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On (k,n) power quasinormal operators

Salah MECHERI[1,*], **Aissa Nasli BAKIR**[2]
[1]Faculty of Mathematics and informatics, Mohamed El Bachir El Ibrahimi University, Bordj Bordj Arreridj, Algeria
[2]Department of Mathematics,Faculty of Exact Sciences and Informatics,
Hassiba Benbouali University of Chlef, Algeria

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Abstract: The aim of this paper is to present certain basic properties of some classes of nonnormal operators defined on a complex separable Hilbert space. Both of the normality of their integer powers and their relations with isometries are established. The ascent of such operators as well as other important related results are also established. The decomposition of such operators, their restrictions on invariant subspaces, and some spectral properties are also presented.

Key words: m-isometry, quasinormal operator, n power quasinormal, finite ascent

1. Introduction

Let H be an infinite dimensional complex separable Hilbert space, and let $B(H)$ be the Banach algebra of all bounded linear operators on H. Denote by $N(T)$ and $R(T)$ respectively, for the null space and the range of an operator T in $B(H)$. An operator $T \in B(H)$ is said to be normal if $TT^\star = T^\star T$[22], isometry if $T^\star T = I$ where I is the identity operator on H, coisometry if $T^\star$ is an isometry, unitary if T is an isometry and invertible. An operator $T \in B(H)$ is said to be m-isometric for some integer $m \geq 1$, if

$$\sum_{k=0}^{m}(-1)^k \begin{pmatrix} m \\ k \end{pmatrix} T^{\star m-k}T^{m-k} = 0$$

[1–3].

The authors in [25, 27] introduce a new class of nonnormal operators as follows:

An operator $T \in B(H)$ is said to be n power quasinormal for some integer n, if

$$T^nT^\star T = T^\star T^{n+1}$$

A 1 power quasinormal operator is quasinormal.

Let (N), (QN), and (nQN) denote respectively, the classes of normal, quasinormal, and n power quasinormal operators. Obviously,

$$(N) \subset (QN) \text{ and } (N) \subset (nQN)$$

*Correspondence: mecherisalah@hotmail.com
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In Section 2, we present some basic properties of n power quasinormal operators namely their ascent, their reduced subspaces and relations with m-isometries, while Section 3 is devoted to a generalization of this class to a big class called (k, n) power quasinormal operators where we establish some results related to the matrix representation, the restriction on an invariant subspace, and other important results.

2. Main results

2.1. Class of n-power quasinormal operators

Definition 2.1 *[25, 27] An operator $T \in B(H)$ is said to be n power quasinormal for some integer n, if*

$$T^n T^\star T = T^\star T^{n+1}$$

Examples 1 and 2 given in [25] show that the classes (nQN) do not coincide for different values of n.

We investigate in this section, some additive properties related to this class of operators.

Proposition 2.2 *Let S_α be the right weighted shift on the usual space ℓ_2 defined by $S_\alpha e_k = \alpha_k e_{k+1}$, where $\alpha = (\alpha_k)_k$ is a complex sequence. Then, S_α is n power quasinormal if and only if, at least, one of the following conditions holds:*

i. $|\alpha_k| = |\alpha_{k+n}|$, *for all* k.

ii. $\alpha_{k+l} = 0$ *for all* $0 \leq l \leq n-1$, *and for some integer* k.

Proof For all k, and by simple computations,

$$S_\alpha^\star e_k = \overline{\alpha_{k-1}} e_{k-1}$$

Hence,

$$S_\alpha^n S_\alpha^\star S_\alpha e_k = |\alpha_k|^2 \alpha_k \alpha_{k+1} \alpha_{k+2} \ldots . \alpha_{k+n-1} e_{k+n}$$

and

$$S_\alpha^\star S_\alpha S_\alpha^{\star n+1} e_k = |\alpha_{k-n-1}|^2 \overline{\alpha_{k-1} \alpha_{k-2} \ldots . \alpha_{k-n}} e_{k-n}$$

Thus, S_α is n power quasinormal if and only if at least, the condition (i) or (ii) holds. $\square$

Remark. The inclusion $(nQN) \subset (N)$ is in general false. In fact, the operator S_α is not normal whenever $(\alpha_k)_k$ is a positive real sequence.

Lemma 2.3 *[25] If $T \in B(H)$ is n power quasinormal operator, then T is $2n$ power quasinormal.*

Theorem 2.4 *Let T be an n power quasinormal operator in $B(H)$. If $T^\star$ is one-to-one, then T^n is a normal operator.*

Proof Since $T^\star$ is one-to-one, $N(T^\star) = 0$. Then, $\overline{R(T)} = H$. Hence, equality

$$(T^n T^\star - T^\star T^n) T = 0$$

implies that

$$T^n T^\star = T^\star T^n$$

on the whole space H. By iteration, we get

$$T^n T^{\star n} = T^n T^\star T^{\star n-1} = T^\star T^n T^{\star n-1} = T^\star T^n T^\star T^{\star n-2} = ... = T^{\star n} T^n$$

$\square$

Definition 2.5 *[1, 2] An operator $T \in B(H)$ is said to be m-isometric for some integer $m \geq 1$, if*

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*m-k} T^{m-k} = 0$$

where $\binom{m}{k}$ denotes the binomial coefficient.

A 1-isometric operator is an isometry. Clearly, an isometric operator is m-isometric. The case $m = 2$ is object of intensive study in [23] where basic and spectral properties have been shown. Reader is also refered to [1–3, 14] and [16] for more details and applications.

Theorem 2.6 *Let T be an n power quasinormal operator in $B(H)$. If T is a co-isometry, then T is unitary.*

Proof According to the hypotheses, $T^n T^\star T = T^\star T^{n+1}$, and $TT^\star = I$. Then,

$$T^n T^\star = T^{n-1} = T^\star T^n$$

Hence, and by iteration

$$T = T^\star T^2$$

Thus,

$$I = T^\star T$$

The operator T is then unitary. The proof is achieved. $\square$

Theorem 2.7 *Let T be an n power quasinormal operator in $B(H)$. If $R \in B(H)$ is unitarily equivalent to T, then R is also n power quasinormal.*

Proof There exists a unitary operator $U \in B(H)$ satisfying $T = U^\star RU$. Since T is n power quasinormal, $T^n T^\star T = T^\star T^{n+1}$. Then,

$$U^\star R^n U U^\star R^\star U U^\star R U = U^\star R^\star U U^\star R^{n+1} U$$

Since U is unitary,

$$R^n R^\star R = R^\star R^{n+1}$$

This shows that R is n power quasinormal. $\square$

Let $T \in B(H)$.

Definition 2.8 *[5] The smallest integer k satisfying $N(T^k) = N(T)$ is said to be the ascent of T and is denoted by $\alpha(T)$. If such integer does not exist, we shall write $\alpha(T) = \infty$.*

Similarly,

Definition 2.9 *[5] The smallest integer k for which $R(T^k) = R(T)$ is said to be the descent of T and is denoted by $\delta(T)$. If no such integer exists, we shall set $\delta(T) = \infty$.*

According to [5], $\alpha(T) = \delta(T)$ whenever $\alpha(T)$ and $\delta(T)$ are both finite. A deep study on this property can be found in [5]. We have then the following important result

Theorem 2.10 *Let $T \in B(H)$ be n power quasinormal. Then, $N(T^n) = N(T)$.*

Proof For $n = 1$. It suffices to show that $N(T^2) \subset N(T)$. Let $T^2x = 0$. Then $TT^\star Tx = 0$. Implying $(T^\star T)^2 x = 0$. Thus, for all $y \in H$

$$\langle (T^\star T)^2 x, y\rangle = \langle T^\star Tx, T^\star Ty\rangle = 0$$

Hence,

$$\langle T^\star Tx, T^\star Tx\rangle = 0$$

Thus, $T^\star Tx = 0$. Consequently, for all $z \in H$

$$\langle T^\star Tx, z\rangle = \langle Tx, Tz\rangle = 0$$

For $z = x$, we get $Tx = 0$. Then $x \in N(T)$.

Likewise for $n = 2$, we have

$$\begin{aligned} T^3x = 0 \quad &\Rightarrow \quad T^\star T^3 x = T^2 T^\star Tx = 0 \\ &\Rightarrow \quad T^\star x \in N(T^2) \\ &\Rightarrow \quad T^\star Tx \in N(T) \\ &\Rightarrow \quad TT^\star Tx = 0 \\ &\Rightarrow \quad (T^\star T)^2 x = 0 \\ &\Rightarrow \quad Tx = 0 \end{aligned}$$

by the previous arguments.

Suppose now that

$$N(T^k) \subset N(T) \tag{2.1}$$

for some k, and let $x \in N(T^{k+1})$. Thus,

$$\begin{aligned} T^{k+1}x = 0 \quad &\Rightarrow \quad T^\star T^{k+1} x = 0 = T^k T^\star Tx \\ &\Rightarrow \quad T^\star Tx \in N(T^k) \\ &\Rightarrow \quad T^\star Tx \in N(T) \\ &\Rightarrow \quad Tx = 0 \end{aligned}$$

By (2.1). Therefore, $N(T^{k+1}) \subset N(T)$. Consequently, $N(T^{*n}) \subset N(T^*)$. □

Now, we will extend the previous result to $(T - \lambda)$ for all complex scalar $\lambda, \lambda \neq 0$. First, we need the following important result

Theorem 2.11 *If $T \in B(H)$ is a quasinormal operator, then $(T^\star T)^3 = T^{\star 3} T^3$.*

Proof Since T is quasinormal,

$$TT^{\star}T = T^{\star}T^{2} \tag{2.2}$$

Equality $(T^{\star}T)^{2} = T^{\star 2}T^{2}$ is satisfied by multiplying (2.2) on the left by $T^{\star}$.
We have then,

$$\begin{aligned}(T^{\star}T)^{3} = T^{\star}TT^{\star}TT^{\star}T = T^{\star}T^{\star}T^{2}T^{\star}T &= T^{\star 2}T^{2}T^{\star}T\\ &= T^{\star 2}T^{\star}T^{3}\\ &= T^{\star 3}T^{3}\end{aligned}$$

by Lemma 2.3. □

Theorem 2.12 *For a quasinormal operator $T \in B(H)$, and for each nonzero complex scalar λ, $N(T-\lambda) \subset N(T-\lambda)^{\star}$.*

Proof Let $x \in H$ be such that $(T-\lambda)x = 0$. Since T is quasinormal,

$$TT^{\star}T = T^{\star}T^{2}$$

Hence,

$$\langle TT^{\star}Tx, x\rangle = \langle T^{\star}T^{2}x, x\rangle$$

Then,

$$\langle T^{\star}Tx, T^{\star}x\rangle = \langle T^{2}x, Tx\rangle$$

Since $\lambda \neq 0$,

$$\|T^{\star}x\| = |\lambda|\|x\| \tag{2.3}$$

On the other side, and by Theorem 2.11,

$$(T^{\star}T)^{2} = T^{\star 2}T^{2}$$

Hence,

$$\langle T^{\star}TT^{\star}Tx, x\rangle = \langle T^{\star 2}T^{2}x, x\rangle$$

Then,

$$\langle T^{\star}x, x\rangle = \overline{\lambda}\|x\|^{2} \tag{2.4}$$

Finally,

$$\begin{aligned}\|(T-\lambda)^{\star}x\|^{2} &= \|T^{\star}x\|^{2} + |\lambda|^{2}\|x\|^{2} - 2Re(\lambda\langle T^{\star}x, x\rangle)\\ &= \|T^{\star}x\|^{2} + |\lambda|^{2}\|x\|^{2} - 2|\lambda|^{2}\|x\|^{2}\\ &= \|T^{\star}x\|^{2} - |\lambda|^{2}\|x\|^{2}\\ &= 0\end{aligned}$$

by (2.3) and (2.4). The proof is achieved. □

Theorem 2.13 *Let T be quasinormal in $B(H)$. Then, $(T-\lambda)$ has finite ascent for all $\lambda \in \mathbb{C}, \lambda \neq 0$.*

Proof From Theorem 2.12, $N(T-\lambda)$ reduces $(T-\lambda)$. Then, $(T-\lambda)$ can be written according to the decomposition $H = (N(T-\lambda))^{\perp} \oplus N(T-\lambda)$

$$(T-\lambda) = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$$

Let $x \in H$. Hence, $x = u + v$ where $u \in (N(T-\lambda)^{\perp}, v \in N(T-\lambda)$. Thus, and using the above Theorem,

$$\begin{aligned} (T-\lambda)^2 x = 0 \quad &\Rightarrow \quad (T-\lambda)^2 v = 0 \\ &\Rightarrow \quad (T-\lambda)^2 v = 0 \\ &\Rightarrow \quad (T-\lambda)v \in N(T-\lambda) \subset N(T-\lambda)^{\star} \\ &\Rightarrow \quad (T-\lambda)v \in N(T-\lambda) \cap \overline{(R(T-\lambda)} \\ &\Rightarrow \quad (T-\lambda)v \in N(T-\lambda) \cap (N(T-\lambda)^{\star})^{\perp} = \{0\} \\ &\Rightarrow \quad (T-\lambda)u = 0 = (T-\lambda)x \end{aligned}$$

This shows that $x \in N(T-\lambda)$, and therefore, $N(T-\lambda)^2 \subset N(T-\lambda)$. This completes the proof. □

As a consequence of Theorems 2.11 and 2.13, we have the following Corollary

Corollary 2.14 *For a quasinormal operator $T \in B(H)$, $\alpha(T-\lambda) = 1$ for all $\lambda \in \mathbb{C}$.*

Definition 2.15 *[5] An operator T in $B(H)$ is said to have the Single Valued Extension Property, (SVEP) at a complex number α, if for each open neighborhood V of α, the unique analytic function $f\colon V \to H$ that satisfies*

$$\forall \lambda \in V : (T-\lambda)f(\lambda) = 0$$

is the function $f \equiv 0$ on V. If furthermore, T has SVEP at every $\alpha \in \mathbb{C}$, we say that T has SVEP.

Detailed results on this important property can be seen in [5, 20, 21]. The author in [10] presented a hard study of SVEP on Banach spaces.

Corollary 2.16 *If $T \in B(H)$ is an n power quasinormal operator, then $(T-\lambda I)$ has SVEP.*

Proof The result derives by Corollary 2.14 and [5, Theorem 3.8]. □

2.2. Class of (k,n)-power quasinormal operators

In this section, we introduce a new class of operators generalizing the class of n power quasinormal operators as follows:

Definition 2.17 *An operator $T \in B(H)$ is said to be (k,n) power quasinormal for some positive integers n, k if*

$$T^{\star k}(T^nT^\star T - T^\star T^{n+1})T^k = 0$$

A $(0,n)$ power quasinormal operator is n power quasinormal.

Denote by $((k,n)QN)$ for the class of such operators. Then,

$$(nQN) \subset ((k,n)QN)$$

By the same arguments in Lemma 2.3 and Theorem 2.7, we can state the following result:

Lemma 2.18 *Let $T \in B(H)$ be a (k,n) power quasinormal operator. Then,*

i. T is $(k,2n)$ power quasinormal.

ii. Unitarily equivalent operators to T are also (k,n) power quasinormal.

iii. If T is invertible, then its inverse is also (k,n) power quasinormal.

Theorem 2.19 *Let $T \in B(H)$ be a (k,n) power quasinormal operator such that T^k has a dense range. Then T is n power quasinormal.*

Proof Let $x \in H$. Since $\overline{R(T^k)} = H$, there exists a sequence $(x_n)_n \in H$ for which $x = \lim_{n\to\infty} T^k x_n$. Since T is (k,n) power quasinormal,

$$T^{*k}(T^nT^\star T - T^\star T^{n+1})T^k x_n = 0$$

for all n. Hence,

$$\begin{aligned} 0 &= \langle T^{*k}(T^nT^\star T - T^\star T^{n+1})T^k x_n, x_n\rangle \\ &= \langle (T^nT^\star T - T^\star T^{n+1})T^k x_n, T^k x_n\rangle \end{aligned}$$

By the continuity of the scalar product,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \langle (T^nT^\star T - T^\star T^{n+1})T^k x_n, T^k x_n\rangle \\ &= \langle (T^nT^\star T - T^\star T^{n+1})x, x\rangle \end{aligned}$$

Thus,

$$(T^nT^\star T - T^\star T^{n+1})x = 0$$

This finishes the proof since x is arbitary in H. □

Corollary 2.20 *Let T be a (k,n) power quasinormal operator but not n power quasinormal. Then T is not invertible.*

Theorem 2.21 *Let $T \in B(H)$ be a (k,n) power quasinormal operator, and let $M \subset H$ be a closed invariant subspace for T. Then, the restriction $T|_M$ of T to M is also (k,n) power quasinormal.*

Proof Under the decomposition $H = M \oplus M^{\perp}$, we can write

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

Hence, for all integer $m \geq 1$,

$$T^m = \begin{pmatrix} A^m & \sum\limits_{p=0}^{m-1} A^{m-p-1}BC^p \\ 0 & C^m \end{pmatrix}$$

By simple calculations, and since T is (k,n) power quasinormal,

$$\begin{aligned} 0 &= T^{\star k}(T^nT^{\star}T - T^{\star}T^{n+1})T^k \\ &= \begin{pmatrix} A^{\star k}(A^nA^{\star}A - A^{\star}A^{n+1})A^k & E \\ F & G \end{pmatrix} \end{aligned}$$

for some bounded linear operators E, F and G in $B(H)$. Thus,

$$A^{\star k}(A^nA^{\star}A - A^{\star}A^{n+1})A^k = 0$$

This shows that $A = T|_M$ is (n,k) power quasinormal. $\square$

Theorem 2.22 *Let $T \in B(H)$ be a (k,n) power quasinormal operator such that $R(T^k)$ be not dense in H. If*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $H = \overline{R(T^k)} \oplus N(T^{\star k})$, then T_1 is an n power quasinormal operator. Moreover, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$, where $\sigma(T)$ denotes the spectrum of T.

Proof Since T is (k,n) power quasinormal,

$$T^{\star k}(T^nT^{\star}T - T^{\star}T^{n+1})T^kx = 0$$

for all $x \in H$. Hence,

$$\begin{aligned} 0 &= \langle T^{\star k}(T^nT^{\star}T - T^{\star}T^{n+1})T^kx, x\rangle \\ &= \langle (T^nT^{\star}T - T^{\star}T^{n+1})T^k, T^kx\rangle \end{aligned}$$

Thus, for all $y \in \overline{R(T^k)}$

$$\langle (T^nT^{\star}T - T^{\star}T^{n+1})y, y\rangle = 0$$

Therefore,

$$(T^nT^{\star}T - T^{\star}T^{n+1})\Big|_{\overline{R(T^k)}} = (T_1^nT_1^{\star}T_1 - T_1^{\star}T_1^{n+1}) = 0$$

Thus, T_1 is n power quasinormal.

Let now P be the orthogonal projection on $\overline{R(T^k)}$. For all $x = x_1 + x_2, y = y_1 + y_2 \in H$, we have

$$\left\langle T_3^kx_2, y_2\right\rangle = \left\langle T^k(I-P)x, (I-P)y\right\rangle = \left\langle (I-P)x, T^{*k}(I-P)y\right\rangle = 0$$

Thus, $T_3^k = 0$.
Furthermore, $\sigma(T_1) \cup \sigma(T_3) = \sigma(T) \cup \Omega$, where Ω is the union of holes in $\sigma(T)$ which happen to be a subset of $\sigma(T_1) \cap \sigma(T_3)$ by [13, Corollary 7], with the interior of $\sigma(T_1) \cap \sigma(T_3) = \emptyset$, and T_3 is nilpotent. Thus, $\sigma(T) = \sigma(T_1) \cup \{0\}$. ◻

Corollary 2.23 *Let $T \in B(H)$ be an (k,n) power quasinormal operator. If the restriction $T_1 = T\Big|_{\overline{R(T^k)}}$ is invertible, then T is similar to the direct sum of an n power quasinormal operator and a nilpotent operator.*

Proof Let

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } H = \overline{R(T^k)} \oplus N(T^{*k})$$

Then, T_1 is n power quasinormal operator by the above Theorem. Since T_1 is invertible, $0 \notin \sigma(T)$. Hence, $\sigma(T_1) \cap \sigma(T_3) = \emptyset$. Thus, there exists $X \in B(H)$ for which $T_1 X - X T_3 = T_2$ by Rosenblum's Corollary [24]. Finally,

$$T = \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 \oplus T_3 \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$$

◻

3. Weyl's theorem

An operator $T \in B(H)$ is called Fredholm if $R(T)$ is closed, $\alpha(T) = \dim N(T) < \infty$ and $\beta(T) - \dim H \setminus R(T) < \infty$. Moreover, if $i(T) = \alpha(T) - \beta(T) = 0$, then T is called Weyl. The Weyl spectrum $w(T)$ of T is defined by

$$w(T) : \{\lambda \in \mathbb{C} : T - \lambda I \text{is not Weyl}\}$$

According to [15], we say that Weyl's theorem holds for T if

$$\sigma(T) \setminus w(T) = \pi_{00}(T),$$

where $\pi_{00}(T) : \{\lambda \in iso\sigma(T) : \ 0 < \dim N(T - \lambda I < \infty$. In [23], Patel showed that Weyl's theorem holds for 2-isometric operator, which has been extended to 2-expansive operators with $0 \notin \sigma(T^*T)$ [12] and to nonnormal operators [18, 19]. In this section, we obtain that Weyl's theorem holds for (k,n) power quasinormal operators without any additional conditions. An operator $T \in B(H)$ has the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open neighborhood G of λ_0, the only analytic function $f : G \to H$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in G$ is the function $f \equiv 0$. An operator T has SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$ for more details see ([20, 21]).

Theorem 3.1 *Weyl's theorem holds for (k,n) power quasinormal operator.*

Proof Suppose that T is a (k,n)-power quasinormal operator, then T has SVEP at zero. Either $\sigma(T_1) \subseteq \partial D$ or $\sigma(T_1) = \overline{D}$. If $\sigma(T_1) \subseteq \partial D$, then T has SVEP everywhere: else $\sigma(T_1) = \overline{D}$. T has SVEP at $\sigma(T) \setminus w(T)$.

then $< 0\dim(T-\lambda) < \infty$. We have $\lambda \in \sigma_p(T) \subseteq \partial D \cup \{0\}$, where D denotes the open unit disc. An operator such that its point spectrum has empty interior has SVEP [5, Remark 2.4(d)]; hence, T has SVEP. Again, if $\sigma(T_1) = \sigma(T) = \overline{D}$, then $iso\sigma(T) = \emptyset$; if $\sigma(T_1) \subset \partial D$, then T_1; hence, T is polaroid (i.e. isolated points of the spectrum are poles). This guarantees Weyl's theorem for T. □

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