ and check the type $\varkappa$ of the thirteen descendants with step size $s = 2$ of $\langle 81, 3\rangle$. It turns out that only three of them are relevant, $\langle 729, 10\rangle$ with $\varkappa = (044; 4)$, $\langle 729, 11\rangle$ with $\varkappa = (044; 0)$ and $\langle 729, 12\rangle$ with $\varkappa = (444; 4)$. Indeed, $\langle 729, 10\rangle$ has the child $\langle 2187, 139\rangle$ with $\varkappa = (444; 4)$, and $\langle 729, 11\rangle$ has two descendants $\langle 6561, i\rangle$, $i \in \{179, 181\}$, with $\varkappa = (444; 4)$. Clearly, all descendants of $\langle 729, 12\rangle$ have $\varkappa = (444; 4)$, since this type is not able to shrink further. Consequently, as mentioned in the Introduction, § 1, the type $\varkappa$ is by far too insufficient in order to identify G_2, since there are infinitely many candidates. Here the Galois action of S_3 on G_2 is definitely required: $\langle 729, 10\rangle$ and $\langle 729, 11\rangle$ are forbidden, because they have only an action by $C_2 \times C_2 = \langle 4, 2\rangle$, and $\langle 729, 12\rangle$ can be omitted, since only $C_6 = \langle 6, 2\rangle$ acts on it. The same is true for all descendants of these three parents. Finally, $\langle 243, 22\rangle$ is discouraged, due to its poor action by $C_2 = \langle 2, 1\rangle$ only. So the unique remaining candidate for G_2 is $\langle 81, 4\rangle$, as claimed.

The abelian type invariants $\alpha = [(9, 3)^3; (9, 3)]$ which are common to the metabelian 3-groups $\langle 81, 4\rangle$ and $\langle 243, 22\rangle$, cannot occur for any other finite 3-group of type $(9, 3)$ which must be descendant of the metabelian root $R = \langle 81, 3\rangle$ with pc-presentation $\langle x, y, s_2 \mid x^9 = 1,\ y^3 = 1,\ s_2 = [y, x]\rangle$ and $\alpha(R) = [(9, 3)^3, (3, 3, 3)]$. Consequently, at least one component of $\alpha(D)$ will always be of rank three, for any descendant D of the root R. This argument shows that there cannot be a nonmetabelian 3-group G of type $(9, 3)$ with second derived quotient G/G'' isomorphic to either $\langle 81, 4\rangle$ or $\langle 243, 22\rangle$, since G would necessarily be required to have $\alpha(G) = [(9, 3)^3); (9, 3)]$, which is not compatible with being a descendant of R. According to the Artin reciprocity law of class field theory, applied to $k_3^{(2)}$, the 3-class field tower of k must therefore have precise length $\ell_3(k) = 2$. Eventually, the unique candidate $G = \langle 81, 4\rangle$ for $G_2 = G_\infty$ satisfies the inequalities $2 \le d_2 \le 5$ for the relation rank in Theorem 5.5, since it has $d_2 = 3$, and the automorphism group $\mathrm{Aut}(Q)$ of its Frattini quotient $Q = G/\Phi(G)$ contains a subgroup isomorphic to $S_3 = \langle 6, 1\rangle$, as required. □

5.2.3. Harmonically balanced capitulation

We shall see that harmonically balanced capitulation $\varkappa = (123; 4)$ occurs in two variants with distinct fourth components $(9, 3, 3)$, resp. $(9, 9, 3)$, in the abelian type invariants α. It turns out that the first variant leads to sporadic groups outside of coclass trees, and the second variant is connected with periodic groups on coclass trees.

Proposition 5.10 *A power commutator presentation of the finite metabelian* 3*-group* $\langle 729, i\rangle$ *of class* 3 *in* *terms of the commutators* $s_2 = [y, x]$, $s_3 = [s_2, x]$, $t_3 = [s_2, y]$ *is given by*

$$\begin{cases} \langle x, y, s_2, s_3, t_3 \mid x^9 = t_3,\ y^3 = s_3 \rangle & \text{if } i = 17, \\ \langle x, y, s_2, s_3, t_3 \mid x^9 = t_3^2,\ y^3 = s_3 \rangle & \text{if } i = 20. \end{cases} \tag{5.2}$$

The groups are sporadic of class 3 *and coclass* 3.

Proof Presentations of these groups are part of the SmallGroups database, implemented in Magma [12]. □

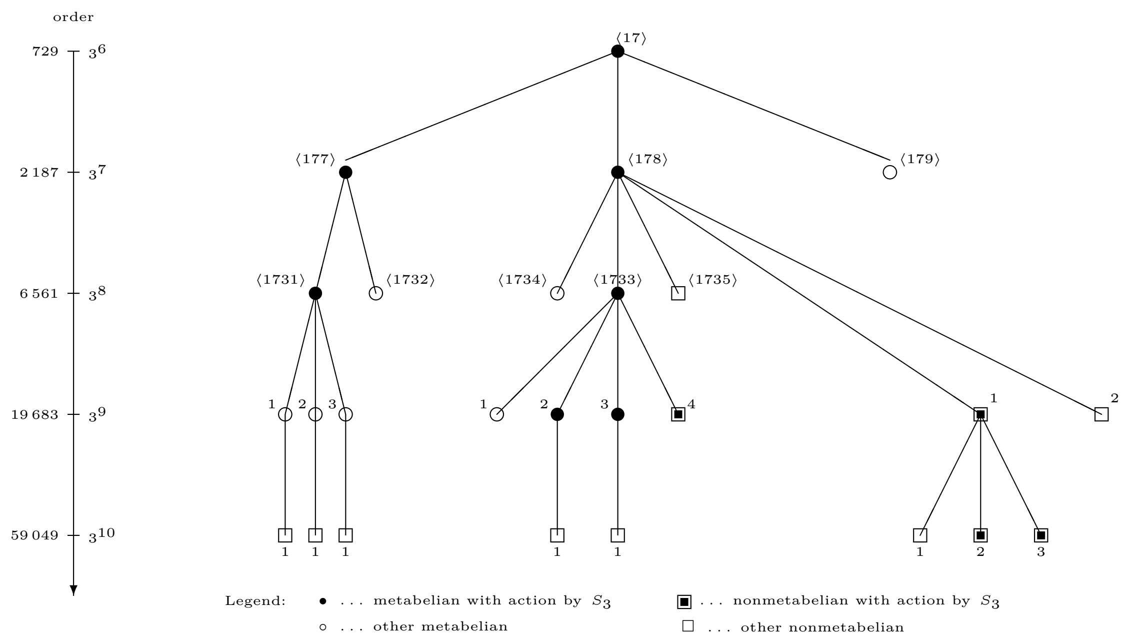

Figure 4. Descendant tree of $\langle 729, 17\rangle$ with stable $\varkappa = (123; 4)$, $\alpha = [(27, 3)^3; (9, 3, 3)]$.

Theorem 5.11 *For a pure metacyclic field* $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ *with* $p \equiv 1 \,(\mathrm{mod}\, 9)$ *having harmonically balanced capitulation* $\varkappa(k) = (123; 4)$ *and the first variant of* $\alpha = [(27, 3)^3; (9, 3, 3)]$, *the sporadic Galois group* G_2 *of* $k_3^{(2)}$, *the second Hilbert* 3*-class field of* k, *is given by* $\mathrm{Gal}(k_3^{(2)}/k) \simeq$

$$\begin{cases} \langle 729, \ell\rangle & \text{if } \varkappa_2(k) = ((C_{k,3})^3; H_{4,3}),\ \alpha_2(k) = [(9, 3)^3; (9, 3, 3)], \\ \langle 2187, m\rangle & \text{if } \varkappa_2(k) = ((C_{k,3})^3; H_{4,3}),\ \alpha_2(k) = [(9, 9)^3; (9, 9, 3)], \end{cases} \tag{5.3}$$

where $\ell \in \{17, 20\}$, $m \in \{177, 178, 187, 188\}$. *See Figures 4 and 5.*

Proof All vertices of the entire descendant trees of the roots $\langle 729, i\rangle$ with $i \in \{17, 20\}$ share the required Artin pattern $(\varkappa, \alpha)$ with harmonically balanced capitulation $\varkappa = (123; 4)$ and the first variant of $\alpha = [(27, 3)^3; (9, 3, 3)]$. Since the trees are isomorphic as structured graphs, we focus on $\langle 729, 17\rangle$, which gives rise to a finite mainline, standing out through an action by the direct product $S_3 \times C_2 \simeq \langle 12, 4\rangle$. The metabelian vertices of this finite mainline are $\langle 729, 17\rangle$, $\langle 2187, 178\rangle$, $\langle 6561, 1733\rangle$, and $\langle 6561, 1733\rangle - \#1; 2$. The other two

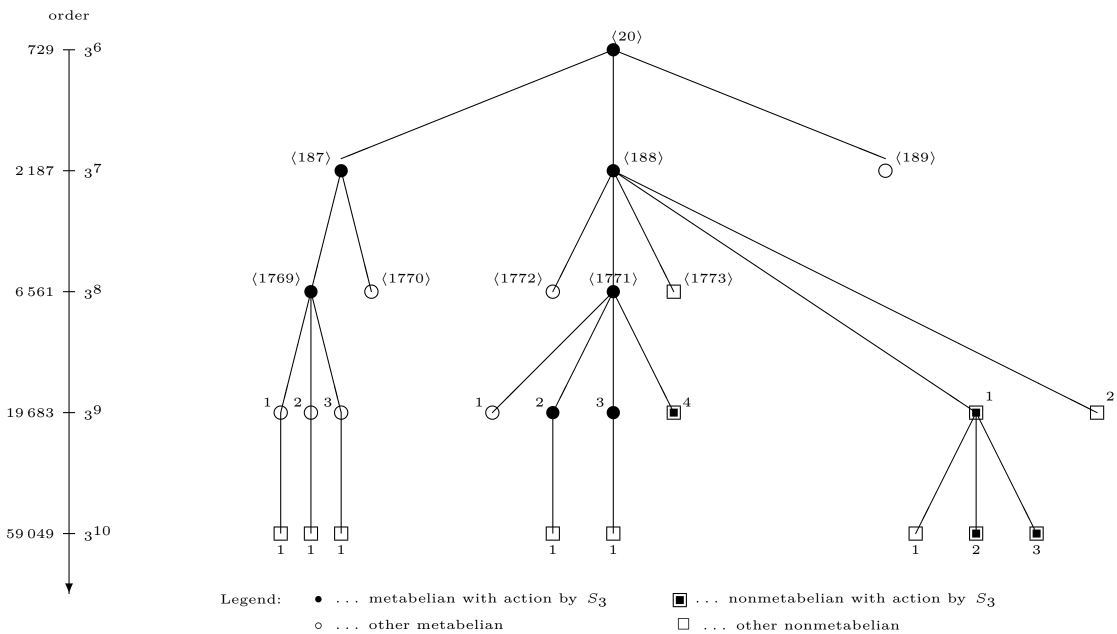

Figure 5. Descendant tree of $\langle 729, 20\rangle$ with stable $\varkappa = (132; 4)$, $\alpha = [(27,3)^3; (9,3,3)]$.

immediate descendants of the root $\langle 729, 17\rangle$ are $\langle 2187, 177\rangle$ with action by S_3 and $\langle 2187, 179\rangle$ with action by C_2 only. There are exactly two further candidates for G_2 with action by S_3, namely the metabelian groups $\langle 6561, 1731\rangle$ and $\langle 6561, 1733\rangle - \#1; 3$. However, $\langle 6561, n\rangle$ with $n \in \{1731, 1733\}$ and $\langle 6561, 1733\rangle - \#1; s$ with $s \in \{2, 3\}$ share the forbidden second layer $\varkappa_2 = (H_{1,3}, H_{2,3}, H_{3,3}; H_{4,3})$, $\alpha_2 = [(27,9)^3; (9,9,3)]$. □

Corollary 5.12 *For the fields k with harmonically balanced capitulation $\varkappa = (123; 4)$ and first variant of $\alpha = [(27,3)^3; (9,3,3)]$ (Theorem 5.11) the 3-class tower of k must stop at the second stage, that is, $k_3^{(2)} = k_3^{(\infty)}$ is the maximal unramified pro-3-extension of k.*

Proof The groups G_2 in Theorem 5.11 are not second derived quotients G/G'' of nonmetabelian 3-groups G. □

Proposition 5.13 *A power commutator presentation of the finite metabelian 3-group $\langle 2187, i\rangle$ in terms of the commutators $s_2 = [y, x]$, $s_3 = [s_2, x]$, $s_4 = [s_3, x]$, $t_3 = [s_2, y]$ is given by*

$$\begin{cases} \langle x, y, s_2, s_3, s_4, t_3 \mid x^9 = t_3,\ y^3 = s_3^2,\ s_2^3 = s_4^2\rangle & \text{if } i = 180, \\ \langle x, y, s_2, s_3, s_4, t_3 \mid x^9 = t_3^2,\ y^3 = s_3^2,\ s_2^3 = s_4^2\rangle & \text{if } i = 190. \end{cases} \tag{5.4}$$

The groups are periodic of class 4 and coclass 3.

Proof Presentations of these groups are part of the SmallGroups database, implemented in Magma [12]. □

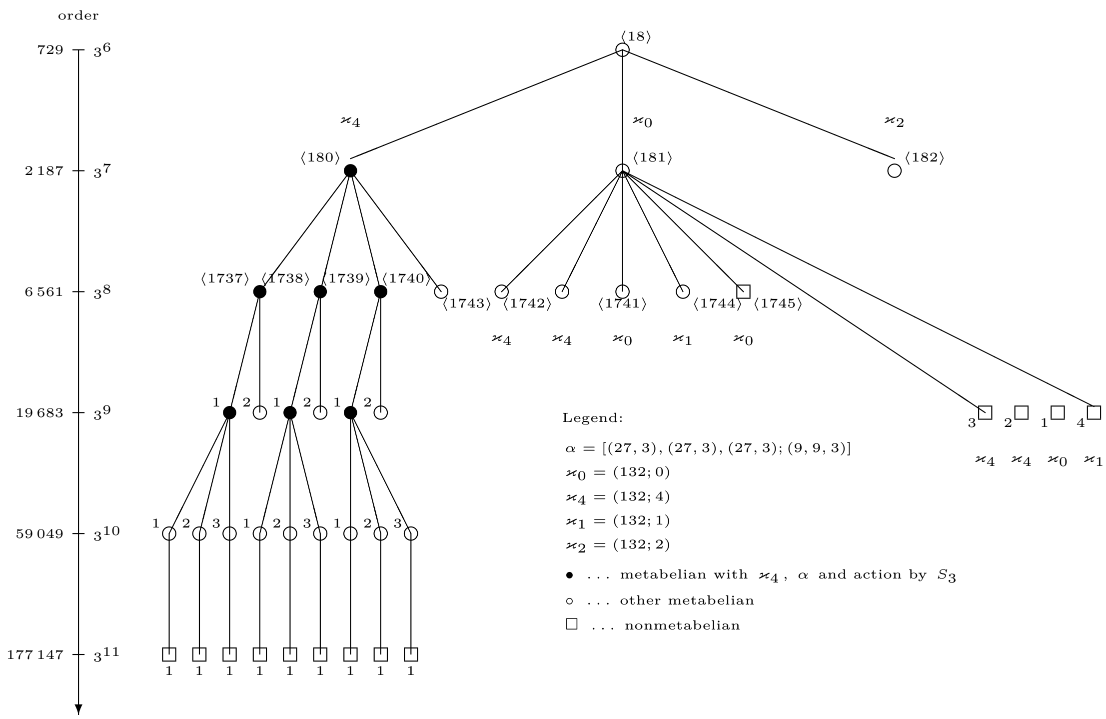

Figure 6. Descendant tree of $\langle 729, 18\rangle$.

Theorem 5.14 *For a pure metacyclic field* $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ *with* $p \equiv 1 \pmod 9$ *having harmonically balanced capitulation* $\varkappa(k) = (123; 4)$ *and the second variant of* $\alpha = [(27, 3)^3; (9, 9, 3)]$*, the periodic Galois group* G_2 *of the second Hilbert* 3*-class field* $k_3^{(2)}$ *is given by* $\mathrm{Gal}(k_3^{(2)}/k) \simeq$

$$\begin{cases} \langle 2187, m\rangle & \textit{if } \varkappa_2(k) = ((C_{k,3})^3; H_{4,3}),\ \alpha_2(k) = [(9, 9)^3; (9, 9, 3)], \\ \langle 6561, n\rangle & \textit{if } \varkappa_2(k) = ((C_{k,3})^3; H_{4,3}),\ \alpha_2(k) = [(27, 9)^3; (9, 9, 9)], \end{cases} \tag{5.5}$$

where $m \in \{180, 190\}$ *and* $n \in \{1737, 1738, 1739, 1775, 1776, 1777\}$*. See Figures 6 and 7.*

Proof The required Artin pattern $(\varkappa, \alpha)$ with harmonically balanced capitulation $\varkappa = (123; 4)$ and second variant of $\alpha = [(27, 3)^3; (9, 9, 3)]$ cannot occur for descendants of the roots $\langle 729, i\rangle$ with $i \in \{17, 20\}$, because on the entire descendant trees of these sporadic roots $\alpha = [(27, 3)^3; (9, 3, 3)]$ remains stable.

The only possibility are vertices of the coclass trees with roots $\langle 729, i\rangle$ for $i \in \{18, 21\}$. Since the trees are isomorphic as structured graphs, we focus on $\langle 729, 21\rangle$, which has three immediate descendants, $\langle 2187, 190\rangle$ with $\varkappa = (123; 4)$, $\alpha = [(27, 3)^3; (9, 9, 3)]$, the mainline group $\langle 2187, 191\rangle$ with host type $\varkappa = (123; 0)$ like the parent $\langle 729, 21\rangle$, and $\langle 2187, 192\rangle$ with inadequate $\varkappa = (123; 2)$. Due to the antitony principle for the components of the Artin pattern $(\varkappa, \alpha)$, all descendants of $\langle 2187, 191\rangle$ can be eliminated, because they have $\alpha \geq [(27, 3)^3; (27, 9, 3)]$. The group $\langle 2187, 190\rangle$ has the required action by $S_3 = \langle 6, 1\rangle$, and this is also true for three of its immediate descendants $\langle 6561, n\rangle$ with $1775 \leq n \leq 1777$ but not for $n = 1778$ with action by $C_3 = \langle 3, 1\rangle$ only. Each of the three former has an immediate descendant $\langle 6561, n\rangle - \#1; 1$ with $1775 \leq n \leq 1777$ and action by S_3. The other descendant $\langle 6561, n\rangle - \#1; 2$ has action by C_3, and three further descendants

$\langle 6561, n\rangle - \#1; 1 - \#1; i$ with $1 \le i \le 3$ have only an action by $C_2 = \langle 2, 1\rangle$. Further suitable candidates for G_2 are impossible. Finally, the groups $\langle 6561, n\rangle - \#1; 1$ with $n \in \{1775, 1776, 1777\}$ are discouraged by a wrong transfer kernel in the second layer with $\varkappa_2 = (H_{1,3}, H_{2,3}, H_{3,3}; H_{4,3})$, $\alpha_2 = [(27, 27)^3; (9, 9, 9)]$. □

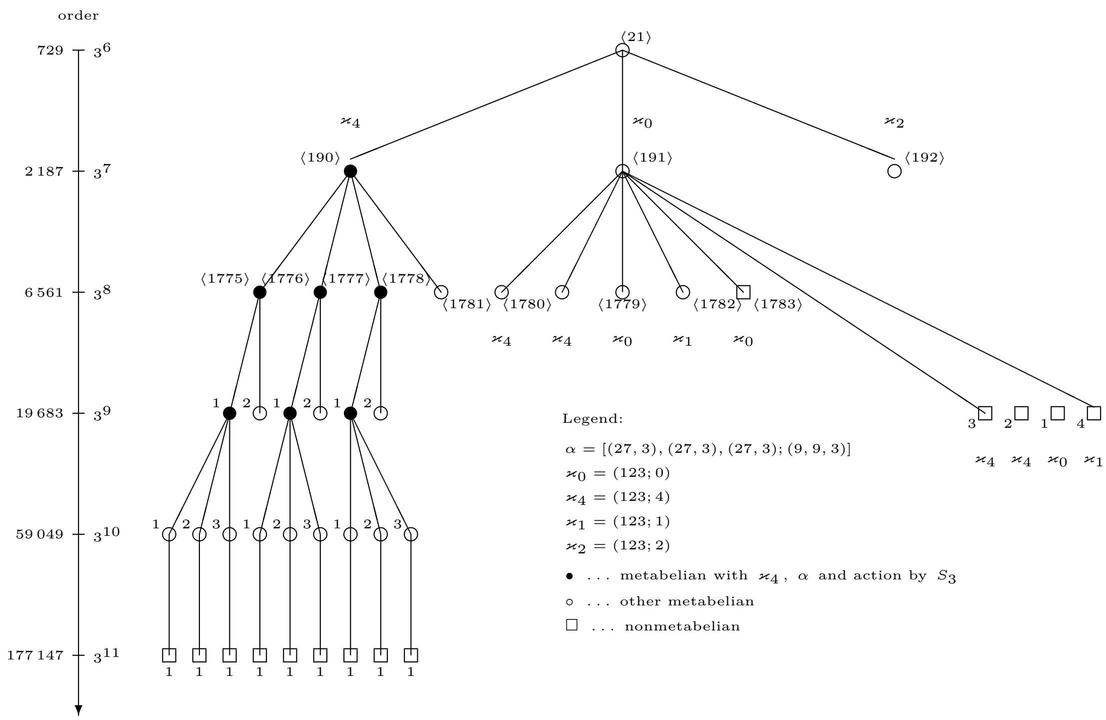

Figure 7. Descendant tree of $\langle 729, 21\rangle$.

Corollary 5.15 *For the fields k with harmonically balanced capitulation $\varkappa = (123; 4)$ and second variant of $\alpha = [(27, 3)^3; (9, 9, 3)]$ (Theorem 5.14) the 3-class tower of k must stop at the second stage, that is, $k_3^{(2)} = k_3^{(\infty)}$ is the maximal unramified pro-3-extension of k.*

Proof According to the antitony principle, there cannot exist nonmetabelian 3-groups G whose metabelianization G/G'' is isomorphic to one of the 14 candidates for G_2 in Theorem 5.14. Thus $k_3^{(2)} = k_3^{(\infty)}$. □

Figures **??** illustrate the location in descendant trees of all metabelian groups M and certain nonmetabelian groups G which occur in subsection § 5.2.

The proofs of the Corollaries 5.12 and 5.15 are based on the following fact. All the candidates for G_2 in Theorems 5.11 and 5.14 satisfy the inequalities $2 \le d_2 \le 5$ for the relation rank in Theorem 5.5, since they even satisfy the more severe estimates $3 \le d_2 \le 4$. So there is no reason which precludes a metabelian tower with length $\ell_3(k) = 2$.

Remark 5.16 *We emphasize that the strict limitation $\ell_3(k) = 2$ for the length of the 3-class field tower of k in Corollary 5.12 is only due to item 5. of Theorem 4.2, i.e. the requirement $\ker(T_{K_{1,9}/k}) = C_{k,3}$ for the second layer.*

Although we also had $\ell_3(k) = 2$ if $G_2 = \langle 6561, n\rangle$, $n \in \{1731, 1769\}$, were admissible, $\ell_3(k) = 3$ would be enabled for $r \in \{1733, 1771\}$, and $\langle 6561, r\rangle \simeq G/G''$ with $G = \langle 6561, r\rangle - \#1; 4$, resp. $\langle 6561, r\rangle - \#1; s \simeq G/G''$ with $G = \langle 2187, m\rangle - \#2; 1 - \#1; s$, $m \in \{178, 188\}$, $s \in \{2, 3\}$. The groups G are nonmetabelian with action by S_3, in contrast to $\langle 6561, r\rangle$ with $r \in \{1735, 1773\}$. On the other hand, it is known that the descendant trees of the roots $\langle 729, i\rangle$ with $i \in \{17, 20\}$ contain vertices with unbounded derived length, whence any finite value $\ell_3(k) \geq 4$ would also be possible. (Note that the tree continues at the nonmetabelian vertex $\langle 2187, m\rangle - \#2; 1 - \#1; 2$ with $m = 178$, resp. $m = 188$.)

5.2.4. Total capitulation

Due to the wealth of metabelian groups M of low orders $\#M \leq 3^8$ in the descendant tree of the root $\langle 729, 9\rangle$, we restrict ourselves to immediate descendants of the root with action by S_3.

Proposition 5.17 *A power commutator presentation of the finite metabelian 3-group $\langle 729, 9\rangle$ is given by*

$$\langle x, y, s_2, s_3, t_3 \mid x^9 = 1,\ y^3 = 1,\ s_2 = [y, x],\ s_3 = [s_2, x],\ t_3 = [s_2, y]\rangle. \tag{5.6}$$

The group is periodic of class 3 and coclass 3.

Proof The presentation of $\langle 729, 9\rangle$ is part of the SmallGroups database, implemented in Magma [12]. □

Theorem 5.18 *For a pure metacyclic field $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ with $p \equiv 1 \pmod 9$ having total capitulation $\varkappa(k) = (000; 4)$ and abelian type invariants $\alpha(k) = [(9, 3, 3)^3; (3, 3, 3, 3)]$, the smallest possible Galois groups G_2 of the second Hilbert 3-class field $k_3^{(2)}$ are given by*

$$\mathrm{Gal}(k_3^{(2)}/k) \simeq \begin{cases} \langle 729, 9\rangle & \text{if } \alpha_2(k) = [(3,3,3)^3; (3,3,3,3)], \\ \langle 2187, 123\rangle & \text{if } \alpha_2(k) = [(3,3,3,3)^3; (3,3,3,3,3)], \\ \langle 2187, 124\rangle & \text{if } \alpha_2(k) = [(9,3,3)^3; (3,3,3,3,3)], \\ \langle 6561, i\rangle & \text{if } \alpha_2(k) = [(3,3,3,3)^3; (3,3,3,3,3,3)], \\ \langle 6561, 109\rangle & \text{if } \alpha_2(k) = [(3,3,3,3)^3; (9,3,3,3,3)], \\ \langle 6561, j\rangle & \text{if } \alpha_2(k) = [(9,3,3)^3; (9,3,3,3,3)], \end{cases} \tag{5.7}$$

where $i \in \{103, 105\}$ and $j \in \{110, 111\}$.

Corollary 5.19 *For the fields k with total capitulation $\varkappa(k) = (000; 4)$ in Theorem 5.18, the length of the 3-class field tower $k_3^{(\infty)}$ is given by*

1. *$\ell_3(k) \geq 2$, if $G_2 \in \{\langle 729, 9\rangle, \langle 2187, 124\rangle, \langle 6561, 109\rangle, \langle 6561, 110\rangle, \langle 6561, 111\rangle\}$,*
2. *$\ell_3(k) \geq 3$, if $G_2 \in \{\langle 2187, 123\rangle, \langle 6561, 103\rangle, \langle 6561, 105\rangle\}$.*

Proof (Proof of Theorem 5.18 and Corollary 5.19) Since the root $\langle 729, 9\rangle$ has nuclear rank $\nu = 3$, it has descendants of step sizes $s \in \{1, 2, 3\}$. The 37 children with $s = 3$ are of order $3^9 = 19683$ and possess abelian quotient invariants α beyond the threshold $[(9, 9, 3)^3; (3, 3, 3, 3)]$. Among the 15, resp. 61, children with $s = 1$, resp. $s = 2$, and order $3^7 = 2187$, resp. $3^8 = 6561$, only the two, resp. five, mentioned possess an action by

S_3. Concerning the length of 3-class field towers, the groups G_2 in item *1.* of the corollary have relation ranks $4 \le d_2 \le 5$, thus admitting a two-stage tower, whereas those in item *2.* have $6 \le d_2 \le 7$, which definitely excludes $\ell_3(k) = 2$. □

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