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# On Sense of Yamakawa family of meromorphic bi-univalent and bi-subordinate functions

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**Abstract:** This study offers three different univalent function families of bi-meromorphic and bi-subordinate functions defined on $\Delta = \{z : z \in \mathbb{C}, 1 < |z| < \infty\}$. The estimations of the first three coefficients $|b_0|$, $|b_1|$, $|b_2|$ and extra $|b_0 b_1 + 2b_2|$ are obtained for the functions of these families. We also point out some closely related cases for our results.

**Key words:** Univalent function, bi-univalent function, meromorphic function, subordiation

## 1. Introduction

One of the major branches of complex analysis is univalent function theory: the study of one-to-one analytic functions. Analytical functions are widely used in thermodynamics, electricity and magnetism and also quantum physics. In electricity, current and impedance equations can be expressed in a complex plane, and basic electrical relationships become complex functions [10]. There are numerous mathematical descriptions of the electromagnetic field. The subordination results are interpreted in the context of electromagnetic cloaking for possible practical applications. By the Riemann mapping theorem, both the regions are equivalent to conformal maps on the unit disc $\sqcup$ [12].

Let $\mathcal{A}$ indicate the family of functions written as

$$F(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\sqcup = \{z \in \mathbb{C} : |z| < 1\}$. Also $\mathcal{S}$ shows the subfamily of the normalized analytic function family $\mathcal{A}$ covering all univalent functions in $\sqcup$.

Since univalent functions are one-to-one, they are invertible and these functions do not need to be defined on the all unit disk $\sqcup$. Actually, the Koebe on-quarter theorem [3] confirms that the image of $\sqcup$ includes a disk of radius $1/4$ for every univalent function $F \in \mathcal{S}$. Therefore, every function $F \in \mathcal{S}$ has an $F^{-1}$ defined by

$$F^{-1}(F(z)) = z, \qquad (z \in \sqcup)$$

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and

$$F\left(F^{-1}(w)\right)=w \qquad \left(|w|<r_0(F);\, r_0(F)\geq\frac{1}{4}\right).$$

A function $F$ in $\mathcal{A}$ is considered as bi-univalent in the open unit disk $\sqcup$ if both the function and its inverse are univalent in $\sqcup$. A function $F\in\mathcal{A}$ is said to be bi-univalent in the open unit disk $\sqcup$ if both the inverse of function and itself are univalent in $\sqcup$. Let $\sigma$ show the family of bi-univalent functions in $\sqcup$ presented by Taylor–Maclaurin expansion in (1.1). One can see [20] for a brief history and some examples of functions in $\sigma$. Actually, the review of various subfamilies of the bi-univalent function family $\sigma$ was essentially made by Srivastava et al.[20] in recent years. This was followed by papers published by Frasin and Aouf [6], Srivastava et al. [21],[19], Xu et al. [23],[24] and others (see also,[7], [4], [2], [11], [1], [15]). In current study, the notion of

bi-univalency is enlarged to the family of meromorphic functions defined on $\Delta=\{z: z\in\mathbb{C}, 1<|z|<\infty\}$. All meromorphic and univalent function familes in $\Delta$ is denoted by $\Sigma'$ and the family of $g$ functions

$$g(z)=z+b_0+\sum_{k=1}^{\infty}\frac{b_k}{z^k}. \tag{1.2}$$

Every univalent function $g$ with an inverse $g^{-1}$ satisfies the following expansion

$$g^{-1}(w)=h(w)=w+D_0+\sum_{k=1}^{\infty}\frac{D_k}{w^k}, \tag{1.3}$$

where $0<M<|w|<\infty$. Similar to the bi-univalent functions, a function $g\in\Sigma'$ is considered as meromorphic and bi-univalent if both of $g$ and $g^{-1}$ are univalent meromorphic in $\Delta$ as given by (1.2). The family of all meromorphic and bi-univalent functions are shown by $\Sigma'_\sigma$. An elementary calculation gives $h(w)$ as expressed:

$$h(w)=g^{-1}(w)=w-b_0-\frac{b_1}{w}-\frac{b_2+b_0b_1}{w^2}-\frac{b_3+2b_0b_2+b_0^2b_1+b_1^2}{w^3}+\cdots. \tag{1.4}$$

Examples of the meromorphic bi-univalent functions are as follows:

$$z+\frac{1}{z},\quad z-1,\quad -\frac{1}{\log\left(1-\frac{1}{z}\right)}.$$

Coefficient estimates on the inverses of meromorphic univalent functions were frequently investigated in the studies. For instance, it was shown by Schiffer [16] that if $g$ given by (1.2) is in $\Sigma'$ with $b_0=0$, then $|b_2|\leq 2/3$. The inequality $|b_n|\leq 2/(n+1)$ for $g\in\Sigma'$ with $b_k=0$, $1\leq k<n/2$ was calculated by Duren [5] in 1971. It was also shown by Springer [18] that the following expression holds for $g^{-1}$ .

$$|D_3|\leq 1 \qquad \textit{and} \qquad |D_3+\frac{1}{2}D_1^2|\leq\frac{1}{2},$$

and following inequality was also conjuctered

$$|D_{2n-1}|\leq\frac{(2n-2)!}{n!(n-1)!} \qquad (n=1,2,\cdots).$$

It was confirmed by Kubota [9] in 1977 that the Springer conjecture is valid for $n = 3, 4, 5$ and Schober [17] later achieved sharp bounds for $D_{2n-1}$, $(1 \leq n \leq 7)$.

The coefficient estimates for inverses of meromorphic starlike functions with positive order $\alpha$ were recently obtained by Kapoor and Mishra [8].

Motivated by the work of [22] and [14], the main aim of current investigation is to introduce certain families of meromorphic bi-univalent function families $\Sigma'$, sense of Yamakawa, and obtain estimates for coefficients $|b_0|$, $|b_1|$ and $|b_2|$ and extra $|b_0 b_1 + 2b_2|$ belonging to these families.

The current coefficients were obtained from the positive real part of functions. An analytic function $p$ in the form of $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is known that a function with positive real part in $\sqcup$ if $Rep(z) > 0$ for all $z \in \sqcup$. The family of all functions having positive real part is denoted by $\mathcal{P}$. The following lemma [13] is required to derive our main results.

**Lemma 1.1** If $\varphi(z) \in \mathcal{P}$, the family of analytic functions in $\sqcup$ with positive real part, given by:

$$\varphi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \qquad (z \in \sqcup),$$

then $|c_n| \leq 2$ for each $n \in \mathbb{N}$.

## 2. Main results

Firstly, lets consider Yamakawa's family $T(n, \alpha)$ (for the case of $p = 1$ in [22]).

**Definition 2.1** [22] A function $F \in T(n)$ is said to be a member of the family $T(n, \alpha)$ if it holds the inequality

$$Re\left\{\frac{F(z)}{zF'(z)}\right\} > \alpha \qquad (z \in \sqcup; 0 \leq \alpha \leq 1).$$

where $F(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \qquad (a_k \geq 0; n \in \mathbb{N} = 1, 2, 3, ...)$. Since

$$Re\left\{\frac{F(z)}{zF'(z)}\right\} > \alpha \Rightarrow Re\left\{\frac{zF'(z)}{F(z)}\right\} > 0 \qquad (z \in \sqcup; 0 \leq \alpha \leq 1).$$

$T(n, \alpha)$ is a subfamily of $T_0(n)$.

We call $\mathcal{T}\Sigma'_\sigma(\alpha)$ the family of meromorphically bi-univalent functions sense of Yamakawa defined as follows:

**Definition 2.2** A function $g \in \Sigma'_\sigma$ given by (1.2) is said to be in the family $\mathcal{T}\Sigma'_\sigma(\alpha)$, $(0 < \alpha \leq 1)$ if the inequalities are satisfied.

$$\left|arg\left(\frac{g(z)}{zg'(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \Delta; 0 < \alpha \leq 1),$$

$$and \qquad (2.1)$$

$$\left|arg\left(\frac{h(w)}{zh'(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \Delta; 0 < \alpha \leq 1)$$

where $h$ is an extension of $g^{-1}$ to $\Delta$ given in (1.4).

We now derive the estimates on coefficients for meromorphically bi-univalent function family $\mathcal{T}\Sigma'_{\sigma}(\alpha)$ defined in Definition 2.2.

**Theorem 2.1** *Let the function $g$ given by (1.2) be in the function family $\mathcal{T}\Sigma'_{\sigma}(\alpha)$, $0 < \alpha \leq 1$. Then*

$$|b_0| \leq 2\alpha \tag{2.2}$$

$$|b_1| \leq \alpha \tag{2.3}$$

$$|b_2| \leq \frac{4}{9}\alpha(\alpha-1)(\alpha-2) + \frac{2}{3}\alpha(2\alpha-1) + \alpha^2 \tag{2.4}$$

$$|b_0 b_1 + 2b_2| \leq \frac{8}{9}\alpha(\alpha-1)(\alpha-2) + 4\alpha(2\alpha-1) \tag{2.5}$$

**Proof**

It follows from (2.1) that

$$\frac{g(z)}{zg'(z)} = [s(z)]^{\alpha} \qquad \text{and} \qquad \frac{h(w)}{zh'(w)} = [t(w)]^{\alpha} \qquad (z, w \in \Delta), \tag{2.6}$$

respectively, where $s(z)$ and $t(z)$ are functions having positive real part in $\Delta$ and have the forms

$$s(z) = 1 + \frac{s_1}{z} + \frac{s_2}{z^2} + \cdots, \tag{2.7}$$

and

$$t(w) = 1 + \frac{t_1}{w} + \frac{t_2}{w^2} + \cdots, \tag{2.8}$$

Now, upon equating the coefficient in (2.6), we get

$$b_0 = \alpha s_1 \tag{2.9}$$

$$2b_1 = \frac{\alpha(\alpha-1)}{2}s_1^2 + \alpha s_2 \tag{2.10}$$

$$b_0 b_1 + 3b_2 = \frac{\alpha(\alpha-1)(\alpha-2)}{6}s_1^3 + \alpha(\alpha-1)s_1 s_2 + \alpha s_3 \tag{2.11}$$

and

$$-b_0 = \alpha t_1 \tag{2.12}$$

$$-2b_1 = \frac{\alpha(\alpha-1)}{2}t_1^2 + \alpha t_2 \tag{2.13}$$

$$-2b_0 b_1 - 3b_2 = \frac{\alpha(\alpha-1)(\alpha-2)}{6}t_1^3 + \alpha(\alpha-1)t_1 t_2 + \alpha t_3. \tag{2.14}$$

From (2.9) and (2.12)

$$s_1^2 = t_1^2 \tag{2.15}$$

$$2b_0^2 = \alpha^2(s_1^2 + t_1^2). \tag{2.16}$$

We subtract (2.12) from (2.9) to obtain first coefficient inequalities

$$2b_0 = \alpha(s_1 - t_1). \tag{2.17}$$

Hence, applying Lemma 1.1,

$$|b_0| \leq 2\alpha. \tag{2.18}$$

Also, from (2.10) and (2.13) and using (2.15) we obtain

$$4b_1 = \frac{\alpha(\alpha-1)}{2}(s_1^2 - t_1^2) + \alpha(s_2 - t_2). \tag{2.19}$$

Hence, applying Lemma 1.1,

$$|b_1| \leq \alpha. \tag{2.20}$$

From the equalities (2.11) and (2.14) we get,

$$3b_0b_1 + 6b_2 = \frac{\alpha(\alpha-1)(\alpha-2)}{6}(s_1^3 - t_1^3) + \alpha(\alpha-1)(s_1s_2 - t_1t_2) + \alpha(s_3 - t_3). \tag{2.21}$$

When we use the equalities (2.17) and (2.19) we obtain,

$$6b_2 = \frac{\alpha(\alpha-1)(\alpha-2)}{6}(s_1^3 - t_1^3) + \alpha(\alpha-1)(s_1s_2 - t_1t_2) + \alpha(s_3 - t_3)$$

(2.22)

$$-\frac{3\alpha^2(\alpha-1)}{16}(s_1^2 - t_1^2)(s_1 - t_1) + \frac{3\alpha^2}{8}(s_1 - t_1)(s_2 - t_2). \tag{2.23}$$

Hence, applying Lemma 1.1,

$$|b_2| \leq \frac{4}{9}\alpha(\alpha-1)(\alpha-2) + \frac{2}{3}\alpha(2\alpha-1) + \alpha^2. \tag{2.24}$$

Also, from the equality (2.11), we can easily get

$$|b_0b_1 + 2b_2| \leq \frac{8}{9}\alpha(\alpha-1)(\alpha-2) + 4\alpha(2\alpha-1).$$

So, we reached the expected inequalities. □

We call $\mathcal{T}\Sigma'_\sigma(\beta)$ the family of meromorphically bi-univalent functions sense of Yamakawa defined as follows:

**Definition 2.3** A function $g \in \Sigma'_\sigma$ given by (1.2) is said to be in the family $\mathcal{T}\Sigma'_\sigma(\beta)$ if the conditions given below are satisfied:

$$\Re\left(\frac{g(z)}{zg'(z)}\right) > \beta \qquad (z \in \Delta; 0 \le \beta < 1),$$

*and*

$$\Re\left(\frac{h(w)}{zh'(w)}\right) > \beta \qquad (w \in \Delta; 0 \le \beta < 1)$$

where $h$ is an extension of $g^{-1}$ to $\Delta$ in (1.4).

Following theorem gives us the estimates on coefficients for meromorphically bi-univalent function family $\mathcal{T}\Sigma'_\sigma(\beta)$ defined in Definition 2.3.

**Theorem 2.2** *Let the function $g$ given by the series expansion (1.2) be in the function family $\mathcal{T}\Sigma'_\sigma(\beta)$, $0 \le \beta < 1$. Then*

$$|b_0| \le 2(1-\beta) \tag{2.25}$$

$$|b_1| \le (1-\beta) \tag{2.26}$$

$$|b_2| \le (1-\beta)(\beta - \frac{1}{3}). \tag{2.27}$$

$$|b_0 b_1 + 2b_2| \le \frac{4}{3}(1-\beta). \tag{2.28}$$

**Proof** Let $g \in \mathcal{T}\Sigma'_\sigma(\beta)$. Then from Definition 2.3 we can write the equalities as follows:

$$\frac{g(z)}{zg'(z)} = \beta + (1-\beta)s(z) \qquad \textit{and} \qquad \frac{h(w)}{zh'(w)} = \beta + (1-\beta)t(w) \qquad (z, w \in \Delta), \tag{2.29}$$

respectively, where $s(z)$ and $t(z)$ are functions with positive real part in $\Delta$ and have the forms in (2.7)and (2.8).

Now, when we equate the coefficient in (2.29), we get

$$b_0 = (1-\beta)s_1 \tag{2.30}$$

$$2b_1 = (1-\beta)s_2 \tag{2.31}$$

$$b_0 b_1 + 3b_2 = (1-\beta)s_3 \tag{2.32}$$

and

$$-b_0 = (1-\beta)t_1 \tag{2.33}$$

$$-2b_1 = (1-\beta)t_2 \tag{2.34}$$

$$-2b_0b_1 - 3b_2 = (1-\beta)t_3. \tag{2.35}$$

From (2.30) and (2.33), we obtain

$$s_1 = -t_1 \tag{2.36}$$

and

$$2b_0^2 = (1-\beta)^2(s_1^2 + t_1^2). \tag{2.37}$$

Applying Lemma 1.1 for the coefficients $s_1$ and $t_1$, we directly have

$$|b_0| \leq 2(1-\beta).$$

This gives the bound on $|b_0|$ as asserted in Theorem 2.2. Next, to have the bound on $|b_1|$, by using the equation (2.31) and (2.34), we obtain

$$4b_1 = (1-\beta)(s_2 - t_2). \tag{2.38}$$

Applying Lemma 1.1 one again,

$$|b_1| \leq (1-\beta).$$

Lastly, to obtain the bound on $|b_2|$, by using the equation (2.32) and (2.35), we obtain

$$3b_0b_1 + 6b_2 = (1-\beta)(s_3 - t_3). \tag{2.39}$$

substituting $b_0$ and $b_1$ given by (2.37) and (2.38)in (2.39), we obtain

$$6b_2 = (1-\beta)(s_3 - t_3) - \frac{(1-\beta)^2}{8}(s_2 - t_2)\sqrt{\frac{s_1^2 + t_1^2}{2}}.$$

Applying Lemma 1.1 we obtain,

$$|b_2| \leq (1-\beta)(\beta - \frac{1}{3}).$$

Also from the equality (2.39), we can easily get

$$|b_0b_1 + 2b_2| \leq \frac{4}{3}(1-\beta).$$

So, the proof is completed. □

We know that $s(z) \in \mathcal{P} \quad (z \in \sqcup) \Leftrightarrow s(\frac{1}{z}) \in \mathcal{P} \quad (z \in \Delta)$.
Define the functions $s$ and $t$ in $\mathcal{P}$ given by

$$s(z) = \frac{1 + k(z)}{1 - k(z)} = 1 + \frac{s_1}{z} + \frac{s_2}{z^2} + \cdots$$

and

$$t(z) = \frac{1 + l(z)}{1 - l(z)} = 1 + \frac{t_1}{z} + \frac{t_2}{z^2} + \cdots,$$

where $k(z) = 1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_n}{z^n} + \cdots$, $|k(z)| < 1, \quad (z \in \Delta)$ and $l(z) = 1 + \frac{d_1}{z} + \frac{d_2}{z^2} + \cdots + \frac{d_n}{z^n} + \cdots$, $|l(z)| < 1, \quad (z \in \Delta)$ are Schwarz functions [13]. It follows that

$$k(z) = \frac{s(z) - 1}{s(z) + 1} = \frac{s_1}{2}\frac{1}{z} + \frac{1}{2}\left(s_2 - \frac{s_1^2}{2}\right)\frac{1}{z^2} + \cdots$$

and

$$l(z) = \frac{t(z) - 1}{t(z) + 1} = \frac{t_1}{2}\frac{1}{z} + \frac{1}{2}\left(t_2 - \frac{t_1^2}{2}\right)\frac{1}{z^2} + \cdots \quad .$$

An analytic function $g$ is subordinate to an analytic function $h$, written by $g \prec h$, provided that there is an analytic function $w$ defined on $\sqcup = \{z \in \mathbb{C} : |z| < 1\}$. with $w(0) = 0$ and $|w(z)| < 1$ satisfying $F(z) = g(w(z))$. After that, it is supposed that $\psi$ is an analytic function having positive real part in $\sqcup$, satisfying $\psi(0) = 1$, $\psi'(0) > 0$, and $\psi(\sqcup)$ is symmetric with respect to the real axis. Such a function is known to be real with the series expansion $\psi(z) = 1 + \Lambda_1 z + \Lambda_2 z^2 + \Lambda_3 z^3 + \cdots$ where $\Lambda_1, \ \Lambda_2$ are real and $\Lambda_1 > 0$.

We call $\mathcal{T}\Sigma'_\sigma(\varphi)$ the family of meromorphically bi-subordinate functions sense of Yamakawa defined as follows:

**Definition 2.4** A function $g \in \Sigma'_\sigma$ given by (1.2) is said to be in the family $\mathcal{T}\Sigma'_\sigma(\varphi)$ if conditions given below are satisfied:

$$\frac{g(z)}{zg'(z)} \prec \varphi(z) \qquad (z \in \Delta),$$

*and*

$$\frac{h(w)}{zh'(w)} \prec \varphi(w) \qquad (w \in \Delta)$$

where the function $h$ is an extension of $g^{-1}$ to $\Delta$ in (1.4).

We now derive the estimates on the coefficients for meromorphically bi-subordinate function family $\mathcal{T}\Sigma'_\sigma(\varphi)$ defined in Definition 2.4.

**Theorem 2.3** *Let the function $g$ given by (1.2) be in the function family $\mathcal{T}\Sigma'_\sigma(\psi)$. Then,*

$$|b_0| \le \Lambda_1 \tag{2.40}$$

$$|b_1| \le \frac{\Lambda_1}{2} \tag{2.41}$$

$$|b_2| \le \Lambda_1 + 2\Lambda_2 + \Lambda_3 \tag{2.42}$$

$$|b_0 b_1 + 2b_2| \le \frac{2}{3}(\Lambda_1 + \Lambda_2 + \Lambda_3) \tag{2.43}$$

**Proof** Let $\mathcal{T}\Sigma'_\sigma(\psi)$. Then, there are analytic functions $k, l : \Delta \to \mathbb{C}$ with $k(\infty) = l(\infty) = 0$, satisfying

$$\frac{g(z)}{zg'(z)} = \psi(k(z)) \ \ (z \in \Delta) \quad \textit{and} \quad \frac{g(w)}{wg'(w)} = \psi(l(w)) \ \ (w \in \Delta).$$

Then, equating the coefficients in (2)

$$b_0 = \Lambda_1 a_1 \tag{2.44}$$

$$2b_1 = \Lambda_1 a_2 + \Lambda_2 a_1^2 \tag{2.45}$$

$$b_0 b_1 + 3b_2 = \Lambda_1 a_3 + 2\Lambda_2 a_1 a_2 + \Lambda_3 a_1^3 \tag{2.46}$$

and

$$-b_0 = \Lambda_1 d_1 \tag{2.47}$$

$$-2b_1 = \Lambda_1 d_2 + \Lambda_2 d_1^2 \tag{2.48}$$

$$-2b_0 b_1 - 3b_2 = \Lambda_1 d_3 + 2\Lambda_2 d_1 d_2 + \Lambda_3 d_1^3 \tag{2.49}$$

Now, considering (2.44) and (2.47), we get

$$a_1^2 = d_1^2,$$

and

$$2b_0^2 = \Lambda_1^2(a_1^2 + d_1^2),$$

in the light of inequalities $|a_n| \leq 1$ and $|d_n| \leq 1$ and taking modulus it yields

$$|b_0| \leq \Lambda_1.$$

Since, we reach the desired first estimate on $|b_0|$ given in (2.40). In addition, comparing the coefficients of (2.45) and (2.49), we get

$$4b_1 = \Lambda_1(a_2 - d_2) + \Lambda_2(a_1^2 - d_1^2),$$

from $a_1^2 = d_1^2$, applying inequalities $|a_n| \leq 1$ and $|d_n| \leq 1$, and taking modulus it yields

$$|b_1| \leq \frac{\Lambda_1}{2}$$

which is the bound on $|b_1|$.

Next, to find the bound on $|b_2|$, by further computations from (2.46) and (2.49), we get

$$3b_2 = \Lambda_1(2a_3 + d_3) + 2\Lambda_2(2a_1 a_2 + d_1 d_2) + \Lambda_3(2a_1^3 + d_1^3).$$

Applying inequalities $|a_n| \leq 1$ and $|d_n| \leq 1$, and taking modulus it yields

$$|b_2| \leq \Lambda_1 + 2\Lambda_2 + \Lambda_3$$

Furthermore, when we subtract (2.49) from (2.46)

$$3b_0 b_1 + 6b_2 = \Lambda_1(a_3 - d_3) + 2\Lambda_2(a_1 a_2 - d_1 d_2) + \Lambda_3(a_1^3 - d_1^3)$$

Applying inequalities $|a_n| \leq 1$ and $|d_n| \leq 1$, and taking modulus it yields

$$|b_0 b_1 + 2b_2| \leq \frac{2}{3}(\Lambda_1 + \Lambda_2 + \Lambda_3).$$

$\square$

## 3. Conclusion

**Example 3.1** For the function

$$\psi(z) = \left(\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right)^{\theta} = 1 + \frac{2\theta}{z} + \frac{2\theta^2}{z^2} + \frac{2\theta^3}{z^3} + \cdots, (0 < \theta \leq 1, \ \ z \in \Delta),$$

we have the family $\mathcal{T}\Sigma'_\sigma(\psi) = \mathcal{T}\Sigma'_\sigma\left(\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right)^{\theta}$ and the following corollary:

**Corollary 3.1** *Let $g$ given by (1.2) be in the family $\mathcal{T}\Sigma'_\sigma\left(\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right)^{\theta}$, $0 < \theta \leq 1$. Then*

$$|b_0| \leq 2\theta$$

$$|b_1| \leq \theta$$

$$|b_2| \leq 2\theta + 4\theta^2 + 2\theta^3$$

$$|b_0 b_1 + 2b_2| \leq \frac{4\theta}{3}(1 + \theta + \theta^2).$$

**Remark 3.3** For the special cases of $\theta = 1$ we get function $\psi(z) = \left(\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right) = 1 + \frac{2}{z} + \frac{2}{z^2} + \frac{2}{z^3} + \cdots, z \in \Delta),$ and following coefficients

$$|b_0| \leq 2$$

$$|b_1| \leq 1$$

$$|b_2| \leq 8$$

$$|b_0 b_1 + 2b_2| \leq 4.$$

**Example 3.4** For the function

$$\psi(z) = \left(\frac{1+\frac{1-2\xi}{z}}{1-\frac{1}{z}}\right) = 1 + \frac{2(1-\xi)}{z} + \frac{2(1-\xi)}{z^2} + \frac{2(1-\xi)}{z^3} + \cdots, (0 \leq \xi < 1, \ \ z \in \Delta),$$

we have the family $\mathcal{T}\Sigma'_\sigma(\psi) = \mathcal{T}\Sigma'_\sigma\left(\frac{1+\frac{1-2\xi}{z}}{1-\frac{1}{z}}\right)$ and the following corollary:

**Corollary 3.2** *Let $g$ given by (1.2) be in the family $\mathcal{T}\Sigma'_\sigma\left(\frac{1+\frac{1-2\xi}{z}}{1-\frac{1}{z}}\right)$, $0 \leq \xi < 1$. Then*

$$|b_0| \leq 2(1-\xi)$$

$$|b_1| \leq (1-\xi)$$

$$|b_2| \leq 8(1-\xi)$$

$$|b_0 b_1 + 2b_2| \leq 4(1-\xi).$$

**Example 3.5** For the function

$$\psi(z) = \frac{1+\frac{K}{z}}{1+\frac{L}{z}} = 1 + \frac{K-L}{z} - \frac{L(K-L)}{z^2} + \frac{L^2(K-L)}{z^3} + \cdots, (-1 \leq K \leq L < 1, \ \ z \in \Delta),$$

we have the family $\mathcal{T}\Sigma'_\sigma(\psi) = \mathcal{T}\Sigma'_\sigma\left(\frac{1+\frac{K}{z}}{1+\frac{L}{z}}\right)$ and the following corollary:

$$|b_0| \leq K - L$$

$$|b_1| \leq \frac{K-L}{2}$$

$$|b_2| \leq (K-L)(1-2L+L^2)$$

$$|b_0 b_1 + 2b_2| \leq \frac{2}{3}(K-L)(1-L+L^2).$$

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