A contiguous extension of Dixon’s theorem for a terminating $4F_3(1)$ series with applications

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Abstract: We derive a summation formula for the terminating hypergeometric series

$$4F_3\left[\begin{array}{c}
-m, a, b, 1 + c \\
1 + a + m, 1 + a - b, c \\
\end{array}; 1 \right],$$

where $m$ denotes a nonnegative integer. Using this summation formula, we establish a reduction formula for the Srivastava–Daoust double hypergeometric function with arguments $z$ and $-z$. Special cases of this reduction formula lead to several reduction formulas for the hypergeometric functions $p+1F_p$ with quadratic arguments when $p = 2, 3$ and 4 by employing series rearrangement techniques. A general double series identity involving a bounded sequence of arbitrary complex numbers is also given.

Key words: Hypergeometric summation theorems, Srivastava–Daoust double hypergeometric function, bounded sequence, series rearrangement technique

1. Introduction

In our investigations, we shall use the following standard notation: $\mathbb{N} := \{1, 2, 3, \cdots \}; \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \mathbb{Z}^- := \mathbb{Z} \cup \{0\} = \{0, -1, -2, -3, \cdots \}$. The symbols $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Z}$ denote the sets of complex numbers, real numbers, natural numbers and integers, respectively. The well-known Pochhammer symbol (or the shifted factorial) is given by $(\alpha)_n = \alpha(\alpha + 1)\cdots(\alpha + n - 1) = \Gamma(\alpha + n)/\Gamma(\alpha)$, it being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the gamma quotient exists.

A natural generalization of the Gaussian hypergeometric series $2F_1[\alpha, \beta; \gamma; z]$ is accomplished by introducing an arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$pF_q\left[\begin{array}{c}
(\alpha_p) \\
(\beta_q) \\
\end{array}; z \right] = pF_q\left[\begin{array}{c}
\alpha_1, \alpha_2, \ldots, \alpha_p \\
\beta_1, \beta_2, \ldots, \beta_q \\
\end{array}; z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_q)_n} \frac{z^n}{n!}$$

is known as the generalized hypergeometric function. Here $p$ and $q$ are nonnegative integers, the variable $z \in \mathbb{C}$ and we write $(\alpha_p) = (\alpha_1, \alpha_2, \ldots, \alpha_p)$. The numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator

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parameters $\beta_1, \beta_2, \ldots, \beta_q$ can, in general, take on complex values, provided that
\[
\beta_j \neq 0, -1, -2, \ldots, \quad (j = 1, 2, \ldots, q).
\]
Assuming that none of the numerator and denominator parameters is zero or a negative integer, the $_pF_q(z)$ function defined by Equation (1.1) converges for $|z| < \infty$ ($p \leq q$), $|z| < 1$ ($p = q + 1$) and $|z| = 1$ ($p = q + 1$ and $\Re(s) > 0$), where $s$ is the parametric excess defined by
\[
s = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j.
\] (1.2)

In an earlier paper [13, p.199], Srivastava and Daoust defined a generalization of the Kampé de Fériet function [2, p.150] by means of the double hypergeometric series (see also [14, 15]):
\[
_F^A: B; D; D' \left[ \frac{((\alpha_A) : \vartheta; \varphi) ; ((\beta_B') : \psi')}{((\gamma_C) : \xi; \epsilon)} ; ((\delta_D') : \eta') \right] x, y
\]
\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{A} (\alpha_j)_m \psi_j n^m}{\prod_{j=1}^{B} (\beta_j)'_m \psi_j n^m} \frac{\prod_{j=1}^{B'} (\beta_j')_n \psi_j n^m}{\prod_{j=1}^{D} (\delta_j)_n \psi_j n^m} \frac{\prod_{j=1}^{D'} (\delta_j')_n \psi_j n^m}{m! n!},
\] (1.3)
where the coefficients
\[
\begin{cases}
\vartheta_1, \ldots, \vartheta_A; \varphi_1, \ldots, \varphi_A; \psi_1, \ldots, \psi_B; \psi'_1, \ldots, \psi'_B; \xi_1, \ldots, \xi_C; \\
\epsilon_1, \ldots, \epsilon_C; \eta_1, \ldots, \eta_D; \eta'_1, \ldots, \eta'_D
\end{cases}
\]
are real and positive. The double power series in (1.3) converges for all complex values of $x$ and $y$ when $\Delta_1 > 0$, $\Delta_2 > 0$; for suitably constrained values of $|x|$ and $|y|$ when $\Delta_1 = \Delta_2 = 0$; and diverges (except in the trivial case $x = y = 0$) when $\Delta_1 < 0$, $\Delta_2 < 0$, where
\[
\Delta_1 = 1 + \sum_{j=1}^{C} \epsilon_j + \sum_{j=1}^{D} \eta_j - \sum_{j=1}^{A} \vartheta_j - \sum_{j=1}^{B} \psi_j,
\]
\[
\Delta_2 = 1 + \sum_{j=1}^{C} \epsilon_j + \sum_{j=1}^{D'} \eta'_j - \sum_{j=1}^{A} \varphi_j - \sum_{j=1}^{B'} \psi'_j.
\]

Motivated by the studies of Miller [4, 5], Miller and Paris [6–8], Miller and Srivastava [9], we obtain a summation formula for a terminating series $_4F_3(1)$ in Section 3. In Section 4, this summation formula is used to derive a reduction formula for the Srivastava–Daoust double hypergeometric function defined in (1.3) with arguments $z$ and $-z$. The consideration of special cases of this last result enables a few reduction formulas for the generalised hypergeometric function $_p+1F_p$ ($p = 2, 3, 4$) with quadratic arguments to be deduced using a series rearrangement technique. In the final section, we specify a general double-series identity involving a bounded sequence of complex numbers.

It should be observed that throughout we tacitly exclude any values of the parameters and arguments in Sections 3 to 5 leading to results that do not make sense.
2. Preliminaries

In this section we present some preliminary results necessary for our investigation. First, we state Cauchy’s double series identity [11, p. 56], [16, p. 100]

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Theta(m, n) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \Theta(m - n, n),
\]

(2.1)

provided that the associated double series are absolutely convergent.

Our second result is Dixon’s theorem [10, p. 535, Entry 21]:

\[
_{3}F_{2}\left[\begin{array}{l}
a, b, c \\
1 + a - b, 1 + a - c;
\end{array}\right] = \frac{\Gamma(1 + a - b) \Gamma(1 + a - c) \Gamma(1 + \frac{1}{2}a) \Gamma(1 + \frac{1}{2}a - b - c)}{\Gamma(1 + \frac{1}{2}a - b) \Gamma(1 + \frac{1}{2}a - c) \Gamma(1 + a) \Gamma(1 + a - b - c)},
\]

(2.2)

where \( \Re(a - 2b - 2c) > -2 \) and \( 1 + a - b, 1 + a - c \in \mathbb{C} \setminus \mathbb{Z}_0 \). When \( c = -m \) in (2.2), the terminating form of Dixon’s theorem is given by

\[
_{3}F_{2}\left[\begin{array}{l}
-m, a, b \\
1 + a + m, 1 + a - b;
\end{array}\right] = \frac{(1 + a)_m (1 + \frac{1}{2}a - b)_m}{(1 + a - b)_m (1 + \frac{1}{2}a)_m},
\]

(2.3)

where \( a, b, 1 + a - b \in \mathbb{C} \setminus \mathbb{Z}_0 \) and \( m \in \mathbb{N}_0 \).

The contiguous extension of Dixon’s theorem is [10, p. 535, Entry 22] (see also [3, p. 13, Eq.(4.7)])

\[
_{3}F_{2}\left[\begin{array}{l}
a, b, c \\
2 + a - b, 2 + a - c;
\end{array}\right] = \frac{\Gamma(2 + a - b) \Gamma(2 + a - c)}{2(b-1)(c-1)\Gamma(a) \Gamma(2 + a - b - c)} \times
\]

\[
\times \left\{ \frac{\Gamma(\frac{1}{2}a) \Gamma(2 + \frac{1}{2}a - b - c)}{\Gamma(1 + \frac{1}{2}a - b) \Gamma(1 + \frac{1}{2}a - c)} - \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}) \Gamma(\frac{5}{2} + \frac{1}{2}a - b - c)}{\Gamma(\frac{5}{2} + \frac{1}{2}a - b) \Gamma(\frac{5}{2} + \frac{1}{2}a - c)} \right\},
\]

(2.4)

where \( \Re(a - 2b - 2c) > -4 \) and \( 2 + a - b, 2 + a - c \in \mathbb{C} \setminus \mathbb{Z}_0 \) and \( b \neq 1, c \neq 1 \). When \( c = -m \), the terminating contiguous form of (2.4) is given by

\[
_{3}F_{2}\left[\begin{array}{l}
-m, a, b \\
2 + a + m, 2 + a - b;
\end{array}\right] = \frac{\Gamma(2 + a + m) \Gamma(2 + a)}{2(b-1)(-m-1)\Gamma(a) \Gamma(2 + a - b + m)} \times
\]

\[
\times \left\{ \frac{\Gamma(\frac{1}{2}a) \Gamma(2 + \frac{1}{2}a - b + m)}{\Gamma(1 + \frac{1}{2}a - b) \Gamma(1 + \frac{1}{2}a + m)} - \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}) \Gamma(\frac{5}{2} + \frac{1}{2}a - b + m)}{\Gamma(\frac{5}{2} + \frac{1}{2}a - b) \Gamma(\frac{5}{2} + \frac{1}{2}a + m)} \right\}
\]

(2.5)

\[
_{3}F_{2}\left[\begin{array}{l}
-m, a, b \\
2 + a + m, 2 + a - b;
\end{array}\right] = \frac{\Gamma(2 + a + m) \Gamma(2 + a)}{2(1-b)(m+1)\Gamma(2 + a) \Gamma(a)(2 + a - b)_m} \times
\]

\[
\times \left\{ \frac{\Gamma(\frac{1}{2}a) \Gamma(2 + \frac{1}{2}a - b + m)}{\Gamma(1 + \frac{1}{2}a - b) \Gamma(1 + \frac{1}{2}a + m)} - \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}) \Gamma(\frac{5}{2} + \frac{1}{2}a - b + m)}{\Gamma(\frac{5}{2} + \frac{1}{2}a - b) \Gamma(\frac{5}{2} + \frac{1}{2}a + m)} \right\}
\]

(2.6)
Using the identity \( \Gamma(z + 1) = z\Gamma(z) \) (see [11]), after simplification we can write

\[
_{3}F_{2} \left[ -m, \ a, \ b \atop 2 + a + m, \ 2 + a - b \right] = \frac{a(1 + a)(2 + a)m}{2(1 - b)(m + 1)(2 + a - b)m} \times
\]

\[
\frac{1}{a(1 + \frac{1}{2}a)m} \times \left\{ \frac{(a - 2b + 2)(2 + \frac{1}{2}a - b)m}{(2 + a)^{m}} - \frac{(a - 2b + 3)(\frac{5}{2} + \frac{1}{2}a - b)m}{(2 + a - b)^{m}} \right\}, \tag{2.5}
\]

where \( a, \ b, \ 2 + a - b \in \mathbb{C}\setminus\mathbb{Z}^{-} \), \( b \neq 1 \) and \( m \in \mathbb{N}_{0} \). The summation formulas (2.3) and (2.5) will play an important role in our subsequent analysis.

We have the closed-form evaluations of the Gauss hypergeometric function (see [1, p. 185, Ex. (39)], [11, p. 70, Ex. (10)], [12, p.19, Eq.(1.5.20))):

\[
_{2}F_{1} \left[ \alpha, \frac{\alpha - \frac{1}{2}}{\alpha} \atop \frac{2}{1 + \sqrt{(1 - z)}} \right]^{2\alpha - 1}, \tag{2.6}
\]

and

\[
_{2}F_{1} \left[ \alpha, \frac{\alpha + \frac{1}{2}}{\alpha} \atop \frac{2}{1 + \sqrt{(1 - z)}} \right]^{2\alpha - 1}, \tag{2.7}
\]

where \( 2\alpha \in \mathbb{C}\setminus\mathbb{Z}^{-} \) and \( |\arg(1 - z)| < \pi \). These last two results enable us to obtain the following lemma:

**Lemma 2.1** We have the closed-form evaluation of Clausen’s function given by

\[
_{3}F_{2} \left[ \alpha + 1, \beta, \frac{\beta - \frac{1}{2}}{\alpha} \atop \alpha, \frac{2\alpha}{2\beta} \right] = \left( \frac{2}{1 + \sqrt{(1 - z)}} \right)^{2\beta - 1} \left[ 1 + \frac{(2\beta - 1)z}{2\alpha(1 - z + \sqrt{(1 - z)})} \right], \tag{2.8}
\]

where \( \alpha, \ 2\beta \in \mathbb{C}\setminus\mathbb{Z}^{-} \) and \( |\arg(1 - z)| < \pi \).

**Proof:** We have

\[
_{3}F_{2} \left[ \alpha + 1, \beta, \frac{\beta - \frac{1}{2}}{\alpha} \atop \alpha, \frac{2\alpha}{2\beta} \right] = \sum_{r=0}^{\infty} \frac{(\beta)_{r} (\beta - \frac{1}{2})_{r} z^{r}}{(2\beta)_{r} r!} \left( 1 + \frac{r}{\alpha} \right)
\]

\[
= \_{2}F_{1} \left[ \beta, \frac{\beta - \frac{1}{2}}{2\beta} ; z \right] + \frac{(2\beta - 1)z}{4\alpha} \_{2}F_{1} \left[ \beta + \frac{1}{2}, \frac{\beta + 1}{2\beta} ; z \right]. \tag{2.9}
\]

Using the closed forms (2.6) and (2.7) in the right-hand side of (2.9), we obtain after some simplification the required result (2.8).
3. A summation formula

In this section, we derive a summation formula for a terminating $4F_3$ series with positive unit argument, which we believe is not in the literature. This takes the following form:

**Theorem 3.1** The following result holds true:

\[
4F_3 \left[ \begin{array}{c}
-m, a, b, 1 + c \\
1 + a + m, 1 + a - b, c
\end{array} ; 1 \right] = \frac{(1 + a)_m}{(1 + a - b)_m} \left\{ \left( 1 - \frac{a}{2c} \right) \frac{(1 + \frac{a}{2} a - b)_m}{(1 + \frac{a}{2} a)_m} + \frac{a}{2c} \frac{(1 + \frac{a}{2} a - b)_m}{(1 + \frac{a}{2} a)_m} \right\}, \tag{3.1}
\]

where $m \in \mathbb{N}_0$ and $a, b, c, 1 + a - b \in \mathbb{C}\backslash\mathbb{Z}_0^−$.

**Proof.** Let

\[
H := 4F_3 \left[ \begin{array}{c}
-m, a, b, 1 + c \\
1 + a + m, 1 + a - b, c
\end{array} ; 1 \right] = \sum_{r=0}^{m} \frac{(-m)_r (a)_r (b)_r (1 + c)_r}{(1 + a + m)_r (1 + a - b)_r c_r!}.
\]

Replacing $r$ by $r + 1$, we obtain the second term on the right-hand side of (3.2) in the form

\[
\frac{1}{c} \sum_{r=0}^{m-1} \frac{(-m)_{r+1} (a)_{r+1} (b)_{r+1}}{(1 + a + m)_{r+1} (1 + a - b)_{r+1} c_{r+1}!} = -\frac{mab}{c(1 + a + m)(1 + a - b)} \sum_{r=0}^{m-1} \frac{(-m)_r (a)_r (b + 1)_r}{(2 + a + m)(2 + a - b) c_r!}.
\]

Identification of this last sum as the $3F_2(1)$ series with parameters augmented by unity then leads to the result

\[
H = 3F_2 \left[ \begin{array}{c}
-m, a, b \\
1 + a + m, 1 + a - b
\end{array} ; 1 \right] \frac{mab}{c(1 + a + m)(1 + a - b)} 3F_2 \left[ \begin{array}{c}
-(m - 1), a + 1, b + 1 \\
2 + a + m, 2 + a - b
\end{array} ; 1 \right]. \tag{3.3}
\]

Use of the results stated in (2.3) and (2.5) in the first and second hypergeometric series on the right-hand side of (3.3), then leads to

\[
H = \frac{(1 + a)_m (1 + \frac{a}{2} a - b)_m}{(1 + a - b)_m (1 + \frac{a}{2} a)_m} \times \frac{(a)_{m+2}}{2c(1 + a + m)(1 + a - b)_m} \times \left\{ \frac{(1 + a - 2b)(\frac{3}{2} + \frac{1}{2} a - b)_{m-1}}{(1 + a)(\frac{3}{2} + \frac{1}{2} a)_{m-1}} - \frac{(2 + a - 2b)(2 + \frac{1}{2} a - b)_{m-1}}{(2 + a)(2 + \frac{1}{2} a)_{m-1}} \right\}.
\]

Finally, employing the fact that $(\alpha)_{m-1} = (\alpha - 1)_{m}/(\alpha - 1)$ and after some straightforward simplification, we obtain the required result (3.1).

**Corollary 1.** If we set $c = \frac{1}{2} a$ in (3.1) then we recover the known summation formula

\[
4F_3 \left[ \begin{array}{c}
-m, a, b, 1 + \frac{1}{2} a \\
\frac{1}{2} a, 1 + a + m, 1 + a - b
\end{array} ; 1 \right] = \frac{(1 + a)_m (1 + \frac{1}{2} a - b)_m}{(\frac{1}{2} + \frac{1}{2} a)_m (1 + a - b)_m} \tag{3.4}
\]

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Corollary 2. If we set $c = \frac{1}{2}b$ in (3.1) then we obtain the summation formula

$$4F_3 \left[ \begin{array}{c} -m, a, b, 1 + \frac{1}{2}b \\ \frac{1}{2}b, 1 + a + m, 1 + a - b \end{array} ; 1 \right] = \frac{(1 + a)_m}{(1 + a - b)_m} \left\{ \frac{(1 - a)}{b} \left( 1 + \frac{1}{2}a - b \right)_m + \frac{(a)}{b} \left( \frac{1}{2} + \frac{1}{2}a - b \right)_m \right\} + (\alpha \frac{2}{2})^A \sum_{B+2} \left[ (a_A), 1 + \alpha, 1 + \frac{1}{2} \alpha - \beta \right. (b_B), 1 + \alpha - \beta, 1 + \frac{1}{2} \alpha ; z \right] \right\}, \quad (3.5)

4. An application of Theorem 3.1 to the Srivastava–Daoust function

Here we establish a result concerning the reducibility of the Srivastava–Daoust double hypergeometric function defined in (1.3) given in the following theorem:

Theorem 4.1 The following result holds true:

$$F_{A+1: 0; 3}^{A+1: 0; 2} \left[ \begin{array}{c} (a_A) : 1, 1, 1 + [a : 1], [b : 1], [1 + \gamma : 1]; \\ (b_B) : 1, 1, 1 + [a : 1, 2], [1 + \gamma : 1], z, -z \end{array} \right] = \left( 1 - \frac{\alpha}{2\gamma} \right)^A \sum_{B+2} \left[ (a_A), 1 + \alpha, 1 + \frac{1}{2} \alpha - \beta \right. (b_B), 1 + \alpha - \beta, 1 + \frac{1}{2} \alpha ; z \right] \right\}, \quad (4.1)

where $b_1, b_2, ..., b_B, \alpha, \beta, \gamma, 1 + \alpha - \beta \in \mathbb{C} \setminus \mathbb{Z}_0$. When $A \leq B$ both sides of (4.1) are convergent for $|z| < \infty$, but when $A = B + 1$ the two sides are convergent for suitably constrained values of $|z|$.

Proof: Let

$$F := F_{A+1: 0; 3}^{A+1: 0; 2} \left[ \begin{array}{c} (a_A) : 1, 1, 1 + [a : 1], [b : 1], [1 + \gamma : 1]; \\ (b_B) : 1, 1, 1 + [a : 1, 2], [1 + \gamma : 1], z, -z \end{array} \right] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1)_{m+n} \ldots (a_A)_{m+n} (1 + \alpha)_m \ldots (\alpha)_m (\beta)_n (1 + \gamma)_n z^m (-z)^n}{(b_1)_{m+n} \ldots (b_B)_{m+n} (1 + \alpha - \beta)_m \ldots (\alpha - \beta)_m (\gamma)_n m! n!}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1)_{m+n} \ldots (a_A)_{m+n} (\alpha) \ldots (\alpha) \ldots (\beta) \ldots (1 + \gamma)_n \gamma (1 + \alpha + m + n)_n (1 + \alpha - \beta)_n m! m! n!}{(b_1)_{m+n} \ldots (b_B)_{m+n} (1 + \alpha + m + n)_n (1 + \alpha - \beta)_n (\gamma)_n m! n!}. \quad (4.2)

Replacing $m$ by $m - n$ in (4.2), we find upon application of (2.1) that

$$F = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(a_1)_m \ldots (a_A)_m (\alpha) \ldots (\alpha) \ldots (\beta) \ldots (1 + \gamma)_n (1 + \alpha + m + n)_n (1 + \alpha - \beta)_n \gamma (m - n)_n m! m! n!}{(b_1)_m \ldots (b_B)_m (1 + \alpha + m + n)_n (1 + \alpha - \beta)_n (\gamma)_n m! n!}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(-m)_n (\alpha) \ldots (\alpha) \ldots (\beta) \ldots (1 + \gamma)_n}{(b_1)_m \ldots (b_B)_m m!} \sum_{n=0}^{m} \frac{(-m)_n (\alpha) \ldots (\alpha) \ldots (\beta) \ldots (1 + \gamma)_n}{(1 + \alpha + m + n)_n (1 + \alpha - \beta)_n (\gamma)_n n!} m! m! n!$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(a_1)_m \ldots (a_A)_m z^m}{(b_1)_m \ldots (b_B)_m m!} \sum_{n=0}^{m} \frac{(-m)_n (\alpha) \ldots (\alpha) \ldots (\beta) \ldots (1 + \gamma)_n}{(1 + \alpha + m + n)_n (1 + \alpha - \beta)_n (\gamma)_n n!} m! m! n!$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(a_1)_m \ldots (a_A)_m z^m}{(b_1)_m \ldots (b_B)_m m!} \sum_{n=0}^{m} \frac{(-m)_n (\alpha) \ldots (\alpha) \ldots (\beta) \ldots (1 + \gamma)_n}{(1 + \alpha + m + n)_n (1 + \alpha - \beta)_n (\gamma)_n n!} m! m! n!.$$
Finally, employing the summation formula (3.1), we arrive at the right-hand side of (4.1) after some routine simplification.

Corollary 3. If we take \( \gamma = \frac{1}{2} \alpha \) in (4.1) we obtain another reduction formula:

\[
F_{B+1; 0; \frac{1}{2}}^A \left[ \begin{array}{c} (a_A); 1, 1; [1 + \alpha ; 1, 1]; \cdots ; [\alpha ; 1, \beta_1; [1 + \frac{1}{2} \alpha ; 1]; \cdots ; [\frac{3}{2} \alpha ; 1]; z, -z \\
(b_B); 1, 1; [1 + \alpha ; 1, 2]; \cdots ; [1 + \alpha - \beta ; 1], [\frac{3}{2} \alpha ; 1] 
\end{array} \right]
= A + 2 F_{B+2} \left[ \begin{array}{c} (a_A); 1, \alpha, \frac{1}{2} + \frac{1}{2} \alpha - \beta \\
(b_B); 1, \alpha - \beta, \frac{1}{2} + \frac{1}{2} \alpha \end{array} \right],
\]

where \( b_1, b_2, \ldots, b_B, \alpha, \beta, 1 + \alpha - \beta \in \mathbb{C} \setminus \mathbb{Z} \). When \( A \leq B \) both sides of (4.3) converge for \( |z| < \infty \), but when \( A = B + 1 \) both sides converge for suitably constrained values of \(|z|\).

In the following corollaries we present some cases where the Srivastava–Daoust function in (4.1) reduces to a generalised hypergeometric function with a quadratic argument which can be expressed in terms of lower-order hypergeometric functions with linear argument. At this point it will be convenient to introduce the variable

\[
Z := \frac{z}{1 + \sqrt{1 - z}} = \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}}
\]

Corollary 4. In (4.1) put \( A = 2, B = 1, a_1 = \frac{1}{2} + \frac{1}{2} \alpha, a_2 = 1 + \frac{1}{2} \alpha, b_1 = 1 + \alpha \) to yield:

\[
\frac{1}{\sqrt{1 - z}} \left( \frac{2}{1 + \sqrt{1 - z}} \right)^{\alpha} _{3}F_{2} \left[ \begin{array}{c} \alpha, \beta, 1 + \gamma \\
1 + \alpha - \beta, \gamma 
\end{array} \right]_{1 + \alpha - \beta; -Z}
= \left( 1 - \frac{\alpha}{2} \right) _{2}F_{1} \left[ \begin{array}{c} \frac{1}{2} + \frac{1}{2} \alpha, 1 + \frac{1}{2} \alpha - \beta \\
1 + \alpha - \beta; z 
\end{array} \right] + \left( \frac{\alpha}{2} \right) _{2}F_{1} \left[ \begin{array}{c} 1 + \frac{1}{2} \alpha, \frac{1}{2} + \frac{1}{2} \alpha - \beta \\
1 + \alpha - \beta; z 
\end{array} \right],
\]

where \(|Z| < 1, |z| < 1\) and \( \alpha, \beta, 1 + \alpha - \beta \in \mathbb{C} \setminus \mathbb{Z} \).

**Proof:** With the stated parameter values the left-hand side of (4.1) takes the form

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\frac{1}{2} + \frac{1}{2} \alpha)_{m+n}(1 + \frac{1}{2} \alpha)_{m+n}(\alpha)_{n}(1 + \gamma)_{n}z^{m}(-z)^{n}}{(1 + \alpha m + 2n)(1 + \alpha - \beta)_{n}(\gamma)_{n}m!n!}.
\]

Using the identities \((a)_{m+n} = (a)_{n}(a+n)\) and \((a)_{2n} = 2^{n}(\frac{1}{2}a)_{n}(\frac{1}{2} + \frac{1}{2}a)_{n}\) (see [11]), we can write the above double sum as

\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2} + \frac{1}{2} \alpha)_{n}(1 + \frac{1}{2} \alpha)_{n}(\alpha)_{n}(1 + \gamma)_{n}(-z)^{n}}{(1 + \alpha)_{2n}(1 + \alpha - \beta)_{n}(\gamma)_{n}n!} \sum_{m=0}^{\infty} \frac{(\frac{1}{2} + \frac{1}{2} \alpha + n)_{m}(1 + \frac{1}{2} \alpha + n)_{m}z^{m}}{(1 + \alpha + 2n)_{m}m!}
\]

\[
= \sum_{n=0}^{\infty} \frac{(\alpha)_{n}(\beta)_{n}(1 + \gamma)_{n}(-z)^{n}}{2^{2n}(1 + \alpha - \beta)_{n}(\gamma)_{n}n!} _{3}F_{2} \left[ \begin{array}{c} \frac{1}{2} + \frac{1}{2} \alpha + n, 1 + \frac{1}{2} \alpha + n \\
1 + \alpha + 2n 
\end{array} ; z \right]
\]

\[
= \frac{1}{\sqrt{1 - z}} \left( \frac{2}{1 + \sqrt{1 - z}} \right)^{\alpha} \sum_{n=0}^{\infty} \frac{(\alpha)_{n}(\beta)_{n}(1 + \gamma)_{n}(-z)^{n}}{(1 + \alpha - \beta)_{n}(\gamma)_{n}n!} \left( \frac{-z}{1 + \sqrt{1 - z}} \right)^{n},
\]

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upon use of (2.7). Identification of the sum over \( n \) as a \( _3F_2 \) function and simplification of the right-hand side of (4.1) then yields the result stated in (4.4).

**Corollary 5.** In (4.1) put \( A = 2, \ B = 1, \ a_1 = \frac{1}{2} + \frac{1}{2} \alpha, \ a_2 = \frac{1}{2} \alpha, \ b_1 = 1 + \alpha \) to yield upon application of (2.6):

\[
\left( \frac{2}{1 + \sqrt{1-z}} \right)^\alpha \ _3F_2 \left[ \begin{array}{c} \frac{1}{2} \alpha, 1 + \frac{1}{2} \alpha ; \ z \\ 1 + \frac{1}{2} \alpha, 1 + \alpha - \beta, \gamma ; -Z \end{array} \right] = \left( 1 - \frac{\alpha}{2 \gamma} \right) \ _3F_2 \left[ \begin{array}{c} \frac{1}{2} \alpha, 1 + \frac{1}{2} \alpha ; \ z \\ 2, \alpha, 1 + \alpha - \beta, \gamma ; -Z \end{array} \right] \]

where \( |Z| < 1, \ |z| < 1 \) and \( \alpha, \beta, \gamma, 1 + \frac{1}{2} \alpha, 1 + \alpha - \beta \in C \backslash Z_0^- \).

**Corollary 6.** In (4.1) put \( A = 3, \ B = 2, \ a_1 = 2 \alpha, \ a_2 = \frac{1}{2} + \frac{1}{2} \alpha, \ a_3 = \frac{1}{2} \alpha, \ b_1 = 2 \alpha - 1, \ b_2 = 1 + \alpha \) to yield upon application of (2.8):

\[
\left( \frac{2}{1 + \sqrt{1-z}} \right)^\alpha \ _5F_4 \left[ \begin{array}{c} \frac{1}{2} \alpha, 2 \alpha, 1 + \alpha - \beta, \gamma \alpha z \end{array} \right] = \left( 2(2 \alpha - 1)(1-z + \sqrt{1-z}) \right) \ _3F_2 \left[ \begin{array}{c} \alpha, 1 + \alpha - \beta, \gamma ; -Z \end{array} \right] \]

where \( |Z| < 1, \ |z| < 1 \) and \( \alpha, \beta, \gamma, 2 \alpha - 1, 1 + \frac{1}{2} \alpha, 1 + \alpha - \beta \in C \backslash Z_0^- \).

The manipulation of the Srivastava–Daoust function in Corollaries 5 and 6 is similar to that in Corollary 4 and so will be omitted. Corollaries 4–6 have been derived on the assumption that \( |Z| < 1, \ |z| < 1 \). However, these results may be extended by analytic continuation to all \( z \in C \) such that \( |\text{arg}(1-z)| < \pi \) and \( z \neq 1 \) in (4.4), (4.6) and \( z = 1 \) in (4.5) (since the parametric excess (see (1.2)) of the hypergeometric functions on the right-hand sides is \( s = -\frac{1}{2} \) and \( s = \frac{1}{2} \), respectively).

5. A second application of Theorem 3.1 to a general double series

**Theorem 5.1.** Let \( \{ \Phi(p) \}_{p=1}^\infty \) be a bounded sequence of essentially arbitrary numbers (real or complex) such that \( \Phi(0) \neq 0 \). Then, the following general double-series identity holds true:

\[
\sum_{m=0}^\infty \sum_{n=0}^\infty \Phi(m+n) \frac{(1+\alpha)_{m+n} (\alpha)_n (1+\gamma)_n z^m (-z)^n}{(1+\alpha)_{m+2n} (1+\alpha-\beta)_n \gamma_n \ m! \ n!} = \left( 1 - \frac{\alpha}{2 \gamma} \right) \sum_{m=0}^\infty \Phi(m) \frac{(1+\alpha)_m (1+\frac{1}{2} \alpha - \beta)_m \ z^m}{(1+\alpha-\beta)_m (1+\frac{1}{2} \alpha)_m \ m!} + \left( \frac{\alpha}{2 \gamma} \right) \sum_{m=0}^\infty \Phi(m) \frac{(1+\alpha)_m (\frac{1}{2} + \frac{1}{2} \alpha - \beta)_m \ z^m}{(1+\alpha-\beta)_m (\frac{1}{2} + \frac{1}{2} \alpha)_m \ m!},
\]

where \( 1 + \alpha, \beta, \gamma, 1 + \alpha - \beta \in C \backslash Z_0^- \) and provided that the infinite series occurring on both sides of (5.1) are absolutely convergent.

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Proof: Let

\[ G := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Phi(m+n) \frac{(1+\alpha)_{m+n} (\alpha)_{n} (\beta)_{n} (1+\gamma)_{n} z^{m} (-z)^{n}}{(1+\alpha)_{m+2n} (1+\alpha-\beta)_{n} (\gamma)_{n} m! n!} \]

\[ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Phi(m+n) \frac{(\alpha)_{n} (\beta)_{n} (1+\gamma)_{n} (-1)^{n} z^{m+n}}{(1+\alpha+m+n)_{n} (1+\alpha-\beta)_{n} (\gamma)_{n} m! n!}. \] (5.2)

Replacing \( m \) by \( m-n \) in (5.2) and making use of (2.1), we obtain

\[ G = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \Phi(m) \frac{(\alpha)_{n} (\beta)_{n} (1+\gamma)_{n} (-1)^{n} z^{m}}{(1+\alpha+m)_{n} (1+\alpha-\beta)_{n} (\gamma)_{n} (m-n)! n!} \]

\[ = \sum_{m=0}^{\infty} \Phi(m) \frac{z^{m}}{m!} \sum_{n=0}^{m} \frac{(-m)_{n} (\alpha)_{n} (\beta)_{n} (1+\gamma)_{n}}{(1+\alpha+m)_{n} (1+\alpha-\beta)_{n} (\gamma)_{n} n!} \]

\[ = \sum_{m=0}^{\infty} \Phi(m) \frac{z^{m}}{m!} 4F_{3} \left[ \frac{-m, \alpha, \beta, 1+\gamma}{1+\alpha+m, 1+\alpha-\beta, \gamma}; 1 \right]. \]

By using the summation formula (3.1), we obtain the required result (5.1).

**Remark 1.** All the results (2.3), (2.5), (2.8), (3.1), (3.4), (3.5), (4.4), (4.5) and (4.6) have been verified numerically by taking suitable values of the parameters and arguments given below:

**Numerical proof of (2.3):** Taking left-hand side of (2.3) and setting \( m = 3, a = \frac{5}{2}, b = \frac{3}{2} \), we get

\[ _{3}F_{2}\left[ -3, \frac{5}{2}, \frac{3}{2}; 1 \right] = \sum_{r=0}^{3} \frac{(-3)_{r} (\frac{5}{2})_{r} (\frac{3}{2})_{r}}{(\frac{15}{2})_{r} (2)_{r} r!} \]

\[ = 1 + \frac{(-3)(\frac{5}{2})(\frac{3}{2})}{(\frac{15}{2})(2)} + \frac{(-3)(\frac{5}{2})(\frac{3}{2})^{2}}{(\frac{15}{2})(2)(2)} + \frac{(-3)(\frac{5}{2})(\frac{3}{2})^{3}}{(\frac{15}{2})(2)(3) 3!} \]

\[ = 1 + \frac{(-3)(\frac{5}{2})(\frac{3}{2})}{(\frac{15}{2})(2)} + \frac{(-3)(\frac{5}{2})(\frac{3}{2})(\frac{5}{2})}{(\frac{15}{2})(2)(2)} + \frac{(-3)(\frac{5}{2})(\frac{3}{2})^{3}}{(\frac{15}{2})(2)(3) 2} + \frac{(-3)(\frac{5}{2})(\frac{3}{2})^{3}}{(\frac{15}{2})(2)(2)(2)(3)(4) 6} \],

after simplification, we find

\[ _{3}F_{2}\left[ -3, \frac{5}{2}, \frac{3}{2}; 1 \right] = 5929 \]

14144.

Now taking right-hand side of (2.3) and setting \( m = 3, a = \frac{5}{2}, b = \frac{3}{2} \), we get

\[ \frac{(\frac{7}{2})_{3} (\frac{3}{2})_{3}}{(2)_{3} (\frac{3}{2})_{3}} = \frac{(\frac{7}{2})_{3} (\frac{3}{2})_{3} (\frac{15}{2})_{3} (\frac{15}{2})_{3}}{(2)(3)(4)(\frac{15}{2})(\frac{15}{2})} = \frac{5929}{14144}. \]

Hence L.H.S=R.H.S

Similarly, we can verify the remaining results numerically.
6. Conclusion
We conclude our present investigation by observing that several further interesting hypergeometric summation formulas for terminating series $4F_3(1)$, reduction formulas for the Gaussian hypergeometric functions $3F_2$, $4F_3$ and $5F_4$ with the argument $-Z$ and general double-series identity (which is the generalization of a reduction formula for Srivastava–Daoust double hypergeometric function with arguments $z$ and $-z$) can be obtained in an analogous manner. Moreover, it is hoped that the results derived in this paper will find useful applications in a wide range of problems of mathematics, statistics and the physical sciences.

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References
