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Weak and injective dimensional analogues of Kaplansky's and Auslander's lemmas for purity

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Abstract: We prove a stronger form of an analogue of a Kaplansky lemma on homological dimensions by showing that in a pure-exact sequence $0 \to A \to B \to C \to 0$, the weak dimensions of the modules satisfy $\text{w.d.}B = \max\{\text{w.d.}A, \text{w.d.}C\}$. We also show that the same equality holds for the injective dimensions whenever the ring is noetherian. In addition, a version of Auslander's lemma for chains of pure submodules M_ρ is proved: the weak dimension of the union of the chain equals the supremum of the weak dimensions of the factor modules $M_{\rho+1}/M_\rho$ in the chain. The same holds for injective dimensions if the ring is noetherian.

Key words: Tor, pure submodules, weak dimension, injective dimension

1. Introduction

In homological algebra, two most frequently used results on projective dimensions (p.d.) are Kaplansky's and Auslander's lemmas on the projective dimensions of modules in short exact sequences and on the projective dimensions of the unions of chains, respectively.

Recall that Kaplansky's lemma (see e.g., [6, Theorem B, p. 124]) states that if $0 \to A \to B \to C \to 0$ is a short exact sequence of modules, then their projective dimensions, if finite, satisfy the following:

(i) if $\text{p.d.}A < \text{p.d.}B$, then $\text{p.d.}C = \text{p.d.}B$;

(ii) if $\text{p.d.}A > \text{p.d.}B$, then $\text{p.d.}C = \text{p.d.}A + 1$;

(iii) if $\text{p.d.}A = \text{p.d.}B$, then $\text{p.d.}C \leq \text{p.d.}B + 1$.

Analogous theorems hold for weak dimensions (w.d.) and for injective dimensions (i.d.). We now wonder if more specific results are available for pure-exact sequences, i.e. when A is a pure submodule of B. Browsing in the literature, we have found related results only in the Enochs–Jenda book [4] for w.d. and i.d. Their Lemmas 9.1.4 and 9.1.5 state that if A is pure in B, then $\text{w.d.}A \leq \text{w.d.}B$, and in the noetherian case $\text{i.d.}A \leq \text{i.d.}B$. Our Theorems 2.2 and 2.3 will prove much stronger results:

$$\text{w.d.}B = \max\{\text{w.d.}A, \text{w.d.}C\}$$

and if the ring is noetherian, then also

$$\text{i.d.}B = \max\{\text{i.d.}A, \text{i.d.}C\}.$$

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Auslander's lemma (see [1]) is concerned with a continuous well-ordered ascending chain

$$0 = M_0 < M_1 < \cdots < M_\rho < \cdots < M_\tau = M \tag{1}$$

of submodules of a module M, where τ is a limit ordinal (continuity means that $M_\sigma = \cup_{\rho<\sigma} M_\rho$ whenever σ is a limit ordinal). One version of the lemma claims that if p.d. $M_{\rho+1}/M_\rho \leq m$ holds for some integer $m \geq 0$ and for all $\rho < \tau$, then p.d. $M \leq m$ as well. The other version asserts that p.d. $M_\rho \leq m$ for all $\rho < \tau$ implies p.d. $M \leq m+1$. We are going to show in Theorem 3.2 that

$$\text{w.d.}M = \sup_{\rho<\tau}(\text{w.d.}M_{\rho+1}/M_\rho)$$

and in the noetherian case also

$$\text{i.d.}M = \sup_{\rho<\tau}(\text{i.d.}M_{\rho+1}/M_\rho).$$

We note that a weaker result on p.d. was proved in the very special case of valuation domains by Dimitrić–Fuchs [3]: if in the chain (1) the modules are pure and torsion-free, and $\tau = \omega_m$ for some $m \geq 0$, then p.d. $M_\rho \leq m$ for all $\rho < \omega_m$ implies p.d. $M \leq m+1$. (ω_m denotes the first ordinal of cardinality $\aleph_m$.)

2. Dimensions in pure-exact sequences

The rings R in this note are associative with identity, and the modules over them are unital. A short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules is called *pure-exact* if the induced sequence $0 \to A \otimes_R X \to B \otimes_R X \to C \otimes_R X \to 0$ is exact for all left R-modules X. The next result is our crucial starting point.

Proposition 2.1 *Let $0 \to A \to B \to C \to 0$ be a pure-exact sequence of right R-modules. Then for every left R-module X and for every integer $n \geq 1$, the induced sequence*

$$0 \to \text{Tor}_n^R(A, X) \to \text{Tor}_n^R(B, X) \to \text{Tor}_n^R(C, X) \to 0 \tag{2}$$

is pure-exact.

Proof It is well known that a sequence (2) is exact if and only if the sequence of character modules

$$0 \to \text{Tor}_n^R(C, X)^\flat \to \text{Tor}_n^R(B, X)^\flat \to \text{Tor}_n^R(A, X)^\flat \to 0 \tag{3}$$

is exact for all left modules X. (Recall that the character module of a right R-module M is defined as the left module $M^\flat = \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$.) The sequence (3) is equivalent to the sequence

$$0 \to \text{Ext}_R^n(X, C^\flat) \to \text{Ext}_R^n(X, B^\flat) \to \text{Ext}_R^n(X, A^\flat) \to 0 \tag{4}$$

in view of the well-known isomorphism

$$\text{Hom}_S(\text{Tor}_n^R(N, M), E) \cong \text{Ext}_R^n(M, \text{Hom}_S(N, E)) \quad (n \geq 1)$$

valid for all ${}_RM$, ${}_SN_R$ and injective ${}_SE$ over any rings R, S (see [2, Proposition 5.1, p. 120] or [7, before Theorem 10.66, p. 668]). The pure-exactness of the given sequence is equivalent to the splitting of the exact sequence $0 \to C^\flat \to B^\flat \to A^\flat \to 0$ (see e.g. [5, Proposition 8.6, Chapter I]). Hence, we conclude that (4) is a splitting exact sequence. Consequently, (3) is a splitting exact sequence; this guarantees the pure-exactness of (2). □

We can now prove one of our main results. Recall that w.d. B is the smallest n for which $\mathrm{Tor}^R_{n+1}(B,X) = 0$ holds for all left R-modules X.

Theorem 2.2 *If $0 \to A \to B \to C \to 0$ is a pure-exact sequence of right R-modules, then*

$$\mathrm{w.d.}B = \max\{\mathrm{w.d.}A, \mathrm{w.d.}C\}.$$

Proof Since the sequence (3) is splitting, we have

$$\mathrm{Tor}^R_n(B,X)^\flat \cong \mathrm{Tor}^R_n(A,X)^\flat \oplus \mathrm{Tor}^R_n(C,X)^\flat \qquad \text{for all } n \geq 1.$$

It suffices to observe that, for a fixed n, the left side vanishes for all left modules X if and only if so does the right side, and the character module is 0 if and only if the module is 0. □

We are going to establish a similar result on the injective dimensions of modules in noetherian rings. We will make use of the natural isomorphism

$$\mathrm{Hom}_S(\mathrm{Ext}^n_R(M,N),E) \cong \mathrm{Tor}^R_n(\mathrm{Hom}_S(N,E),M) \tag{5}$$

that holds for all $n \geq 1$ provided R is left noetherian, ${}_RM$ is finitely generated, ${}_RN_S$ is a bimodule, and E_S is injective (see [2, Proposition 5.3, p. 120] for $n = 1$ or [7, p. 669] for all n).

Theorem 2.3 *Assume R is a left noetherian ring, and $0 \to A \to B \to C \to 0$ is a pure-exact sequence of left R-modules. Then*

$$\mathrm{i.d.}B = \max\{\mathrm{i.d.}A, \mathrm{i.d.}C\}.$$

Proof In our special case, isomorphism (5) yields

$$(\mathrm{Ext}^n_R(R/I,B))^\flat \cong \mathrm{Tor}^R_n(B^\flat, R/I) \tag{6}$$

where I is a left ideal of R and B is a left R-module. From the given pure-exact sequence, we obtain $B^\flat \cong A^\flat \oplus C^\flat$. Inserting this direct sum on the right side of (6) and using the same isomorphism for A and C, we obtain the isomorphism

$$(\mathrm{Ext}^n_R(R/I,B))^\flat \cong (\mathrm{Ext}^n_R(R/I,A))^\flat \oplus (\mathrm{Ext}^n_R(R/I,C))^\flat$$

that holds for all $n \geq 1$ and for all (finitely generated) left ideals I. Hence, the claim is obvious, since i.d. B is the smallest n for which $\mathrm{Ext}^{n+1}_R(R/I,B) = 0$ holds for all left ideals I. □

Remark 2.4 *Note that Theorems 2.2 and 2.3 are related. It is well-known that* w.d. $B =$ i.d. $B^\flat$ *for all modules B, while* (5) *implies that, in a noetherian case,* i.d. $B =$ w.d. $B^\flat$ *as well.*

3. Dimensions of unions of chains

Our arguments above lead us to the following general result that is a stronger version of an analogue of Auslander's theorem.

Theorem 3.1 *For an ordinal τ, let*

$$0 = M_0 < M_1 < \cdots < M_\rho < \cdots < M_\tau = M \tag{7}$$

be a continuous well-ordered ascending chain of pure submodules of the right module M. Then

$$\text{w.d.}M = \sup_{\rho<\tau}(\text{w.d.}M_{\rho+1}/M_\rho).$$

Proof The inequality $\leq$ is of course a consequence of Auslander's lemma. For the converse, we prove that $\text{w.d.}M_{\rho+1}/M_\rho \leq \text{w.d.}M$ holds for every ρ. Since the module M_ρ is pure in each module of higher index in the chain, in view of Theorem 2.2 we have $\text{w.d.}M_{\rho+1}/M_\rho \leq \text{w.d.}M_{\rho+1} \leq \text{w.d.}M$ for every ρ. Hence, the claim is evident. □

It is well known that if the cardinality of a ring R is $\kappa \geq \aleph_0$, then every flat R-module M can be written as the union of a continuous well-ordered ascending chain (7) of flat pure submodules M_ρ with flat factor modules $M_{\rho+1}/M_\rho$ of cardinalities $\leq \kappa$. We now extend this result to all weak-dimensions.

Theorem 3.2 *Let $\kappa = \max\{|R|, \aleph_0\}$. Every R-module M of weak dimension n admits a continuous well-ordered ascending chain* (7) *of pure submodules of weak dimension $\leq n$ such that*

$$|M_{\rho+1}/M_\rho| \leq \kappa \quad \textit{and} \quad \text{w.d.}M_{\rho+1}/M_\rho \leq n \quad \textit{for all } \rho < \tau.$$

In the noetherian case, the same holds for injective dimensions.

Proof As every element of M can be embedded in a pure submodule N of cardinality $\leq \kappa$, the arguments above imply both $\text{w.d.}N \leq n$ and $\text{w.d.}M/N \leq n$. A routine transfinite induction completes the proof. □

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