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Generalized Stević–Sharma type operators from Hardy spaces into nth weighted type spaces

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Abstract: In this paper, some characterizations for boundedness, essential norm and compactness of generalized Stević-Sharma type operators from Hardy spaces into nth weighted type spaces are given.

Key words: Essential norm, generalized Stević–Sharma type operators, nth weighted type spaces, Hardy spaces

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$. Every positive and continuous function on $\mathbb{D}$ is called a weight. Suppose that $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$ and μ be a weight. The nth weighted type space $\mathcal{W}_\mu^{(n)}(\mathbb{D}) = \mathcal{W}_\mu^{(n)}$ consists of all analytic functions on $\mathbb{D}$ for which the following statement is finite

$$b_{\mathcal{W}_\mu^{(n)}}(f) := \sup_{z\in\mathbb{D}} \mu(z)|f^{(n)}(z)|.$$

The above statement is just a semi norm and $\mathcal{W}_\mu^{(n)}$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{W}_\mu^{(n)}} = \sum_{i=0}^{n-1} | f^{(i)}(0) | + b_{\mathcal{W}_\mu^{(n)}}(f),$$

See for example [1, 9, 10]. Let $\alpha > 0$. Then $\mathcal{W}^{(1)}_{(1-|z|^2)^\alpha} = \mathcal{B}^\alpha$ (Bloch type space), $\mathcal{W}^{(2)}_{(1-|z|^2)^\alpha} = \mathcal{Z}^\alpha$ (Zygmund type space) and $\mathcal{W}^{(1)}_{(1-|z|^2)\log\frac{2}{1-|z|^2}}$ coincides with the logarithmic Bloch space $\mathcal{B}_{\log}$. Also $\mathcal{W}_\mu^{(0)} = H_\mu$ (weighted type space), $\mathcal{W}_\mu^{(1)} = \mathcal{B}\mu$ (weighted Bloch space) and $\mathcal{W}_\mu^{(2)} = \mathcal{Z}_\mu$ (weighted Zygmund space). For more information about Bloch type spaces or Zygmund type spaces see [8, 15, 16].

For $0 < p < \infty$ a function $f \in H(\mathbb{D})$ belong to the Hardy space H^p if

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}} < \infty.$$

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where $0 < p < \infty$. If $1 \le p < \infty$, H^p is a Banach space and if $0 < p < 1$, H^p is nonlocally convex topological vector space and in this case it is a complete metric space (see [4]).

For Banach spaces X and Y and a continuous linear operator $T : X \to Y$, the essential norm is the distance of T from the space of all compact operators, that is

$$\|T\|_e = \inf\{\|T - K\| : K : X \to Y \text{ is compact}\}.$$

T is compact if and only if $\|T\|_e = 0$.

Let $u, v \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, the set of all analytic self-maps of $\mathbb{D}$. The Stević-Sharma type operator is defined as follows

$$T_{u,v,\varphi} f(z) = u(z) f(\varphi(z)) + v(z) f'(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

Indeed $T_{u,v,\varphi} = uC_\varphi + vC_\varphi D$ where D is the differentiation operator and C_φ is composition operator. More information about this operator can be found in [7, 11, 12].

From the above definition we generalize the Stević-Sharma type operator. Let $m \in \mathbb{N}$, $u, v \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. We denote the generalized Stević–Sharma type operator with $T^m_{u,v,\varphi}$ and define it as follows:

$$T^m_{u,v,\varphi} f(z) = (uC_\varphi f)(z) + (D^m_{\varphi,v} f)(z) \qquad f \in H(\mathbb{D}), \quad z \in \mathbb{D},$$

where $D^m_{\varphi,u}$ is the generalized weighted composition operator. When $v = 0$, then $T^m_{u,0,\varphi} = uC_\varphi$ is the well-known weighted composition operator. If $u = 0,$, then, $T^m_{0,v,\varphi} = D^m_{\varphi,v}$ and for $m = 1$, $T^m_{u,v,\varphi}$ is Stević–Sharma type operator.

For $n, k \in \mathbb{N}_0$ and $k \le n$, the partial Bell polynomials are triangulares

$$B_{n,k}(x_1, x_2, ..., x_{n-k+1}) = \sum \frac{n!}{\prod_{t=1}^{n-k+1} j_t!} \prod_{t=1}^{n-k+1} (\frac{x_t}{t!})^{j_t}.$$

In the above equation we take the sum over all sequences $j_1, j_2, ..., j_{n-k+1}$ of nonnegative integers with the following properties

$$\sum_{t=1}^{n-k+1} j_t = k \quad \text{and} \quad \sum_{t=1}^{n-k+1} t j_t = n.$$

See [3, pp 134].

In this paper, first we obtain some characterizations for boundedness of operator $T^m_{u,v,\varphi} : H^p \to \mathcal{W}^{(n)}_\mu$. Then estimations for the essential norm of these operators are given. Finally some equivalence conditions for compactness of generalized Stević-Sharma type operators from Hardy spaces into nth weighted type spaces are presented. As some applications, we get some characterizations for boundedness, essential norm and compactness of (generalized) weighted composition operators from the Hardy spaces into nth weighted type spaces.

By $A \succeq B$ we mean there exists a constant C such that $A \ge CB$ and $A \approx B$ means that $A \succeq B \succeq A$.

2. Preliminaries

This section is devoted to giving some lemmas we use in the next sections.

Lemma 2.1 ([16], Propositions 7 and 8) *Let $\alpha > 0$ and $H_\alpha^\infty = \mathcal{W}^{(0)}_{(1-|z|^2)^\alpha}$. Then $H_\alpha^\infty = \mathcal{B}^{\alpha+1}$. Moreover, for any $f \in \mathcal{B}^\alpha$ and $n \in \mathbb{N}$,*

$$\|f\|_{\mathcal{B}^\alpha} \approx \sum_{i=0}^{n} |f^{(i)}(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+n} |f^{(n+1)}(z)|.$$

Lemma 2.2 ([5], Lemma 2.1) *Let $\alpha > 0$. The sequence $\{j^{\alpha-1}z^j\}_1^\infty$ is bounded in $\mathcal{B}_0^\alpha$ and*

$$\lim_{j\to\infty} j^{\alpha-1}\|z^j\|_{\mathcal{B}^\alpha} = \left(\frac{2\alpha}{e}\right)^\alpha.$$

Lemma 2.3 ([4]) *Let $0 < p < \infty$, $n \in \mathbb{N}_0$ and $f \in H^p$. Then*

$$|f^{(n)}(z)| \preceq \frac{\|f\|_{H^p}}{(1-|z|^2)^{\frac{1}{p}+n}}, \qquad z \in \mathbb{D}.$$

Let $u \in H(\mathbb{D})$, i and n be integer numbers. For simplicity in calculation, we set

$$I_{i,\varphi}^{n,u}(z) = \begin{cases} \sum_{l=i}^{n} \binom{n}{l} u^{(n-l)}(z) B_{l,i}(\varphi^{'}(z), ..., \varphi^{(l-i+1)}(z)) & 0 \le i \le n \in \mathbb{N}_0 \\ 0 & \text{otherwise} \end{cases}$$

The proof of next lemma resembles to the proof of Lemma 4 [10], therefore it is omitted.

Lemma 2.4 *Let $f, u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and $m, n \in \mathbb{N}_0$. If $T^m_{u,v,\varphi} = uC_\varphi + D^m_{\varphi,v}$, then*

$$\left(T^m_{u,v,\varphi} f\right)^{(n)}(z) = \sum_{i=0}^{m+n} f^{(i)}(\varphi(z)) (I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z).$$

For any $a \in \mathbb{D}$ and $j \in \mathbb{N}$, set

$$f_{j,a}(z) = \frac{(1-|a|^2)^j}{(1-\overline{a}z)^{\frac{1}{p}+j}}. \tag{2.1}$$

One can see that $f_{j,a} \in H^p$, for each $j \in \mathbb{N}$, $\sup_{a\in\mathbb{D}} \|f_{j,a}\|_{H^p} < \infty$ and $f_{j,a}$ converges to 0 as $|a| \to 1$.

Lemma 2.5 *Let $m, n \in \mathbb{N}$ such that $n \ge m$. For any $0 \ne a \in \mathbb{D}$ and $i \in \{0, 1, ..., m+n\}$, there exists a function $g_{i,a} \in H^p$ such that*

$$g^{(k)}_{i,a}(a) = \frac{\overline{a}^k \delta_{ik}}{(1-|a|^2)^{\frac{1}{p}+k}},$$

where δ_{ik} is Kronecker delta. If $i \in \{0, 1, ..., m-1\}$, then $g_{i,a} \in span\{f_{1,a}, ..., f_{m,a}\}$ and for $i \in \{m, ..., n\}$, $g_{i,a} \in span\{f_{m+1,a}, ..., f_{n+1,a}\}$ also when $i \in \{n+1, ..., m+n\}$

$$g_{i,a} \in span\{f_{n+2,a}, ..., f_{m+n+1,a}\}.$$

Proof For any fixed $0 \neq a \in \mathbb{D}$ and coefficients $c_1, ..., c_{m+n+1}$ we set

$$e_{1,a,c_1,...,c_m}(z) = \sum_{j=1}^{m} c_j f_{j,a}(z),$$

$$e_{2,a,c_{m+1},...,c_{n+1}}(z) = \sum_{j=1}^{n-m+1} \frac{c_{j+m}}{\prod_{t=0}^{m-1}(m+j+\frac{1}{p}+t)} f_{j+m,a}(z)$$

$$e_{3,a,c_{n+2},...,c_{m+n+1}}(z) = \sum_{j=1}^{m} \frac{c_{j+1+n}}{\prod_{t=0}^{n}(n+1+j+\frac{1}{p}+t)} f_{j+1+n,a}(z),$$

where $f_{j,a}$ are defined in (2.1). For each $i \in \{0, 1, ..., m+n\}$ the system of linear equations

$$\begin{aligned}
e_{1,a,c_1,...,c_m}(a) &= \frac{1}{(1-|a|^2)^{\frac{1}{p}}} \sum_{j=1}^{m} c_j = \frac{\delta_{i0}}{(1-|a|^2)^{\frac{1}{p}}} \\
&\vdots \\
e_{1,a,c_1,...,c_m}^{(m-1)}(a) &= \frac{\overline{a}^{m-1}}{(1-|a|^2)^{m-1+\frac{1}{p}}} \sum_{j=1}^{m} c_j \prod_{t=0}^{m-2} (j+\frac{1}{p}+t) = \frac{\overline{a}^{m-1}\delta_{i(m-1)}}{(1-|a|^2)^{m-1+\frac{1}{p}}} \\
e_{2,a,c_{m+1},...,c_{n+1}}^{(m)}(a) &= \frac{\overline{a}^{m}}{(1-|a|^2)^{m+\frac{1}{p}}} \sum_{j=1}^{n-m+1} c_{j+m} = \frac{\overline{a}^{m}\delta_{im}}{(1-|a|^2)^{m+\frac{1}{p}}} \\
&\vdots \\
e_{2,a,c_{m+1},...,c_{n+1}}^{(n)}(a) &= \frac{\overline{a}^{n}}{(1-|a|^2)^{n+\frac{1}{p}}} \sum_{j=1}^{n-m+1} c_{j+m} \prod_{t=m}^{n-1} (m+j+\frac{1}{p}+t) = \frac{\overline{a}^{n}\delta_{in}}{(1-|a|^2)^{n+\frac{1}{p}}} \\
e_{3,a,c_{n+2},...,c_{m+n+1}}^{(n+1)}(a) &= \frac{\overline{a}^{n+1}}{(1-|a|^2)^{n+1+\frac{1}{p}}} \sum_{j=1}^{m} c_{j+1+m} = \frac{\overline{a}^{n+1}\delta_{i(n+1)}}{(1-|a|^2)^{n+1+\frac{1}{p}}} \\
&\vdots \\
e_{3,a,c_{n+2},...,c_{m+n+1}}^{(m+n)}(a) &= \qquad \ldots \qquad = \frac{\overline{a}^{m+n}\delta_{i(m+n)}}{(1-|a|^2)^{m+n+\frac{1}{p}}}
\end{aligned}$$

has a unique solution [9, Lemma 2.3] which is independent of the choice of a and therefore it can be shown by $(c_1^i, c_2^i, ..., c_{m+n+1}^i)$. Now we set

$$g_{i,a}(z) = e_{1,a,c_1^i,...,c_m^i}(z) + e_{2,a,c_{m+1}^i,...,c_{n+1}^i}(z) + e_{3,a,c_{n+2}^i,...,c_{m+n+1}^i}(z).$$

□

The proof of the following lemma is similar to the proof of the previous lemma so it is omitted.

Lemma 2.6 *Let $m, n \in \mathbb{N}$ such that $n < m$. For any $0 \neq a \in \mathbb{D}$ and $i \in \{0, ..., n\} \cup \{m, ..., m+n\}$, there exists a function $g_{i,a} \in H^p$ such that*

$$g_{i,a}^{(k)}(a) = \frac{\overline{a}^k \delta_{ik}}{(1-|a|^2)^{\frac{1}{p}+k}}.$$

Also for $i \in \{0,1,...,n\}$ *then* $g_{i,a} \in span\{f_{1,a},...,f_{n+1,a}\}$ *and when* $i \in \{m, m+1,...,m+n\}$

$$g_{i,a} \in span\{f_{m+1,a},...,f_{m+n+1,a}\}.$$

In Sections 3 and 4, $m,n \in \mathbb{N}$, $0 < p < \infty$, $u,v \in H(\mathbb{D})$, μ is a weight and $\varphi \in S(\mathbb{D})$.

3. Boundedness

In this section, we give some necessary and sufficient conditions for the generalized Stević–Sharma type operators to be bounded.

Theorem 3.1 *Let* $u \in \mathcal{W}_\mu^{(n)}$. *If* $n \geq m$, *then the following statements are equivalent.*

(i) The operator $T_{u,v,\varphi}^m : H^p \to \mathcal{W}_\mu^{(n)}$ *is bounded.*

(ii) If $p_j(z) = z^j$, *then* $\sup_{j\geq 1} j^{\frac{1}{p}} \|T_{u,v,\varphi}^m p_j\|_{\mathcal{W}_\mu^{(n)}} < \infty$.

(iii) For each $i \in \{0,1,...,m+n\}$, $\sup_{a\in\mathbb{D}} \|T_{u,v,\varphi}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} < \infty$ *and* $\sup_{z\in\mathbb{D}} \mu(z)\left|(I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(z)\right| < \infty$.

(iv) For each $i \in \{0,1,...,m+n\}$, $\sup_{z\in\mathbb{D}} \frac{\mu(z)\left|(I_{i,\varphi}^{n,u}+I_{i-m,\varphi}^{n,v})(z)\right|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i}} < \infty$.

Proof $(i) \Rightarrow (iii)$ For $i \in \{0,1,...,m+n\}$, $sup_{a\in\mathbb{D}}\|f_{i+1,a}\|_{H^p} < \infty$, so

$$\sup_{a\in\mathbb{D}} \| T_{u,v,\varphi}^m f_{i+1,a} \|_{\mathcal{W}_\mu^{(n)}} \leq \| T_{u,v,\varphi}^m \|_{H^p\to\mathcal{W}_\mu^{(n)}} \sup_{a\in\mathbb{D}} \|f_{i+1,a}\|_{H^p} < \infty.$$

Applying the operator $T_{u,v,\varphi}^m$ to $p_j(z) = z^j$ for $j = 0,1,...,m+n$ respectively and using Lemma 2.4, we obtain the other part of (iii).

$(iii) \Rightarrow (iv)$ For any $i \in \{0,1,...,m+n\}$ and $\varphi(a) \neq 0$, by using Lemmas 2.4 and 2.5, we obtain

$$\frac{\mu(a) \mid \varphi(a) \mid^i \mid (I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(a) \mid}{(1-\mid \varphi(a) \mid^2)^{i+\frac{1}{p}}} \leq \sup_{a\in\mathbb{D}} \|T_{u,v,\varphi}^m g_{i,\varphi(a)}\|_{\mathcal{W}_\mu^{(n)}} \leq \sum_{j=1}^{m+n} \mid c_j^i \mid \sup_{a\in\mathbb{D}} \|T_{u,v,\varphi}^m f_{j,a}\|_{\mathcal{W}_\mu^{(n)}} < \infty.$$

From the previous inequality,

$$\sup_{|\varphi(a)|>\frac{1}{2}} \frac{\mu(a) \mid (I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(a) \mid}{(1-\mid \varphi(a) \mid^2)^{i+\frac{1}{p}}} \preceq \sum_{j=1}^{m+n} \mid c_j^i \mid \sup_{a\in\mathbb{D}} \|T_{u,v,\varphi}^m f_{j,a}\|_{\mathcal{W}_\mu^{(n)}} < \infty,$$

and from (iii), we get

$$\sup_{|\varphi(a)|\leq\frac{1}{2}} \frac{\mu(a) \mid (I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(a) \mid}{(1-\mid \varphi(a) \mid^2)^{i+\frac{1}{p}}} \preceq \sup_{|\varphi(a)|\leq\frac{1}{2}} \mu(a) \mid (I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(a) \mid < \infty.$$

Hence from last inequalities, we get (iv).

$(iv) \Rightarrow (i)$ For any $f \in H^p$, by using Lemmas 2.4 and 2.3, we have

$$\begin{aligned}\mu(z)\Big|(T^m_{u,v,\varphi}f)^{(n)}(z)\Big| \leq& \mu(z)\Big|\sum_{i=0}^{m+n} f^{(i)}(\varphi(z))(I^{n,u}_{i,\varphi}(z)+I^{n,v}_{i-m,\varphi}(z))\Big| \\ \leq& \|f\|_{H^p}\sum_{i=0}^{m+n}\sup_{z\in\mathbb{D}}\frac{\mu(z)\Big|I^{n,u}_{i,\varphi}(z)+I^{n,v}_{i-m,\varphi}(z)\Big|}{(1-\mid\varphi(z)\mid^2)^{\frac{1}{p}+i}}. \quad (3.1)\end{aligned}$$

Also for each $k < n$

$$\Big|(T^m_{u,v,\varphi}f)^{(k)}(0)\Big| \leq \|f\|_{H^p}\sum_{i=0}^{m+k}|(I^{k,u}_{i,\varphi}+I^{k,v}_{i-m,\varphi})(0)|. \quad (3.2)$$

Hence, from (3.1) and (3.2), the operator $T^m_{u,v,\varphi}: H^p \to \mathcal{W}^{(n)}_\mu$ is bounded.

$(ii) \Rightarrow (iii)$ For each $i \in \{0,...,m+n\}$ and $a \in \mathbb{D}$

$$f_{i+1,a}(z) = (1-|a|^2)^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma(\frac{1}{p}+i+1+j)}{j!\Gamma(\frac{1}{p}+i+1)}\bar{a}^j z^j.$$

So,

$$\|T^m_{u,v,\varphi}f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} \preceq (1-|a|^2)^{i+1}\sum_{j=0}^{\infty} j^{i+\frac{1}{p}}|\bar{a}|^j\|T^m_{u,v,\varphi}p_j\|_{\mathcal{W}^{(n)}_\mu} \preceq 2^{i+1}\max\{\|u\|_{\mathcal{W}^{(n)}_\mu}, \sup_{j\geq 1} j^{\frac{1}{p}}\|T^m_{u,v,\varphi}p_j\|_{\mathcal{W}^{(n)}_\mu}\}.$$

Therefore, $\sup_{a\in\mathbb{D}}\|T^m_{u,v,\varphi}f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} < \infty$. The proof of other part is similar to the proof $(i) \Rightarrow (iii)$.

$(iv) \Rightarrow (ii)$ Let $p_j(z) = z^j (j \geq n)$. By using Lemmas 2.1, 2.2 and 2.4, we get

$$\begin{aligned}j^{\frac{1}{p}}\mu(z)|(T^m_{u,v,\varphi}p_j)^{(n)}(z)| \leq& \mu(z)\sum_{i=0}^{m+n} j^{\frac{1}{p}}(1-\mid\varphi(z)\mid^2)^{\frac{1}{p}+i}\frac{j!}{(j-i)!}\times\frac{|\varphi(z)|^{j-i}|(I^{n,u}_{i,\varphi}+I^{n,v}_{i-m,\varphi})(z)|}{(1-\mid\varphi(z)\mid^2)^{\frac{1}{p}+i}} \\ \leq& \sup_j j^{\frac{1}{p}}\|z^j\|_{\mathcal{B}^{\frac{1}{p}+1}}\sum_{i=0}^{m+n}\frac{|(I^{n,u}_{i,\varphi}+I^{n,v}_{i-m,\varphi})(z)|}{(1-\mid\varphi(z)\mid^2)^{\frac{1}{p}+i}} \\ \preceq& \Big(\frac{2(\frac{1}{p}+1)}{e}\Big)^{\frac{1}{p}+1}\sum_{i=0}^{m+n}\frac{|(I^{n,u}_{i,\varphi}+I^{n,v}_{i-m,\varphi})(z)|}{(1-\mid\varphi(z)\mid^2)^{\frac{1}{p}+i}}. \quad (3.3)\end{aligned}$$

For any $k < n$, we have

$$j^{\frac{1}{p}}|(T^m_{u,v,\varphi}p_j)^{(k)}(0)| \preceq \Big(\frac{2(\frac{1}{p}+1)}{e}\Big)^{\frac{1}{p}+1}\sum_{i=0}^{m+k} j^{\frac{1}{p}}\frac{|(I^{k,u}_{i,\varphi}+I^{k,v}_{i-m,\varphi})(0)|}{(1-|\varphi(0)|^2)^{\frac{1}{p}+i}}. \quad (3.4)$$

From (3.3) and (3.4), we obtain (ii). The proof is completed. □

In the same way as in the proof of Theorem 3.1 we can prove the following theorem, just use Lemma 2.6 instead of Lemma 2.5.

Theorem 3.2 *Let $u \in \mathcal{W}^{(n)}_\mu$. If $n < m$, then the following statements are equivalent.*

(i) *The operator* $T^m_{u,v,\varphi} : H^p \to \mathcal{W}^{(n)}_\mu$ *is bounded.*

(ii) *If* $p_j(z) = z^j$, *then* $\sup_{j\geq 1} j^{\frac{1}{p}} \|T^m_{u,v,\varphi,u} p_j\|_{\mathcal{W}^{(n)}_\mu} < \infty$.

(iii) *For each* $i \in \{0, 1, ..., n\} \cup \{m, ..., m+n\}$, $\sup_{a\in\mathbb{D}} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} < \infty$ *and*

$$\sup_{z\in\mathbb{D}} \mu(z)\Big|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z)\Big| < \infty.$$

(iv) *For each* $i \in \{0, 1, ..., n\} \cup \{m, ..., m+n\}$, $\sup_{z\in\mathbb{D}} \frac{\mu(z)\Big|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z)\Big|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i}} < \infty$.

4. Essential norm

In this section, we obtain some estimates for the essential norm of generalized Stević–Sharma type operators from Hardy spaces into nth weighted type spaces. Then we give some equivalence conditions for compactness of such operators.

Theorem 4.1 *Let* $T^m_{u,v,\varphi} : H^p \to \mathcal{W}^{(n)}_\mu$ *is bounded. If* $n \geq m$, *then*

$$\|T^m_{u,v,\varphi}\|_e \approx \max\{A_i\}^{m+n}_{i=0} \approx \max\{B_i\}^{m+n}_{i=0},$$

where

$$A_i = \limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu}, \quad B_i = \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\Big|I^{n,u}_{i,\varphi}(z) + I^{n,v}_{i-m,\varphi}(z)\Big|}{(1-\mid \varphi(z) \mid^2)^{\frac{1}{p}+i}}.$$

Proof For all $i \in \{0, ..., m+n\}$, $\sup_{a\in\mathbb{D}} \|f_{i+1,a}\|_{H^p} < \infty$ and $f_{i+1,a}$ converge to 0 uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Using Lemma 2.10 [13], for any compact operator K from H^p into $\mathcal{W}^{(n)}_\mu$, we get

$$\lim_{|a|\to 1} \|K f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} = 0.$$

Thus, for any $i \in \{0, ..., m+n\}$

$$\begin{aligned}\|T^m_{u,v,\varphi} - K\|_{H^p\to\mathcal{W}^{(n)}_\mu} &\succeq \limsup_{|a|\to 1} \|(T^m_{u,v,\varphi} - K) f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} \\ &\geq \limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} - \limsup_{|a|\to 1} \|K f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} = A_i.\end{aligned}$$

So,

$$\|T^m_{u,v,\varphi}\|_e = \inf_K \|T^m_{u,v,\varphi} - K\|_{H^p\to\mathcal{W}^{(n)}_\mu} \succeq \max\{A_i\}^{m+n}_{i=0}.$$

Now, we prove that

$$\max\{B_i\}^{m+n}_{i=0} \preceq \|T^m_{u,v,\varphi}\|_e. \tag{4.1}$$

Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)|\to 1$ as $j\to\infty$. Since $T^m_{u,v,\varphi}: H^p\to\mathcal{W}^{(n)}_\mu$ is bounded, using Lemmas 2.4 and 2.5 for any compact operator $K: H^p\to\mathcal{W}^{(n)}_\mu$ and $i\in\{0,...,m+n\}$, we obtain

$$\begin{aligned}\|T^m_{u,v,\varphi}-K\|_{H^p\to\mathcal{W}^{(n)}_\mu} &\succeq \limsup_{j\to\infty}\|T^m_{u,v,\varphi}(g_{i,\varphi(z_j)})\|_{\mathcal{W}^{(n)}_\mu}-\limsup_{j\to\infty}\|K(g_{i,\varphi(z_j)})\|_{\mathcal{W}^{(n)}_\mu}\\ &\succeq \limsup_{j\to\infty}\frac{\mu(z_j)\mid\varphi(z_j)\mid^i\mid I^{n,u}_{i,\varphi}(z_j)+I^{n,v}_{i-m,\varphi}(z_j)\mid}{(1-\mid\varphi(z_j)\mid^2)^{i+\frac{1}{p}}}=B_i.\end{aligned}$$

From the last inequality, we get (4.1).

For each $0<r<1$ we consider the compact operator K_r on H^p given by $K_rf(z)=f_r(z)=f(rz)$. Let $\{r_j\}\subset(0,1)$ be a sequence such that $r_j\to 1$ as $j\to\infty$. Since $f_r\to f$ uniformly on compact subsets of $\mathbb{D}$ as $r\to 1$ then for any positive integer j, the operator $T^m_{u,v,\varphi}K_{r_j}: H^p\to\mathcal{W}^{(n)}_\mu$ is compact. So

$$\|T^m_{u,v,\varphi}\|_e\le\limsup_{j\to\infty}\|T^m_{u,v,\varphi}-T^m_{u,v,\varphi}K_{r_j}\|. \tag{4.2}$$

Hence, it is sufficient to prove that

$$\limsup_{j\to\infty}\|T^m_{u,v,\varphi}-T^m_{u,v,\varphi}K_{r_j}\|\preceq\min\{\max\{A_i\}_{i=0}^{m+n},\max\{B_i\}_{i=0}^{m+n}\}.$$

For any $f\in H^p$ such that $\|f\|_{H^p}\le 1$,

$$\begin{aligned}&\|(T^m_{u,v,\varphi}-T^m_{u,v,\varphi}K_{r_j})f\|_{\mathcal{W}^{(n)}_\mu}\le\\ &\sum_{t=0}^{n-1}\underbrace{\left|\left(T^m_{u,v,\varphi}(f-f_{r_j})\right)^{(t)}(0)\right|}_{S_t}+\underbrace{\sup_{|\varphi(z)|\le r_N}\mu(z)\left|\sum_{k=0}^{m+n}(f-f_{r_j})^{(k)}(\varphi(z))\left(I^{n,u}_{k,\varphi}+I^{n,v}_{k-m,\varphi}\right)(z)\right|}_{H_1}\\ &+\underbrace{\sup_{|\varphi(z)|>r_N}\mu(z)\left|\sum_{k=0}^{m+n}(f-f_{r_j})^{(k)}(\varphi(z))\left(I^{n,u}_{k,\varphi}+I^{n,v}_{k-m,\varphi}\right)(z)\right|}_{H_2},\end{aligned}\tag{4.3}$$

where $N\in\mathbb{N}$ such that $r_j\ge\frac{2}{3}$ for all $j\ge N$. Since $(f-f_{r_j})^{(s)}\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$, for any nonnegative integer s, then Theorem 3.1 implies that

$$\limsup_{j\to\infty}H_1=\limsup_{j\to\infty}S_t=0\quad(t=0,...,n-1). \tag{4.4}$$

Also

$$\begin{aligned}H_2&\le\sum_{k=0}^{m+n}\underbrace{\sup_{|\varphi(z)|>r_N}\mu(z)|f^{(k)}(\varphi(z))||(I^{n,u}_{k,\varphi}+I^{n,v}_{k-m,\varphi})(z)|}_{M_{2,k}}\\ &+\sum_{k=0}^{m+n}\underbrace{\sup_{|\varphi(z)|>r_N}\mu(z)|r_j^kf^{(k)}(r_j\varphi(z))||(I^{n,u}_{k,\varphi}+I^{n,v}_{k-m,\varphi})(z)|}_{N_{2,k}}.\end{aligned}\tag{4.5}$$

For $M_{2,k}$, $k \in \{0, \dots, m+n\}$, from Lemmas 2.3, 2.4 and 2.5, we get

$$\begin{aligned} M_{2,k} &= \sup_{|\varphi(z)|>r_N} \mu(z) \frac{(1-|\varphi(z)|^2)^{k+\frac{1}{p}} |f^{(k)}(\varphi(z))|}{|\varphi(z)|^k} \times \frac{|\varphi(z)|^k |(I^{n,u}_{k,\varphi} + I^{n,v}_{k-m,\varphi})(z)|}{(1-|\varphi(z)|^2)^{k+\frac{1}{p}}} \\ &\preceq \|f\|_{H^p} \sup_{|\varphi(z)|>r_N} \|T^m_{u,v,\varphi} g_{k,\varphi(z)}\|_{\mathcal{W}^{(n)}_\mu} \preceq \sum_{j=0}^{m+n} |c^k_{j+1}| \sup_{|a|>r_N} \|T^m_{u,v,\varphi} f_{j+1,a}\|_{\mathcal{W}^{(n)}_\mu} \end{aligned}$$

As $N \to \infty$, we obtain

$$\limsup_{j\to\infty} M_{2,k} \preceq \sum_{i=0}^{m+n} \underbrace{\limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu}}_{A_i} \preceq \max\{A_i\}_{i=0}^{m+n} \quad \text{and} \limsup_{j\to\infty} M_{2,k} \preceq B_k. \tag{4.6}$$

Similarly, we get

$$\limsup_{j\to\infty} N_{2,k} \preceq \sum_{i=0}^{m+n} \underbrace{\limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu}}_{A_i} \preceq \max\{A_i\}_{i=0}^{m+n} \quad \text{and} \limsup_{j\to\infty} N_{2,k} \preceq B_k. \tag{4.7}$$

Thus, by using (4.3), (4.4), (4.5), (4.6) and (4.7), we obtain

$$\limsup_{j\to\infty} \|T^m_{u,v,\varphi} - T^m_{u,v,\varphi} K_{r_j}\|_{H^p \to \mathcal{W}^{(n)}_\mu} = \limsup_{j\to\infty} \sup_{\|f\|_{H^p}\le 1} \|(T^m_{u,v,\varphi} - T^m_{u,v,\varphi} K_{r_j}) f\|_{\mathcal{W}^{(n)}_\mu} \preceq \max\{A_i\}_{i=0}^{m+n}$$

and

$$\limsup_{j\to\infty} \|T^m_{u,v,\varphi} - T^m_{u,v,\varphi} K_{r_j}\|_{H^p \to \mathcal{W}^{(n)}_\mu} \preceq \max\{B_i\}_{i=0}^{m+n}.$$

Hence, from (4.2),

$$\|T^m_{u,v,\varphi}\|_e \preceq \min\{\max\{A_i\}_{i=0}^{m+n}, \max\{B_i\}_{i=0}^{m+n}\}.$$

The proof is completed. □

The proof of the next theorem is similar to the proof of Theorem 4.1, except that we use Lemma 2.6 instead of Lemma 2.5.

Theorem 4.2 *Let $T^m_{u,v,\varphi} : H^p \to \mathcal{W}^{(n)}_\mu$ is bounded. If $n < m$, then*

$$\|T^m_{u,v,\varphi}\|_e \approx \max\{\{A_i\}_{i=0}^{n} \cup \{A_i\}_{i=m}^{m+n}\} \approx \max\{\{B_i\}_{i=0}^{n} \cup \{B_i\}_{i=m}^{m+n}\},$$

where

$$A_i = \limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu}, \quad B_i = \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z)\right|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i}}$$

and $i \in \{0, 1, \dots n\} \cup \{m, \dots, m+n\}$.

Theorem 4.3 *Let $T^m_{u,v,\varphi}: H^p \to \mathcal{W}^{(n)}_\mu$ is bounded. If $n \geq m$, then the following statements are equivalent.*

(i) The operator $T^m_{u,v,\varphi}: H^p \to \mathcal{W}^{(n)}_\mu$ is compact.

(ii) If $p_j(z) = z^j$ then, $\lim_{j\to\infty} j^{\frac{1}{p}} \|T^m_{u,v,\varphi,u} p_j\|_{\mathcal{W}^{(n)}_\mu} = 0$.

(iii) For each $i \in \{0,1,...,m+n\}$, $\limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} = 0$.

(iv) For each $i \in \{0,1,...,m+n\}$ $\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z)\right|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i}} = 0$.

Proof By using Theorem 4.1, $(i), (iii)$ and (iv) are equivalent.

$(ii) \Rightarrow (iii)$ For any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $k \geq N$,

$$k^{\frac{1}{p}} \|T^m_{u,v,\varphi} p_k\|_{\mathcal{W}^{(n)}_\mu} < \epsilon.$$

For any $j \in \{0,1,...,m+n\}$

$$f_{j+1,a}(z) = (1-|a|^2)^{j+1}\Big(\sum_{k=0}^{N-1} + \sum_{k=N}^{\infty}\Big)\frac{\Gamma(\frac{1}{p}+j+1+k)}{k!\Gamma(\frac{1}{p}+j+1)}\overline{a}^k z^k.$$

So,

$$\|T^m_{u,v,\varphi} f_{j+1,a}\|_{\mathcal{W}^{(n)}_\mu} \leq 2\max\{\|u\varphi^k\|_{\mathcal{W}^{(n)}_\mu}\}_{k=0}^{N-1}(1-|a|^2)^j(1-|a|^N)\frac{\Gamma(\frac{1}{p}+N+j)}{N!\Gamma(\frac{1}{p}+j)} + 2^{j+1}\epsilon.$$

Hence, $\limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{j+1,a}\|_{\mathcal{W}^{(n)}_\mu} \leq 2^{j+1}\epsilon$. Since ϵ is arbitrary, we obtain

$$\limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{j+1,a}\|_{\mathcal{W}^{(n)}_\mu} = 0.$$

$(iv) \Rightarrow (ii)$ For any given $\epsilon > 0$ and $k \in \{0,1,...,m+n\}$ there exists a positive constant δ such that $\delta < |\varphi(z)| < 1$,

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|(I^{n,u}_{k,\varphi} + I^{n,v}_{k-m,\varphi})(z)\right|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+k}} < \epsilon. \tag{4.8}$$

Let $p_j(z) = z^j (j \geq n)$. By using Lemma 2.4, we have

$$j^{\frac{1}{p}} \|T^m_{u,v,\varphi} p_j\|_{\mathcal{W}^{(n)}_\mu} \leq$$

$$\sum_{t=0}^{n-1} \underbrace{j^{\frac{1}{p}} \left|\left(T^m_{u,v,\varphi} p_j\right)^{(t)}(0)\right|}_{S_t} + \underbrace{\sup_{|\varphi(z)|\leq r_N} \mu(z) \sum_{k=0}^{m+n} j^{\frac{1}{p}} \frac{j!}{(j-k)!}|\varphi(z)|^{j-k}|(I^{n,u}_{k,\varphi} + I^{n,v}_{k-m,\varphi})(z)|}_{H_1}$$

$$+ \underbrace{\sup_{|\varphi(z)|> r_N} \mu(z) \sum_{k=0}^{m+n} j^{\frac{1}{p}}(1-|\varphi(z)|^2)^{\frac{1}{p}+k} \frac{j!}{(j-k)!}|\varphi(z)|^{j-k} \frac{|(I^{n,u}_{k,\varphi} + I^{n,v}_{k-m,\varphi})(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+k}}}_{H_2}. \tag{4.9}$$

From Theorem 3.1, it is obvious that

$$\limsup_{j\to\infty} H_1 = \limsup_{j\to\infty} S_t = 0 \quad (t = 0, ..., n-1). \tag{4.10}$$

By using Lemma 2.1, 2.2 and (4.8), we obtain

$$H_2 \preceq \Big(\frac{2(\frac{1}{p}+1)}{e}\Big)^{\frac{1}{p}+1} \sum_{k=0}^{m-1} \sup_{|\varphi(z)|>\delta} \frac{\mu(z)|(I^{n,u}_{k,\varphi} + I^{n,v}_{k-m,\varphi})(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+k}} \preceq \epsilon$$

which implies that

$$\limsup_{j\to\infty} H_2 = 0. \tag{4.11}$$

From (4.9), (4.10) and (4.11), we get (ii). The proof is completed. □

Using the same method as in the proof of Theorem 4.3 we can get the following theorem.

Theorem 4.4 *Let $T^m_{u,v,\varphi} : H^p \to \mathcal{W}^{(n)}_\mu$ is bounded. If $n < m$, then the following statements are equivalent.*

(i) The operator $T^m_{u,v,\varphi} : H^p \to \mathcal{W}^{(n)}_\mu$ is compact.

(ii) If $p_j(z) = z^j$ then $\lim_{j\to\infty} j^{\frac{1}{p}} \|T^m_{u,v,\varphi,u} p_j\|_{\mathcal{W}^{(n)}_\mu} = 0$.

(iii) For each $i \in \{0, 1, ...n\} \cup \{m, ..., m+n\}$, $\limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} = 0$.

(iv) For each $i \in \{0, 1, ...n\} \cup \{m, ..., m+n\}$, $\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z)\right|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i}} = 0$.

Remark 4.5 *By putting $v \equiv 0$ in Theorems 3.1, 3.2, 4.1, 4.2, 4.3 and 4.4, we obtain some characterizations for boundedness, the essential norm and compactness of operator $uC_\varphi : H^p \to \mathcal{W}^{(n)}_\mu$ (see [2]).*

Remark 4.6 *By setting $u \equiv 0$ in Theorems 3.1, 3.2, 4.1, 4.2, 4.3 and 4.4, we get some characterizations for boundedness, the essential norm and compactness of generalized weighted composition operator $D^m_{\varphi,u} : H^p \to \mathcal{W}^{(n)}_\mu$ (see [6]).*

Remark 4.7 *By taking $m = 1$ in Theorems 3.1, 3.2, 4.1, 4.2, 4.3 and 4.4, we find some characterizations for boundedness, the essential norm and compactness of Stević-Sharma type operators from Hardy space into nth weighted type spaces (see [14]).*

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