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On some nonlocal inverse boundary problem for partial differential equations of third order

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Abstract: In this paper, we consider the inverse boundary value problem for a partial differential equation of third order with nonlocal boundary conditions, including an integral condition. Using analytical and operator-theoretic methods, as well as the Fourier method, the existence and uniqueness of the classical solution of this problem is proved.

Key words: Inverse boundary value problem, partial differential equation of third-order, integral boundary condition, Riesz basis, contraction operator

1. Introduction

For the equation

$$u_{ttt}(x,t) + u_{xx}(x,t) = a(t)u(x,t) + f(x,t) \tag{1.1}$$

in the region $D_T = \{(x,t) : 0 \leq x \leq 1, 0 \leq t \leq T\}$, we consider the inverse boundary value problem under the conditions

$$u(x,0) = \varphi_0(x), u_x(0,t) = 0,\ du(1,t) + \int_0^1 u(x,t)dx = 0,\ 0 \leq t \leq T, \tag{1.2}$$

$$u_t(x,0) = \varphi_1(x),\ u_{tt}(x,T) = \varphi_2(x),\ 0 \leq x \leq 1, \tag{1.3}$$

$$u(0,t) = h(t),\ 0 \leq t \leq T, \tag{1.4}$$

where $d > 0$ is a given number, $f(x,t)$ and $\varphi_i(x)$, $i = 0, 1, 2,$ are given functions, $u(x,t)$ and $a(t)$ are the required functions.

It is known [9] that when mathematical modeling of various processes of physics, chemistry, ecology, and biology is performed, problems frequently arise, when, instead of classical boundary conditions, a certain connection is established between the values of the unknown function at the boundary of the domain and inside

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it. Problems of this type are called nonlocal problems, and the study of such problems is caused not only by theoretical interests, but also by practical necessity. A systematic study of nonlocal initial-boundary value problems was first carried out in [5]. In particular, spatially nonlocal problems for a certain class of elliptic equations were posed and investigated. Subsequently, in [8, 10] the problem formulated in [5] was called the Bitzadze–Samarsky problem, and methods were proposed for solving problems of this type for general elliptic equations. It should be noted that boundary value problems with nonlocal conditions also arise in the study of certain inverse problems (see [26, 27]).

From the point of view of physical applications, third-order partial differential equations are of great interest. These equations are considered when solving problems of the theory of nonlinear acoustics and in the hydrodynamic theory of cosmic plasma, modeling fluid filtration in porous media. Studies of wave propagation in cold plasma and magnetohydrodynamics also reduce to the partial differential equations of third order (see[6, 24, 32]). To the study of nonlocal boundary value problems (including integral conditions) for partial differential equations of the third order are devoted large number of works (see, for example, [4, 7, 18, 25, 35] and the bibliography therein).

It should be noted that boundary value problems with integral conditions are of particular interest. From physical considerations, the integral conditions are completely natural, and they arise in mathematical modeling in cases where it is impossible to obtain information about the process occurring at the boundary of the region of its flow using direct measurements or when it is possible to measure only some averaged (integral) characteristics of the desired quantity.

Problem (1.1)–(1.4) arises when studying the issues of fluid filtration in porous media [3, 24, 35], heat transfer in a heterogeneous medium [31, 36], moisture transfer in soil [6, 24], propagation of acoustic waves in a weakly inhomogeneous medium [32]. Note that in studying moisture transfer in soils, $u(x,t)$ means soil moisture in fractions of a unit at a depth of x at time t, and the integral in (1.2) is the moisture content in the active soil layer from 0 to 1.

The inverse boundary value problems for a partial differential equations in various formulation were considered in many papers (see, for example, [1, 2, 11–13, 19–23, 26–30, 33, 34, 37]). These problems for elliptic equations with non-local boundary conditions were investigated in [20, 22], for parabolic equations were investigated in [12, 13, 17, 23, 29, 30, 37], for hyperbolic equations were investigated in [1, 21]. The inverse boundary value problems for a partial differential equations of third order with integral condition were investigated only in [2, 23]. In [2], the initial conditions and the condition for redefining the desired function include integrals of the desired function over the time and spatial variables, and in [23], the integral of the desired function over the spatial variable is zero. In these works, when expanding the solution, the well-known systems of functions are used, which form bases in the space $L_2(0,1)$. Note that problem (1.1)–(1.4) is related to a second-order spectral problem with an integral condition, which is equivalent to a spectral problem with a spectral parameter in the boundary conditions. The system of eigenfunctions of this spectral problem forms a basis in $L_2(0,1)$ after removing any function.

The purpose of this paper is to prove the existence and uniqueness of the classical solution of the inverse boundary value problem for a partial differential equation (1.1) with nonlocal boundary conditions (1.2)–(1.4) (with integral boundary condition of the second kind (1.3).

The paper is organized as follows: In Section 2 the problem (1.1)–(1.4) reduces to an equivalent problem. In Section 3, we consider the spectral problem for an ordinary differential operator of second order with a spectral parameter in the boundary condition. We study the basis properties of the system of eigenfunctions of this problem, and using these properties help us obtain some inequalities that we need in the sequel. Here, we also introduced the necessary functional spaces. In Section 4 the existence and uniqueness of the solution of inverse problem (1.1)–(1.4) is proved by using the Fourier method.

2. Reduction of problem (1.1)–(1.4) to an equivalent problem

The classical solution of the inverse boundary value problem (1.1)–(1.4) is a pair $\{u(x,t),\, a(t)\}$ of functions $u(x,t)$ and $a(t)$ having the following properties:

A1. The function $u(x,t)$ is continuous in D_T with all its derivatives entering into equation (1.1) and boundary conditions (1.2) and (1.3);

A2. The function $a(t)$ is continuous on $[0,T]$;

A3. All conditions (1.1)–(1.4) are satisfied in the usual sense.

Consider the problem

$$y'''(t) = a(t)y(t),\ 0 \le t \le T, \tag{2.1}$$

$$y(0) = y'(0) = y''(T) = 0, \tag{2.2}$$

where $a(t) \in C[0,T]$.

Lemma 2.1 *Let the condition*

$$2a_0T^3/3 < 1 \tag{2.3}$$

holds. Then problem (2.1), (2.2) has only the trivial solution.

Proof Note that problem

$$y'''(t) = 0,\ 0 \le t \le T,$$

$$y(0) = y'(0) = y''(T) = 0,$$

has only a trivial solution. Then the boundary value problem (2.1), (2.2) is equivalent to the following integral equation

$$y(t) = \int_0^T K(t,\tau)a(\tau)y(\tau)d\tau, \tag{2.4}$$

where $K(x,t)$ is the Green's function of the differential expression $\ell(h) = h'''$ under the boundary conditions (2.2), which has the form

$$K(t,\tau) = \begin{cases} -\frac{t^2}{2} & \text{if } t \in [0,\tau], \\ -t\tau + \frac{\tau^2}{2} & \text{if } t \in [\tau,T]. \end{cases} \tag{2.5}$$

Denote:

$$Ay(t) = \int_0^T K(t,\tau)a(\tau)y(\tau)d\tau. \tag{2.6}$$

From the properties of the function $K(x,t)$ and the continuity of the function $a(t)$, it follows that the operator $A : C[0,T] \to C[0,T]$ is continuous. By (2.4), (2.6), problem (2.1), (2.2) can be written in the following equivalent form

$$y(t) = Ay(t),\ t \in [0,T]. \tag{2.7}$$

For any $y(t),\ v(t) \in C[0,T]$, we have

$$\begin{aligned} ||Av(t) - Ay(t)||_\infty \le ||a(t)||_\infty ||y(t) - v(t)||_\infty \max_{t\in[0,T]} | \int_0^T |K(t,\tau)|d\tau \le \\ \le \tfrac{2}{3}T^3 a_0 ||y(t) - v(t)||_\infty, \end{aligned} \tag{2.8}$$

since by virtue of (2.5)

$$\int_0^T |K(t,\tau)|d\tau = \int_0^t \left(t\tau + \frac{\tau^2}{2} + \left(Tt - \frac{t^2}{2} \right)\tau \right) d\tau + \int_t^T \frac{t^2}{2} d\tau = \frac{1}{6}t^3 + \frac{1}{2}Tt^2$$

and the function $g(t) = \frac{1}{6}t^3 + \frac{1}{2}Tt^2$ attains its maximum at the point $t = T$ and $g(T) = \frac{2}{3}T^3$. Hence, taking (2.3) into account, we find from (2.8) that the operator A is a contraction in the space $C[0,T]$. Therefore, in the space $C[0,T]$ the operator A has a unique fixed point. Thus, equation (2.7) has a unique solution in the space $C[0,T]$. Since $y \equiv 0$ is a solution of problem (2.1), (2.2), then it is unique. □

Along to the inverse boundary value problem (1.1)–(1.4), we consider the following auxiliary inverse boundary value problem: it is required to determine the pair $\{u(x,t),\ a(t)\}$ of functions $u(x,t)$ and $a(t)$ possessing properties A1 and A2, satisfying conditions (1.1)–(1.3) and the relation

$$h'''(t) + u_{xx}(0,t) = a(t)h(t) + f(0,t), 0 \le t \le T. \tag{2.9}$$

Lemma 2.2 *Let $\varphi_i(x) \in C[0,1]$, $i = 0,1,2$, $f(x,t) \in C(D_T)$, $h(t) \in C^3[0,T]$, $h(t) \in C^3[0,T]$ for $t \in [0,T]$ and the following matching conditions are satisfied:*

$$\varphi_0(0) = h(0), \varphi_1(0) = h'(0), \varphi_2(0) = h''(T). \tag{2.10}$$

Then the following assertions hold:

i) Every classical solution $\{u(x,t), a(t)\}$ of problem (1.1)–(1.4) is also a solution of the problem (1.1)–(1.3), (2.9);

ii) Every solution $\{u(x,t), a(t)\}$ of problem (1.1)–(1.3), (2.9) such that (2.3) holds is a classical solution of (1.1)–(1.4).

Proof Let $\{u(x,t), a(t)\}$ be the classical solution $\{u(x,t), a(t)\}$ of problem (1.1)–(1.4). Assuming $h(t) \in C^3[0,T]$ and differentiating (1.4) three times, we obtain

$$u'''_{t^3}(0,t) = h'''(t),\ \ 0 \le t \le T. \tag{2.11}$$

By virtue of (1.1), we have

$$u_{ttt}(0,t) + u_{xx}(0,t) = a(t)u(0,t) + f(0,t),\ \ 0 < t < T. \tag{2.12}$$

Taking into account (2.11) and (1.4), we obtain from (2.12) that (2.9) is satisfied.

Now let $\{u(x,t), a(t)\}$ be a solution of problem (1.1)–(1.3), (2.9) such that condition (2.3) is satisfied. Then it follows from (2.9) and (2.11) that

$$\frac{d}{dt^3}\left(u(0,t)-h(t)\right)=a(t)\left(u(0,t)-h(t)\right),0<t<T. \tag{2.13}$$

Next, by virtue of (1.2) and (2.10), we have

$$u(0,0)-h(0)=\varphi_0(0)-h(0)=0, \tag{2.14}$$

$$u_t(0,0)-h'(0)=\varphi_1(0)-h'(0)=0, \tag{2.15}$$

$$u_{tt}(0,T)-h''(T)=\varphi_2(0)-h''(T)=0, \tag{2.16}$$

From (2.13)–(2.16), by Lemma 2.1, we conclude that (1.4) is satisfied. □

3. Some properties of the corresponding spectral problems and the introduction of certain necessary spaces

We consider the following eigenvalue problem

$$y''(x)+\lambda y(x)=0,\ 0\le x\le 1, \tag{3.1}$$

$$y'(0)=0,\ y'(1)=d\lambda y(1), \tag{3.2}$$

where λ is a spectral parameter, d is a real constant such that $d>0$. It is easy to verify that problem (3.1), (3.2) has only the eigenfunctions

$$y_k(x)=\sqrt{2}\cos\sqrt{\lambda_k}x,\ k=0,\,1,\,\ldots,$$

with nonnegative eigenvalues λ_k, $k=0,\,1,\,\ldots$, determined by the equation

$$\tan\sqrt{\lambda}=-\sqrt{\lambda}\,.$$

By solving the homogeneous problem corresponding to problem (1.1)–(1.3) by the method of separation of variables, we arrive at the spectral problem for equation (3.1) with boundary conditions

$$y'(0)=0,\ dy(1)+\int_0^1 y(x)dx=0. \tag{3.3}$$

The system of eigenfunctions of this problem is the system $\{y_k(x)\}_{k=1}^{\infty}$, which is obtained from the system of eigenfunctions of problem (3.1), (3.2) without the function $y_0(x)$ corresponding to the eigenvalue λ_0.

Following the corresponding arguments in [14–16], we verify the validity of the following assertions.

Lemma 3.1 *The system* $\{z_k(x)\}_{k=1}^{\infty}$ *adjoint to the system* $\{y_k(x)\}_{k=1}^{\infty}$ *is given by the relation*

$$z_k(x) = \sqrt{2}\,\frac{\cos\sqrt{\lambda_k}x - \cos\sqrt{\lambda_k}}{1 + d\cos^2\sqrt{\lambda_k}}. \tag{3.4}$$

Theorem 3.2 *The system* $\{y_k(x)\}_{k=1}^{\infty}$ *forms a basis in the space* $L_p(0,1)$, $1 < p < \infty$ (*forms a Riesz basis in* $L_2(0,1)$).

Lemma 3.3 *If* $g(x) \in W_2^1(0,1)$ *and* $J(g) \equiv dg(1) + \int\limits_0^1 g(x)dx = 0$, *then*

$$\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\,|g_k|)^2\right)^{\frac{1}{2}} \le M\,\|g'(x)\|_{L_2}; \tag{3.5}$$

if $g(x) \in W_2^2(0,1), J(g) = 0, g'(0) = 0$, *then*

$$\left(\sum_{k=1}^{\infty}(\lambda_k\,|g_k|)^2\right)^{\frac{1}{2}} \le 2m_0\,|g'(1)| + \sqrt{2}M\,\|g''(x)\|_{L_2}; \tag{3.6}$$

and if $g(x) \in W_2^3(0,1), J(g) = 0,\ g'(0) = 0,\ g'(1) + dg''(1) = 0$, *then*

$$\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,|g_k|)^2\right)^{1/2} \le M\,\|g'''(x)\|_{L_2}, \tag{3.7}$$

where $M = \left(N(1+N) + 2 + \frac{1}{9d^2}\right)^{\frac{1}{2}}$, $m_0 = \frac{1}{d}\left(\sum\limits_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{\frac{1}{2}}$ *and* $||\cdot||_{L_2}$ *is the norm of* $L_2(0,1)$.

Denote by $B_{2,T}^{3/2}$ the set of all functions $u(x,t)$ of the form $u(x,t) = \sum\limits_{k=1}^{\infty} u_k(t)y_k(x)$, $(x,t) \in D_T$, such that $u_k(t) \in C[0,T]$ and the following relation holds:

$$\left\{\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]})^2\right\}^{1/2} < +\infty.$$

On this set, we define the norm as follows:

$$\|u(x,t)\|_{B_{2,T}^{3/2}} = \left\{\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{\infty}\right)^2\right\}^{\frac{1}{2}}. \tag{3.8}$$

It is obvious that $B_{2,T}^{3/2}$ is a Banach space with norm (3.8).

The function $u(x,t)$, as an element of the space $B_{2,T}^{3/2}$, in particular, has the following properties:

$$u(x,t),\ u_x(x,t),\ u_{xx}(x,t) \in C(D_T),\ u_{xxx}(x,t) \in C([0,T];L_2(0,1)),$$

$$u_x(0,t)=0,\ du(1,t)+\int\limits_0^1 u(x,t)dx=0,\ 0\le t\le T.$$

We denote by $E_T^{3/2}$ the Banach space consisting of the topological product $B_{2,T}^{3/2}\times C[0,T]$, equipped with the norm $||z||_{E_T^{3/2}}=\|u(x,t)\|_{B_{2,T}^{3/2}}+\|a(t)\|_\infty$, where $z=\{u,\,a\}$.

4. The existence and uniqueness of the classical solution of the inverse boundary value problem

The first component $u(x,t)$ of the solution $\{u(x,t),a(t)\}$ of problem (1.1)–(1.3), (2.9) will be sought in the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)y_k(x), \tag{4.1}$$

where

$$u_k(t)=\int\limits_0^1 u(x,t)z_k(x)dx,\ k=1,\,2,\,\ldots,\ y_k(x)=\sqrt{2}\cos\sqrt{\lambda_k}x,$$

$$z_k(x)=\frac{\sqrt{2}\left(\cos\sqrt{\lambda_k}x-\cos\sqrt{\lambda_k}\right)}{1+d\cos^2\sqrt{\lambda_k}}.$$

To determine the unknown functions $u_k(t),\ k=1,\,2,\,\ldots,$ applying the method of separation of variables from (1.1) and (1.2), we obtain

$$u_k'''(t)-\lambda_k u_k(t)=F_k(t;a,u),\ k=1,\,2,\,\ldots;\,0\le t\le T, \tag{4.2}$$

$$u_k(0)=\varphi_{0,\,k},\ u_k'(0)=\varphi_{1,\,k},\ u_k''(T)=\varphi_{2,\,k},\ k=1,\,2,\,\ldots, \tag{4.3}$$

where

$$F_k(t;u,a)=f_k(t)+a(t)u_k(t),\ f_k(t)=\int\limits_0^1 f(x,t)z_k(x)dx,$$

$$\varphi_{i,\,k}=\int\limits_0^1\varphi_i(x)z_k(x)dx,\ i=0,1,2,\ k=1,2,\ \ldots\ .$$

Solving problem (4.2), (4.3), we find that

$$u_k(t)=\left(\sqrt{3}e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T}+2\sqrt{3}\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T\right)^{-1}\left\{2\varphi_{0,\,k}\left[-e^{\lambda_k^{\frac{1}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}T-\frac{\pi}{3}\right)+\right.\right.$$

$$\left.e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}(3T-t)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}t-\frac{\pi}{6}\right)-e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}(T-t)-\frac{2\pi}{3}\right)\right]+$$

$$\frac{2}{\lambda_k^{\frac{1}{3}}}\varphi_{1k}\left[-e^{\lambda_k^{\frac{1}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}T-\frac{2\pi}{3}\right)+e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}(3T-t)}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}t+\right. \tag{4.4}$$

$$\left.e^{-\frac{1}{2}\lambda_k^{\frac{1}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}(T-t)-\frac{2\pi}{3}\right)\right]+\frac{\sqrt{3}}{\lambda_k^{\frac{1}{3}}}\varphi_{2,\,k}\left[e^{\lambda_k^{\frac{1}{3}}\left(\frac{T}{2}+t\right)}-\right.$$

$$\left.\left.2e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}(T-t)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}t-\frac{\pi}{3}\right)\right]\right\}+\int_0^T G_k(t,\tau)F_k(\tau;u,a)d\tau,$$

where

$$G_k(t,\tau)=\begin{cases}\alpha_k(T,t,\tau) & \text{if } t\in[0,\tau],\\ \beta_k(T,t,\tau) & \text{if } t\in[\tau,T],\end{cases}$$

$$\alpha_k(T,t,\tau)=-\frac{1}{3\lambda_k^{\frac{2}{3}}}\left(\sqrt{3}e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T}+2\sqrt{3}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}T\right)^{-1}$$

$$\left\{e^{\lambda_k^{\frac{1}{3}}\left(\frac{3}{2}T+t-\tau\right)}-2e^{\lambda_k^{\frac{1}{3}}\left(\frac{3}{2}T-\frac{t}{2}-\tau\right)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}t-\frac{\pi}{3}\right)+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}(T-\tau)\cdot\right.$$

$$\left.\left(e^{\lambda_k^{\frac{1}{3}}\left(t+\frac{\tau}{2}\right)}-2e^{-\frac{1}{2}\lambda_k^{\frac{1}{3}}(t-\tau)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}t-\frac{\pi}{3}\right)\right)\right\},$$

$$\beta_k(T,t,\tau)=-\frac{1}{3\lambda_k^{\frac{2}{3}}}\left(\sqrt{3}e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T}+2\sqrt{3}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}T\right)^{-1}$$

$$\left\{-2e^{\lambda_k^{\frac{1}{3}}\left(\frac{3}{2}T-\frac{t}{2}-\tau\right)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}t-\frac{\pi}{3}\right)+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}(T-\tau)\cdot\right.$$

$$\left(e^{\lambda_k^{\frac{1}{3}}\left(t+\frac{\tau}{2}\right)}-2e^{-\frac{1}{2}\lambda_k^{\frac{1}{3}}(t-\tau)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}t-\frac{\pi}{3}\right)\right)+$$

$$2e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}(3T-(t-\tau))}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}(t-\tau)+\frac{\pi}{6}\right)-$$

$$\left.-2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}T\left(e^{\lambda_k^{\frac{1}{3}}(t-\tau)}-2e^{-\frac{1}{2}\lambda_k^{\frac{1}{3}}(t-\tau)}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}(t-\tau)+\frac{\pi}{6}\right)\right)\right\}.$$

After substituting the expressions from (4.4) into (4.1), to determine the components of the classical solution of problem (1.1)–(1.3), (2.9), we obtain

$$u(x,t)=\sum_{k=1}^{\infty}\left\{\left(\sqrt{3}e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T}+2\sqrt{3}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}T\right)^{-1}\left\{2\varphi_{0,\,k}\left[-e^{\lambda_k^{\frac{1}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}T-\frac{\pi}{3}\right)+\right.\right.\right.$$

$$e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}(3T-t)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}t-\frac{\pi}{6}\right)-e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}(T-t)-\frac{2\pi}{3}\right)\Bigg]+$$

$$\frac{2}{\lambda_k^{\frac{1}{3}}}\varphi_{1k}\left[-e^{\lambda_k^{\frac{1}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}T-\frac{2\pi}{3}\right)+e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}(3T-t)}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}t+\right. \tag{4.5}$$

$$\left.e^{-\frac{1}{2}\lambda_k^{\frac{1}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}(T-t)-\frac{2\pi}{3}\right)\right]+\frac{\sqrt{3}}{\lambda_k^{\frac{1}{3}}}\varphi_{2k}\left[e^{\lambda_k^{\frac{1}{3}}\left(\frac{T}{2}+t\right)}-\right.$$

$$\left.\left.2e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}(T-t)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}t-\frac{\pi}{3}\right)\right]\right\}+\int_0^T G_k(t,\tau)F_k(\tau;u,a)d\tau\Bigg\}y_k(x).$$

Now taking into account (4.1), from (2.9) we find

$$a(t)=(h(t))^{-1}\left\{h'''(t)-f(0,t)-\sqrt{2}\sum_{k=1}^{\infty}\lambda_k u_k(t)\right\}. \tag{4.6}$$

In order to obtain the equation for the second component of the solution of problem (1.1)–(1.3), (2.9), substituting expression (4.4) in (4.6), we get

$$a(t)=h^{-1}(t)\left\{h''(t)-f(0,t)-\right.$$

$$-\sqrt{2}\sum_{k=1}^{\infty}\lambda_k\left[\left(\sqrt{3}e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T}+2\sqrt{3}\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T\right)^{-1}\left\{2\varphi_{0,k}\left[-e^{\lambda_k^{\frac{1}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}T-\frac{\pi}{3}\right)+\right.\right.\right.$$

$$\left.e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}(3T-t)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}t-\frac{\pi}{6}\right)-e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}(T-t)-\frac{2\pi}{3}\right)\right]+$$

$$\frac{2}{\lambda_k^{\frac{1}{3}}}\varphi_{1,k}\left[-e^{\lambda_k^{\frac{1}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}T-\frac{2\pi}{3}\right)+e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}(3T-t)}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}t+\right.$$

$$\left.e^{-\frac{1}{2}\lambda_k^{\frac{1}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}(T-t)-\frac{2\pi}{3}\right)\right]+\frac{\sqrt{3}}{\lambda_k^{\frac{1}{3}}}\varphi_{2,k}\left[e^{\lambda_k^{\frac{1}{3}}\left(\frac{T}{2}+t\right)}-\right. \tag{4.7}$$

$$\left.\left.\left.\left.-2e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}(T-t)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}t-\frac{\pi}{3}\right)\right]\right\}+\int_0^T G_k(t,\tau)F_k(\tau;u,a)d\tau\right]\right\}$$

Thus, the solution of problem (1.1)–(1.3), (2.9) is reduced to the solution of (4.5), (4.7) with respect to the unknown functions $u(x,t)$ and $a(t)$.

To study the uniqueness problem for the solution of problem (1.1)–(1.3), (2.9), the following lemma plays an important role.

Lemma 4.1 *If* $\{u(x,t),a(t)\}$ *is any solution of problem (1.1)–(1.3), (2.9), then the functions* $u_k(t)=\int\limits_0^1 u(x,t)z_k(x)dx,\ k=1,2,\ \ldots$ *, satisfy on* $[0,T]$ *system (4.4).*

Proof Let $\{u(x,t),a(t)\}$ be any solution of problem (1.1)–(1.3), (2.9). Then it is obvious that

$$2\int\limits_0^1 u_{ttt}(x,t)z_k(x)dx=\frac{d^3}{dt^3}\int\limits_0^1 u(x,t)z_k(x)dx=u_k'''(t),\ k=1,2,\ \ldots. \tag{4.8}$$

Next, using twice the formula integration of parts and taking into account the boundary condition (1.3), we obtain

$$\begin{gathered}\int\limits_0^1 u_{xx}(x,t)z_k(x)dx=\frac{\sqrt{2}}{\alpha_k}\int\limits_0^1 u_{xx}(x,t)\left(\cos\sqrt{\lambda_k}x-\cos\sqrt{\lambda_k}\right)dx=\\ -\frac{\sqrt{2}}{\alpha_k}\left(d\lambda_k u(1,t)\cos\sqrt{\lambda_k}+\lambda_k\int\limits_0^1 u_{xx}(x,t)\cos\sqrt{\lambda_k}xdx\right)=\\ -\frac{\sqrt{2}}{\alpha_k}\left[\left(\lambda_k du(1,t)+\int\limits_0^1 u(x,t)dx\right)\ \cos\sqrt{\lambda_k}+\right.\\ \left.\lambda_k\int\limits_0^1 u(x,t)\left(\cos(\sqrt{\lambda_k}x)-\cos\sqrt{\lambda_k}\right)dx\right]=-\lambda_k u_k(t),\end{gathered} \tag{4.9}$$

where $\alpha_k=1+d\cos^2\sqrt{\lambda_k}>1$.

Now, multiplying both sides of equation (1.1) by the function $z_k(x)$, $k=1,2,\ \ldots$, integrating the resulting equality over x in the range from 0 to 1 and using (4.8), (4.9) we obtain (4.2).

Similarly, from (1.1) we obtain that the conditions (4.3) are satisfied.

Thus, $u_k(t)$, $k=1,2,\ \ldots$ is a solution of problem (4.2), (4.3); therefore, the functions $u_k(t)$, $k=1,2,\ \ldots$ satisfy system (4.4) on $[0,T]$. □

Corollary 4.2 *Let system (4.5), (4.7) have a unique solution. Then problem (1.1)–(1.3), (2.9), can not have more than one solution, i.e. if problem (1.1)–(1.3), (2.9), has a solution, then it is unique.*

We consider in the space $E_T^{3/2}$ the operator Φ defined by

$$\Phi(u,a)=\{\Phi_1(u,a),\Phi_2(u,a)\}$$

where

$$\Phi_1(u,a)=\tilde{u}(x,t)\equiv\sum_{k=1}^{\infty}\tilde{u}_k(t)\sin\lambda_k x,\ \ \Phi_2(u,a)=\tilde{a}\,(t)$$

and $\tilde{u}_k(t)$, $k=1,2,\ \ldots$, and $a(t)$ are equal to the right-hand sides of (3.4) and (4.3), respectively.

It follows from the inequality

$$\cos x \geq -\frac{1}{4}e^{x} \text{ for } x \geq 0$$

that

$$e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T} + 2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{1}{3}}T \geq \frac{1}{2}e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T}. \tag{4.10}$$

By virtue of inequality (4.10), we have

$$\frac{ch\rho_k t}{ch2\rho_k T + \cos 2\rho_k T} \leq 2,\ \frac{ch\rho_k(2T-t)}{ch2\rho_k T + \cos 2\rho_k T} \leq 2,\ 0 \leq t \leq T,$$

$$\frac{e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}t}}{e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T} + 2\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T} \leq 2,\ \frac{e^{-\frac{1}{2}\lambda_k^{\frac{1}{3}}t}}{e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T} + 2\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T} \leq 2,\ 0 \leq t \leq T,$$

$$\frac{e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}(T-t)}}{e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T} + 2\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T} \leq 2,\ \frac{e^{\lambda_k^{\frac{1}{3}}\left(\frac{T}{2}+t\right)}}{e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T} + 2\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T} \leq 2,\ 0 \leq t \leq T,$$

$$\frac{e^{\lambda_k^{\frac{1}{3}}\left(\frac{3}{2}T+t-\tau\right)}}{e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T} + 2\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T} \leq 2,\ \frac{e^{\lambda_k^{\frac{1}{3}}\left(\frac{3}{2}T-\frac{t}{2}-\tau\right)}}{e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T} + 2\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T} \leq 2,\ 0 \leq t \leq \tau \leq T,$$

$$\frac{e^{\lambda_k^{\frac{1}{3}}\left(t+\frac{\tau}{2}\right)}}{e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T} + 2\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T} \leq 2,\ \frac{e^{-\frac{1}{2}\lambda_k^{\frac{1}{3}}(t-\tau)}}{e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T} + 2\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T} \leq 2,\ 0 \leq t \leq \tau \leq T,$$

$$\frac{e^{\lambda_k^{\frac{1}{3}}\left(\frac{3}{2}T-\frac{t}{2}-\tau\right)}}{e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T} + 2\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T} \leq 2,\ \frac{e^{\lambda_k^{\frac{1}{3}}\left(t+\frac{\tau}{2}\right)}}{e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T} + 2\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T} \leq 2,\ 0 \leq \tau \leq t \leq T,$$

$$\frac{e^{\lambda_k^{\frac{1}{3}}\left(\frac{3}{2}T-\frac{t}{2}-\tau\right)}}{e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T} + 2\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T} \leq 2,\ \frac{e^{-\frac{1}{2}\lambda_k^{\frac{1}{3}}(t-\tau)}}{e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T} + 2\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T} \leq 2,\ 0 \leq \tau \leq t \leq T,$$

$$\frac{e^{\frac{1}{2}\lambda_k^{\frac{1}{3}}(3T-(t-\tau))}}{e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T} + 2\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T} \leq 2,\ \frac{e^{\lambda_k^{\frac{1}{3}}(t-\tau)}}{e^{\frac{3}{2}\lambda_k^{\frac{1}{3}}T} + 2\cos\lambda_k^{\frac{1}{3}}\frac{\sqrt{3}}{2}T} \leq 2,\ 0 \leq \tau \leq t \leq T.$$

Taking these relations into account, we find that

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\left\|\tilde{u}_k(t)\right\|_{\infty}\right)^2\right)^{\frac{1}{2}} \leq 12\sqrt{2}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\left|\varphi_{0,k}\right|\right)^2\right)^{\frac{1}{2}} +$$

$$12\sqrt{2}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\left|\varphi_{1,k}\right|\right)^2\right)^{\frac{1}{2}} + 6\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k\left|\varphi_{2k}\right|\right)^2\right)^{\frac{1}{2}} + \tag{4.11}$$

$$16\sqrt{5}T\left(\int\limits_0^T\sum_{k=1}^{\infty}(\lambda_k\left|f_k(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+16\sqrt{5}T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\left\|u_k(t)\right\|_{\infty}\right)^2\right)^{\frac{1}{2}},$$

$$\left\|\tilde{a}(t)\right\|_{\infty}\le\left\|[h(t)]^{-1}\right\|_{\infty}\{\left\|h'''(t)-f(0,t)\right\|_{\infty}+\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left[4\sqrt{3}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\left|\varphi_{0k}\right|)^2\right)^{\frac{1}{2}}+\right.$$

$$4\sqrt{3}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\left|\varphi_{1,k}\right|\right)^2\right)^{\frac{1}{2}}+6\left(\sum_{k=1}^{\infty}(\lambda_k\left|\varphi_{2,k}\right|)^2\right)^{\frac{1}{2}}+ \tag{4.12}$$

$$\left.16\sqrt{T}\left(\int\limits_0^T\sum_{k=1}^{\infty}(\lambda_k\left|f_k(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+16T\left\|a(t)\right\|_{\infty}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\left\|u_k(t)\right\|_{\infty}\right)^2\right)^{\frac{1}{2}}\right]\Bigg\}.$$

Suppose that the data of problem (1.1)–(1.3), (2.9) satisfy the following conditions:

(i) $\varphi_i(x)\in C^2[0,1],\varphi_i'''(x)\in L_2(0,1),d\varphi_i(1)+\int\limits_0^1\varphi_i(x)dx=0,\varphi_i'(0)=0,$
$\varphi_i'(1)+d\varphi_i''(1)=0,i=0,1;$

(ii) $\varphi_2(x)\in C^1[0,1],\varphi_2''(x)\in L_2(0,1),d\varphi_2(1)+\int\limits_0^1\varphi_2(x)dx=0,\varphi_2'(0)=0;$

(iii) $f(x,t),f_x(x,t)\in C(D_T),\ f_{xx}(x,t)\in L_2(D_T),\ df(1,t)+\int\limits_0^1 f(x,t)dx=0,$
$f_x(1,t)=0,\ 0\le t\le T,$

(iv) $h(t)\in C^3[0,T],h(t)\neq 0,0\le t\le T$.

Then, taking into account (3.6) and (3.7), from (4.11) and (4.12), we obtain, respectively,

$$\left\|\tilde{u}(x,t)\right\|_{B_{2,T}^{3/2}}\le A_1(T)+B_1(T)\left\|a(t)\right\|_{\infty}\left\|u(x,t)\right\|_{B_{2,T}^{3/2}}, \tag{4.13}$$

$$\left\|\tilde{a}(t)\right\|_{\infty}\le A_1(T)+B_2(T)\left\|a(t)\right\|_{\infty}\left\|u(x,t)\right\|_{B_{2,T}^{3/2}}, \tag{4.14}$$

where

$$A_1(T)=12\sqrt{2}M\left\|\varphi_0'''(x)\right\|_{L_2}+12\sqrt{2}M\left\|\varphi_1'''(x)\right\|_{L_2}+6\sqrt{5}\ \ (2m_0\left|\varphi_2''(1)\right|+$$

$$\left.+\sqrt{2}M\left\|\varphi_2''(x)\right\|_{L_2}\right)+16\sqrt{5T}\left(2m_0\left\|f_x(1,t)\right\|_{\infty}+\sqrt{2}M\left\|f_{xx}(x,t)\right\|_{L_2(D_T)}\right),$$

$$B_1(T)=16\sqrt{5}T,$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h'''(t)-f(0,t)\right\|_{C[0,T]}+\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{1/2}\left[4\sqrt{3}M\left\|\varphi_0'''(x)\right\|_{L_2(0,1)}+\right.\right.$$

$$4\sqrt{3}M \left\|\varphi_1'''(x)\right\|_{L_2(0,1)} + 16\left(2m_0 \left|\varphi_2'(1)\right| + \sqrt{2}M \left\|\varphi_2''(x)\right\|_{L_2}\right) +$$

$$+ 16\sqrt{T}\left(m_0 \left\|f_x(1,t)\right\|_\infty + \sqrt{2}M \left\|f_{xx}(x,t)\right\|_{L_2(D_T)}\right)\Big]\Big\},$$

$$B_2(T) = 16\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)\left\|(h(t))^{-1}\right\|_\infty T.$$

From the inequalities (4.13) and (4.14), we conclude that

$$\left\|\tilde{u}(x,t)\right\|_{B_{2,T}^{3/2}} + \left\|a(t)\right\|_\infty \le A(T) + B(T)\left\|a(t)\right\|_\infty \left\|u(x,t)\right\|_{B_{2,T}^{\frac{3}{2}}}, \tag{4.15}$$

where $A(T) = A_1(T) + A_2(T)$, $B(T) = B_1(T) + B_2(T)$.

We denote by B_R the closed ball in the space $E_T^{3/2}$ of radius R centered at the origin.

Theorem 4.3 *Let conditions* (i)-(iv) *and the following condition be satisfied:*

$$(A(T) + 2)^2 B(T) < 1. \tag{4.16}$$

Then problem (1.1)–(1.3), (2.9), has a unique solution in the ball $B_R \subset E_T^{3/2}$*, where* $R = A(T) + 2$.

Proof In the space $E_T^{3/2}$ we consider equation

$$z = \Phi z, \tag{4.17}$$

where $z = \{u, a\}$, the components Φ_1 and Φ_2 of the operator Φ are defined by the right-hand sides of equations (4.5) and (4.7), respectively.

We consider the operator Φ in the ball $B_R \subset E_T^{3/2}$. Analogously to (4.15), we obtain that for any $z = \{u, a\}, z_1 = \{u_1, a_1\}, z_2 = \{u_2, a_2\} \in B_R$ are valid assessments

$$\left\|\Phi z\right\|_{E_T^{3/2}} \le A(T) + B(T)\left\|a(t)\right\|_\infty \left\|u(x,t)\right\|_{B_{2,T}^{3/2}}, \tag{4.18}$$

$$\left\|\Phi z_1 - \Phi z_2\right\|_{E_T^{3/2}} \le 2B(T)R\left(\left\|a_1(t) - a_2(t)\right\|_\infty + \left\|u_1(x,t) - u_2(x,t)\right\|_{B_{2,T}^{3/2}}\right). \tag{4.19}$$

Then, taking into account (4.16), from estimates (4.18), (4.19) we obtain that the operator Φ acts on the ball B_R and is contractive. Consequently, in the ball B_R the operator Φ has a unique fixed point $\{u, a\}$, which is a solution of equation (4.17), i.e. is a unique solution of system (4.5), (4.7) in the ball B_R.

The function $u(x,t)$ as an element of the space $B_{2,T}^{3/2}$ is continuous and has continuous derivatives of $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T.

Taking (3.5) into account from (4.2), we obtain

$$\left(\sum_{k=1}^{\infty}\left(\sqrt{\lambda_k}\left\|u_k'''(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le \sqrt{3}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left\{\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\left\|u_k(t)\right\|_\infty\right)^2\right)^{\frac{1}{2}} +\right.$$

$$+ \left\| \|f_x(x,t)\|_\infty \right\|_{L_2} + \|a(t)\|_\infty \|u(x,t)\|_{B^{\frac{3}{2}}_{2,T}} \Bigg\},$$

which implies that $u_{ttt}(x,t)$ is continuous in D_T.

It is easy to verify that equation (1.1) and conditions (1.2), (1.3) and (2.9) are satisfied in the usual sense. Consequently, $\{u(x,t), a(t)\}$ is a solution of problem (1.1)–(1.3), (2.9). By Corollary 4.2, this solution is unique. □

Using Lemma 2.2 and Theorem 4.3, we can prove the following

Theorem 4.4 *Let all the conditions of Theorem 4.3, the condition*

$$\frac{2}{3}\left(A(T)+2\right)T^3 < 1$$

and the matching conditions

$$\varphi_0(0) = h(0), \varphi_1(0) = h'(0), \varphi_2(0) = h''(T)$$

are satisfied. Then problem (1.1)–(1.4) has a unique classical solution in the ball $B_R \subset E_T^{3/2}$.

5. Conclusion

In this paper, investigated the inverse boundary value problem for a partial differential equation of third order with an integral condition. This problem arises when studying the issues of fluid filtration in porous media, heat transfer in a heterogeneous medium , moisture transfer in soil , propagation of acoustic waves in a weakly inhomogeneous medium. Using analytical and operator-theoretic methods, as well as the method of separation of variables, the existence and uniqueness of the classical solution of this problem is proved.

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