Normalized null hypersurfaces in nonflat Lorentzian space forms satisfying $L_r x = \mathcal{U}x + b$

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Abstract: In the present work, we classify normalized null hypersurfaces $x : (M, g, N) \to Q_1^{n+2}(c)$ immersed into one of the two real standard nonflat Lorentzian space-forms and satisfying the equation $L_r x = \mathcal{U}x + b$ for some field of screen constant matrices $\mathcal{U}$ and some field of screen constant vectors $b \in \mathbb{R}^{n+2}$, where L_r is the linearized operator of the $(r+1)-$mean curvature of the normalized null hypersurface for $r = 0, ..., n$. We show that if the immersion x is a solution of the equation $L_r x = \mathcal{U}x + b$ for $1 \le r \le n$ and the normalization N is quasi-conformal, then M is either an $(r+1)-$maximal null hypersurface, or a totally umbilical (or geodesic) null hypersurface or an almost isoparametric normalized null hypersurface with at most two non-zero principal curvatures. We also show that a null hypersurface M, of a real standard semi-Riemannian nonflat space form $Q_t^{n+2}(c)$, admits a totally umbilical screen distribution (and then M is totally umbilical or totally geodesic) if and only if M is a section of $Q_t^{n+2}(c)$ by a hyperplane of $\mathbb{R}^{n+3}$. In particular a null hypersurface $M \to Q_t^{n+2}(c)$ is totally geodesic if and only if M is a section of $Q_t^{n+2}(c)$ by a hyperplane of $\mathbb{R}^{n+3}$ passing through the origin.

Key words: Normalized null hypersurface, screen quasi-conformal, second order differential operator, screen distribution, Higher order mean curvature

1. Introduction and main results

1.1. Literature review

The last four decades have been marked by a growing interest in the study of submanifolds whose coordinate functions are eigenfunctions of the Laplacian operator and its various generalizations. In [32], Takahashi studied isometric immersions into the Euclidean space whose coordinate functions are eigenfunctions of the Laplacian operator, associated to the same eigenvalue λ. In the particular case where the codimension is one, Takahashi's theorem establishes that if $x : M^n \to \mathbb{R}^{n+1}$ is an (isometrically) immersed hypersurface in the real Euclidean space and Δ denotes its Laplacian operator with respect to the induced metric, then the immersion satisfies $\Delta x = \lambda x$, for some real constant λ, if and only if either $\lambda = 0$ and M is minimal in $\mathbb{R}^{n+1}$ or $\lambda > 0$ and M is an open piece of a round hypersphere of radius $r = \sqrt{n/\lambda}$ centered at the origin of $\mathbb{R}^{n+1}$. In 1990, Garay [22] has established an extension of Takahashi's theorem and proved that if $x : M^n \to \mathbb{R}^{n+1}$ is an immersed hypersurface in Euclidean space with $\Delta x = Ax$, where $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is a constant diagonal matrix then,

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M is minimal hypersurface in $\mathbb{R}^{n+1}$ or an open piece of a round hypersphere or an open piece of generalized right spherical cylinders.

However, Garay's condition is not coordinate-invariant. This issue was remedied by Dillen et al. in [17], where they considered surfaces in $\mathbb{R}^3$ whose immersion satisfies the condition $\Delta x = Ax + b$, where $A \in \mathbb{R}^{3 \times 3}$ is a constant matrix and $b \in \mathbb{R}^3$ is a constant vector, proving that the only such surfaces are minimal surfaces and open pieces of round spheres and circular cylinders. Later on, the result of Dillen et al. has been independently generalized to hypersurfaces in $\mathbb{R}^{n+1}$, using different techniques, by Hasanis and Vlachos [24], and Chen and Petrovic [16].

It is well known that the Laplacian operator of a hypersurface M immersed into $\mathbb{R}^{n+1}$ is a (intrinsic) second-order linear differential operator which arises naturally as the linearized operator of the first variation of the mean curvature for normal variations of the hypersurface. From this point of view, the Laplacian operator Δ can be seen as the first one of a sequence of n operators $L_0 = \Delta, L_1, ..., L_{n-1}$, where L_k stands for the linearized operator of the first variation of the $(k+1)th-$ mean curvature arising from normal variations of the hypersurface. These operators are given for their action on smooth function f on M by $L_k(f) = tr(P_k \circ \nabla^2 f)$, where P_k denotes the $k-$th Newton transformation associated to the second fundamental form of the hypersurface and $\nabla^2 f$ denotes the selfadjoint linear operator metrically equivalent to the Hessian of f. Alías and Gürbüz [3] initiated the study of non-degenerate hypersurfaces in the Lorentz–Minkowski space $\mathbb{R}_1^{n+1}$ satisfying the general condition $L_k x = Ax + b$, where $A \in \mathbb{R}^{(n+1) \times (n+1)}$ is a constant matrix and $b \in \mathbb{R}^{n+1}$ is a constant vector. A first attempt to solve this question has been made by Yang and Liu [33]. They showed that a spacelike hypersurface $x : M \to \mathbb{R}_1^{n+1}$ satisfies the condition $L_r x = Rx + b$, for some constant matrix $R \in \mathbb{R}^{(n+1) \times (n+1)}$ and some constant vector $b \in \mathbb{R}_1^{n+1}$ if and only if M is an $r-$maximal spacelike hypersurface, an open piece of the totally umbilical hypersurface $\mathbb{H}^n(-c)$, or an open piece of hypersurface $\mathbb{H}^k(-c) \times \mathbb{R}^{n-k}$, with $r+1 \leq k \leq n-1$, where $c > 0$ is a constant.

Many other authors solved the equation $L_k x = Ax + b$ for nondegenerate hypersurfaces in flat and nonflat spaces [27–29]. But the case of degenerate (or lightlike or null) hypersurfaces has started to be considered in [8] where the authors showed that: A UCC-normalized null hypersurface $x : (M, g, N) \to \mathbb{R}_1^{n+2}$ satisfies the equation $L_r x = Ax + b$ if and only if (M, g, N) is either an $(r+1)-$maximal null hypersurface or an almost isoparametric normalized null hypersurface with $N = \mathcal{U}_0 x + b_0$, for some field of screen constant matrix $\mathcal{U}_0 \in \mathbb{R}^{(n+2) \times (n+2)}$ and field of screen constant vector $b_0 \in \mathbb{R}^{n+2}$.

1.2. Main results

A rigging or a normalization for a null hypersurface M is a null vector field N defined over M and everywhere transversal to M [21]. The triplet (M, g, N), g being the restriction of the ambient metric on M, is then called a normalized null hypersurface. Orthogonal projection on M being impossible, to study the extrinsic geometry of M geometrical object of the ambient manifold are projected on M parallely to N. Therefore, many properties regarding the extrinsic geometry of M should depend on N, that is on the direction used for projections. That is why it is often necessary to choose a specific normalization or class of normalizations, unless the property in investigation does not depend on the normalization as it is the case for maximality, total umbilicity and total geodesibility.

In the present work, a classification result similar to the one in [8] is established when the ambient

manifold is one the the two real standard nonflat Lorentzian space forms $Q_1^{n+2}(c)$, $c = \pm 1$. As recall, $Q_1^{n+2}(c)$ is either the de Sitter space-time $\mathbb{S}^{n+2}$ when $c = +1$ or the antide Sitter space-time $\mathbb{H}^{n+2}$ when $c = -1$. In fact, we prove the following.

Theorem 1.1 *Let $x : (M, g, N) \to Q_1^{n+2}(c)$ be a normalized null hypersurface isometrically immersed into a real standard Lorentzian nonflat space-form $Q_1^{n+2}(c)$ and furnished with a closed normalization N such that $A_N - \overset{\star}{A}_\xi = \varphi_1 P + \varphi_2 \overset{\star}{A}_\xi$ with $\varphi_1 \neq 0$ or $\varphi_2 \neq -1/2$ and $\tau \equiv 0$. Let L_r be the linearized operator of the $(r+1)-$ mean curvature of M, for some fixed $r = 1, ..., n$. If the immersion x satisfies the equation $L_r x = \mathcal{U} x$, for some field of screen constant matrix $\mathcal{U} \in \mathbb{R}^{(n+3) \times (n+3)}$, then M is either an $(r + 1)-$ maximal null hypersurface, or a section of $Q(c)_1^{n+2}$ with a hyperplane of $\mathbb{R}^{n+3}$ or an almost isoparametric normalized null hypersurface with exactly two nonzero principal curvatures.*

An almost isoparametric normalized null hypersurface is a normalized null hypersurface whose the screen principal curvatures are constant along the screen distribution. To prove the above theorem we first have to classify null hypersurfaces admitting totally umbilical screen distribution in a real standard nonflat space-forms.

Theorem 1.2 *Let $(M, g, \mathscr{S}(N))$ be a null hypersurface of a semi-Riemannian nonflat space form $Q_t^{n+2}(c)$, endow with a screen distribution $\mathscr{S}(N)$.*

The screen distribution is (proper) totally umbilical (and M is totally umbilical or geodesic) if and only if M is a section of $Q_t^{n+2}(c)$ by a hyperplane of $\mathbb{R}^{n+3}$.

M is totally geodesic if and only if M is a section of $Q_t^{n+2}(c)$ by a hyperplane of $\mathbb{R}^{n+3}$ passing through the origin.

The associated metric introduced in [7] is used to define the gradient of smooth functions on M since the induced metric g is degenerate and cannot be used to this ends. The above theorem will also be used in the prove of the following one. A corollary of this following theorem is that the sections of $Q_1^{n+2}(c)$ by hyperplanes of $\mathbb{R}^{n+3}$ passing through the origin are the only null hypersurfaces of $Q_1^{n+2}(c)$ whose coordinate functions are eigenfunctions of the Laplacian operator Δ^η.

Theorem 1.3 *Let $\lambda \in \mathbb{R}$ and $b \in \mathbb{R}^{n+3}$. If a normalized null hypersurface $x : (M, g, N) \to Q_1^{n+2}(c)$ satisfies $\Delta^\eta x = \lambda x + b$ then M is an open piece of a section of $Q_1^{n+2}(c)$ by a hyperplane of $\mathbb{R}^{n+3}$ passing through the origin.*

The rest of the paper is organized as follows: In Section 2 we recall some necessary preliminaries on null hypersurfaces of the standard real nonflat space-forms. Section 3 is concerned with the classification of null hypersurfaces admitting totally umbilical screen distribution in a standard real nonflat space-forms. In Section 4, we recall the definitions of (almost) isoparametric normalized null hypersurface intoduced in [8]. In Section 5 we solve the equation $\Delta x = \lambda x + b$ for $\lambda \in \mathbb{R}$, $b \in \mathbb{R}^{n+3}$ and where the unknown x is a normalized null hypersurface immersed in a nonflat space-form. In Section 6 we introduce the second-order differential operators L_r. Finally in the last section, we solve the equation $L_r x = Ax + b$ for null hypersurfaces in a real standard Lorentzian nonflat space-forms.

2. Necessary background materials on null hypersurfaces of $Q_t^{n+2}(c)$

Let $\mathbb{R}_t^{n+3}$ be the $(n+3)-$dimensional pseudo-Euclidean space of index $t \geq 1$, whose metric tensor $\langle , \rangle$ is given by

$$\langle , \rangle = -\sum_{i=1}^{t} dx_i^2 + \sum_{j=t+1}^{n+3} dx_j^2,$$

where $(x_1, ..., x_{n+3})$ is the cartesian coordinates in $\mathbb{R}^{n+3}$. Recall that, the de Sitter space of index t and radius $r > 0$ is defined by

$$\mathbb{S}_t^{n+2}(r) := \{x \in \mathbb{R}_t^{n+3}; \langle x, x \rangle = r^2\},$$

and the Anti-de Sitter space of index t and radius $-r < 0$ is defined by

$$\mathbb{H}_t^{n+2}(-r) := \{x \in \mathbb{R}_{t+1}^{n+3}; \langle x, x \rangle = -r^2\}.$$

$\mathbb{S}_t^{n+2}(r)$, $\mathbb{R}_t^{n+2}$ and $\mathbb{H}_t^{n+2}(-r)$ are the real standard $(n+2)-$dimensional semi-Riemannian space forms of index t and constant sectional curvature $1/r > 0$, 0 and $-1/r < 0$ respectively.

Throughout this paper, we will consider the case of null hypersurfaces immersed into the pseudo-sphere of index t and radius 1, $\mathbb{S}_t^{n+2} := \mathbb{S}_t^{n+2}(1)$ (which is the de Sitter space-time when $t = 1$), and the ones immersed into the pseudo-hyperbolic space of index t and radius -1, $\mathbb{H}_t^{n+2} := \mathbb{H}_t^{n+2}(-1)$ (which is the antide Sitter space-time when $t = 1$).

We denote by $Q_t^{n+2}(c)$ the space $\mathbb{S}_t^{n+2}$ or $\mathbb{H}_t^{n+2}$, according to $c = 1$ or $c = -1$ (in Section 4 we shall also consider the case $c = 0$, i.e. $\mathbb{R}_t^{n+2}$). We denote by $\mathbb{R}_q^{n+3}$, the corresponding pseudo-Euclidean space where lives $Q_t^{n+2}(c)$. Hence, $q = t$ if $c = 1$ and $q = t+1$ if $c = -1$, and the metric of $\mathbb{R}_q^{n+3}$ can be written as

$$\langle , \rangle = -\sum_{a=0}^{t} dx_a^2 + c dx_{t+1}^2 + \sum_{b=t+2}^{n+2} dx_b^2.$$

Let $x : (M^{n+1}, g) \to Q_t^{n+2}(c)$ be a connected null hypersurface, $g := x^*\langle , \rangle$ being the restriction of the ambient metric on M. Then the normal bundle $TM^\perp$ is a rank 1 vector subbundle of the tangent bundle TM. Let $\mathscr{S}(N)$ be a subbundle of TM such that

$$TM = \mathscr{S}(N) \oplus_{Orth} TM^\perp, \tag{2.1}$$

where $\oplus_{Orth}$ denotes a orthogonal sum. Such a subbundle is called a screen distribution. From [19], it is known that there exists a unique rank 1 vector subbundle $tr(TM)$ of $TQ_t^{n+2}(c)$ over M, such that for any nonzero section ξ of $TM^\perp$ on a coordinate neighborhood $\mathscr{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathscr{U}$ satisfying

$$\langle N, \xi \rangle = 1, \quad \langle N, N \rangle = \langle N, W \rangle = 0, \qquad \forall \, W \in \mathscr{S}(N)|_{\mathscr{U}}. \tag{2.2}$$

Then $TQ_t^{n+2}(c)$ splits as

$$TQ_t^{n+2}(c)|_M \quad = \quad TM \oplus tr(TM) = \{TM^\perp \oplus tr(TM)\} \oplus_{Orth} \mathscr{S}(N). \tag{2.3}$$

We call $tr(TM)$ a (null) transversal vector bundle along M. Conversely, the choice of a null transversal vector field N along M determines uniquely a screen distribution $\mathscr{S}(N) := TM \cap N^{\perp}$, a null transversal vector bundle $tr(TM) := Span(N)$, and a section ξ of $TM^{\perp}$ such that (2.2) and (2.3) hold. See [21] for more detail.

Let $\theta = \langle N, \cdot \rangle$ be the $1-$form metrically equivalent to N and $\eta = x^{\star}\theta$ its restriction to M. The normalized null hypersurface (M, g, N) will be said to be closed if the $1-$form η is closed. It is easy to check that $\mathscr{S}(N) = \ker(\eta)$ and that the screen distribution $\mathscr{S}(N)$ is integrable whenever η is closed. On a normalized null hypersurface (M, g, N), the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B^N(X, Y)N, \tag{2.4}$$

$$\overline{\nabla}_X N = -A_N X + \tau^N(X)N, \tag{2.5}$$

$$\nabla_X PY = \overset{\star}{\nabla}_X PY + C^N(X, PY)\xi, \tag{2.6}$$

$$\nabla_X \xi = -\overset{\star}{A}_\xi X - \tau^N(X)\xi, \tag{2.7}$$

for all $X, Y \in \Gamma(TM)$, where $\overline{\nabla}$ denotes the Levi–Civita connection on $Q_t^{n+2}(c)$, ∇ denotes the connection on M induced from $\overline{\nabla}$ through the projection along the rigging N and $\overset{\star}{\nabla}$ denotes the connection on the screen distribution $\mathscr{S}(N)$ induced from ∇ through the projection morphism P from $\Gamma(TM)$ onto $\Gamma(\mathscr{S}(N))$ with respect to the decomposition (2.1). Now the $(0, 2)$ tensors B^N and C^N are the second fundamental forms on TM and $\mathscr{S}(N)$ respectively, A_N and $\overset{\star}{A}_\xi$ are the shape operators on TM and $\mathscr{S}(N)$ respectively and the rotation $1-$form τ^N a $1-$form on TM defined by

$$\tau^N(X) = \langle \overline{\nabla}_X N, \xi \rangle.$$

For the second fundamental forms B^N and C^N the following holds

$$B^N(X, Y) = g(\overset{\star}{A}_\xi X, Y), \quad C^N(X, PY) = g(A_N X, Y) \quad \forall X, Y \in \Gamma(TM), \tag{2.8}$$

and

$$B^N(X, \xi) = 0, \quad \overset{\star}{A}_\xi \xi = 0. \tag{2.9}$$

It follows from (2.9) that integral curves of ξ are pregeodesics in both $\overline{M}$ and M, as $\overline{\nabla}_\xi \xi = \nabla_\xi \xi = -\tau^N(\xi)\xi$.

The induced connection ∇ is torsion-free, but not necessarily $g-$metric unless M be totally geodesic. In fact we have for all tangent vector fields X, Y and Z in TM,

$$(\nabla_X g)(Y, Z) = B^N(X, Y)\eta(Z) + B^N(X, Z)\eta(Y). \tag{2.10}$$

Also, due to the degeneracy of the induced metric g it cannot be used to do differential calculus on M, especially to define the gradient of a smooth function. We define on M the metric g_η by

$$g_\eta = g + \eta \otimes \eta.$$

Clearly, g_η defines a nondegenerate metric on M which plays an important role in defining the usual differential operators gradient, divergence, Laplacian with respect to degenerate metric g on null hypersurfaces (see [7] for

details). It is called the associated metric to g on the normalized null hypersurface (M, g, N). The following verifications are straightforward,

$$g_\eta(\xi, X) = \eta(X), \quad g_\eta(X, Y) = g(X, Y) \ \ \forall X \in \Gamma(TM), \ \forall Y \in \Gamma(\mathscr{S}(N)). \tag{2.11}$$

In particular $g_\eta(\xi, \xi) = 1$ and the last equality in (2.11) is telling us that restrict to $\mathscr{S}(N)$ the metrics g_η and g coincide.

For every $x \in M$,

$$\mathbb{R}^{n+3} = (\{span\{\xi_x\} \oplus span\{N_x\}\} \oplus_{Orth} \mathscr{S}(N_x)) \oplus_{Orth} span\{x\}. \tag{2.12}$$

Setting $\overset{\circ}{\nabla}$ the flat connection of $\mathbb{R}_q^{n+3}$, we can write the Gauss formula of the immersion $x : (M, g, N) \to Q_t^{n+2}(c) \to \mathbb{R}_q^{n+3}$ as

$$\overset{\circ}{\nabla}_X Y = \overset{\star}{\nabla}_X Y + C(X, Y)\xi + B(X, Y)N - \langle X, Y \rangle cx, \tag{2.13}$$

for every $X, Y \in \Gamma(\mathscr{S}(N))$.

Let us denote by $\overline{R}$ and R the Riemann curvature tensors of $\overline{\nabla}$ and ∇, respectively. Then the following are the Gauss–Codazzi equations [19, p. 93].

$$(\nabla_X B^N)(Y, Z) + \tau^N(X)B^N(Y, Z) = (\nabla_Y B^N)(X, Z) + \tau^N(Y)B^N(X, Z), \tag{2.14}$$

$$c(\langle Y, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle) = \Big\langle R(X, Y)Z, PW \Big\rangle$$
$$+ B^N(X, Z)C^N(Y, PW) - B^N(Y, Z)C^N(X, PW), \tag{2.15}$$

$$C^N(Y, \overset{\star}{A}_\xi X) - C^N(X, \overset{\star}{A}_\xi Y) = 2d\tau^N(X, Y), \quad \forall X, Y, Z, W \in \Gamma(TM|_{\mathscr{U}}), \tag{2.16}$$

$$c(\langle Y, Z \rangle \eta(X) - \langle X, Z \rangle \eta(Y)) = \Big\langle (\nabla_X A_N)Y, PZ \Big\rangle - \Big\langle (\nabla_Y A_N)X, PZ \Big\rangle$$
$$+ \tau^N(Y)\Big\langle A_N X, PZ \Big\rangle - \tau^N(X)\Big\langle A_N Y, PZ \Big\rangle, \tag{2.17}$$

M is said to be totally umbilical (resp. totally geodesic) if there exists a smooth function ρ on M such that $B^N = \rho g$ (resp. B^N vanishes identically on M). These are intrinsic notions on any null hypersurface in the sense that they do not depend on the choice of the normalization. The total umbilicity and the total geodesibility conditions for M can also be written respectively as $\overset{\star}{A}_\xi = \rho P$ and $\overset{\star}{A}_\xi = 0$. Also, the screen distribution $\mathscr{S}(N)$ is totally umbilical (resp. totally geodesic) if $C^N(U, PV) = \lambda g(U, V)$ for all $U, V \in \Gamma(TM)$ (resp. $C^N = 0$), which is equivalent to $A_N = \lambda P$ (resp. $A_N = 0$).

Lemma 2.1 *[19] Let (M, g, N) be a totally umbilical rigged null hypersurface of a $(n+2)-$dimensional pseudo-Riemannian space-form. Then ρ from the above definition satisfies*

$$\xi(\rho) + \rho\tau^N(\xi) - \rho^2 \ = \ 0 \tag{2.18}$$
$$PU(\rho) + \rho\tau^N(PU) \ = \ 0, \tag{2.19}$$

for all $U \in \Gamma(TM)$.

As $\overset{\star}{A}_\xi$ is selfadjoint linear operator on each fiber T_pM with $\overset{\star}{A}_\xi \xi = 0$ and the screen structure $\mathscr{S}(N)$ is Riemannian, $\overset{\star}{A}_\xi$ is diagonalizable and have $(n+1)$ real-valued eigenfunctions $\overset{\star}{k}_0 = 0$, $\overset{\star}{k}_1, ..., \overset{\star}{k}_n$. These are the screen principal curvatures of the null hypersurface (i.e with respect to the shape operator $\overset{\star}{A}_\xi$). Let $\left(\overset{\star}{E}_0 = \xi, \overset{\star}{E}_1, \ldots, \overset{\star}{E}_n \right)$ be a quasi-orthonormal basis of eigenvectors fields.

The $r - th$ mean curvature of the null hypersurface with respect to the shape operator $\overset{\star}{A}_\xi$ is given by

$$\overset{\star}{H}_r = \binom{n+1}{r}^{-1} \sigma_r(\overset{\star}{k}_0, ..., \overset{\star}{k}_n) \quad \text{and} \quad \overset{\star}{H}_0 = 1 \text{ (constant function 1)},$$

where σ_r is the $r - th$ elementary symmetric polynomial. We set $\overset{\star}{S}_r = \sigma_r(\overset{\star}{k}_0, ..., \overset{\star}{k}_n)$ and $\overset{\star}{S}_r^{\,\alpha} = \sigma_r(\overset{\star}{k}_0, ..., \overset{\star}{k}_{\alpha-1} , \overset{\star}{k}_{\alpha+1}, ..., \overset{\star}{k}_n)$.

Definition 2.2 ($r-$maximality) *Let $1 \le r \le n+1$ be an integer. A null hypersurface M with $\overset{\star}{H}_r = 0$ is said to be $r-$maximal.*

For $0 \le r \le n+1$, the $r - th$ Newton transformation $\overset{\star}{T}_r$ with respect to the shape operator $\overset{\star}{A}_\xi$ is the End$(\Gamma(TM))$ element given by

$$\overset{\star}{T}_r = \sum_{a=0}^{r} (-1)^a \, \overset{\star}{S}_a \overset{\star}{A}_\xi^{\,r-a} .$$

Inductively,

$$\overset{\star}{T}_0 = I \quad \text{and} \quad \overset{\star}{T}_r = (-1)^r \overset{\star}{S}_r I + \overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1},$$

where I denotes the identity of $\Gamma(TM)$ and $\overset{\star}{T}_{n+1} = 0$ (from Cayley–Hamilton theorem). By algebraic computations, one shows the following.

Proposition 2.3 ([9])   *1. $\overset{\star}{T}_r$ is selfadjoint and commute with $\overset{\star}{A}_\xi$;*

2. $\overset{\star}{T}_r \overset{\star}{E}_\alpha = (-1)^r \, \overset{\star}{S}_r^{\,\alpha} \overset{\star}{E}_\alpha$;

3. $tr(\overset{\star}{T}_r) = (-1)^r (n+1-r) \, \overset{\star}{S}_r$;

4. $tr\left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1} \right) = (-1)^{r-1} r \, \overset{\star}{S}_r$;

5. $tr\left(\overset{\star}{A}_\xi^{\,2} \circ \overset{\star}{T}_{r-1} \right) = (-1)^{r-1} \left(\overset{\star}{S}_1 \overset{\star}{S}_r - (r+1) \, \overset{\star}{S}_{r+1} \right)$;

6. $tr(\overset{\star}{T}_{r-1} \circ \nabla_X \overset{\star}{A}_\xi) = (-1)^{r-1} X(\overset{\star}{S}_r)$.

The following two lemmas are also proven in [9], the second being derived from the first one by taking $r = 1$.

Lemma 2.4 ([9]) *Let* $x : (M, g, N) \to \omega(c)^{n+2}$ *be a normalized null hypersurface of a semi-Riemannian manifold with constant sectional curvature. Then for* $r = 1, ..., n+1$,

$$(-1)^{r-1}\xi(\overset{\star}{S}_r) + \tau(\xi)tr(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}) - tr(\overset{\star}{A}_\xi^2 \circ \overset{\star}{T}_{r-1}) = 0. \tag{2.20}$$

Lemma 2.5 ([9]) *A null hypersurface* M *of a semi-Riemannian manifold with constant sectional curvature is maximal if and only if* M *is totally geodesic.*

Let $(X_0 = \xi, X_1, ..., X_n)$ be a local $g_\eta-$orthonormal basis of $\Gamma(TM)$ with $\text{span}\{X_1, ..., X_n\} = \mathscr{S}(N)$. The divergence of the operator $\overset{\star}{T}_r$ is the vector field $div^\nabla \left(\overset{\star}{T}_r\right) \in \Gamma(TM)$ define as the trace of the $End(TM)-$valued operator $\nabla \overset{\star}{T}_r$ and given by

$$div^\nabla \left(\overset{\star}{T}_r\right) = tr(\nabla \overset{\star}{T}_r) = \sum_{a,b=0}^{n} g_\eta^{ab}(\nabla \overset{\star}{T}_r)(X_a, X_b) = \sum_{a=0}^{n} (\nabla_{X_a} \overset{\star}{T}_r)X_a. \tag{2.21}$$

3. Totally umbilical and totally geodesic null hypersurfaces in nonflat space-forms $Q_t^{n+2}(c)$

3.1. Sections of $Q_t^{n+2}(c)$ by hyperplanes of $\mathbb{R}_q^{n+3}$

Let T be a hyperplane of $\mathbb{R}^{n+3}$. Then, T can be defined by one of the two (normalized) equations

$$\langle a, x \rangle = c \tag{3.1}$$
$$\langle a, x \rangle = 0, \tag{3.2}$$

where $a \in \mathbb{R}^{n+3}$ is a constant vector. Let M be the section of $Q_t^{n+2}(c)$ by the hyperplane T.

Lemma 3.1 M *is a null hypersurface of* $Q_t^{n+2}(c)$ *if and only if* T *can be defined by Equation* (3.1) *for some* $a \in Q_t^{n+2}(c)$ *or by Equation* (3.2) *for some constant null vector* a.

Proof Let a be constant vector and $f : Q_t^{n+2}(c) \to \mathbb{R}$, $x \mapsto \langle a, x \rangle$. First, we suppose that $M = \{x \in Q_t^{n+2}(c); f(x) = \langle a, x \rangle = c\}$. $\forall X \in \Gamma(TM)$,

$$df(X) = \langle a, X \rangle = \langle a^\top, X \rangle,$$

where $a^\top = a - c\langle a, x \rangle x = a - x$ on M, is the orthogonal projection of a on TQ. Hence, $\overline{\nabla}f = a^\top = a - x$ and $\langle \overline{\nabla}f(x), \overline{\nabla}f(x) \rangle = \langle a, a \rangle - c$ for all $x \in M$. Finally, M is a null hypersurface if and only if $\overline{\nabla}f$ is a null vector field on M, if and only if $\langle a, a \rangle = c$ if and only if $a \in Q_t^{n+2}(c)$.

Secondly, if $M = \{x \in Q_t^{n+2}(c); f(x) = \langle a, x \rangle = 0\}$ then, $a^\top = a$ on M and $\langle \overline{\nabla}f(x), \overline{\nabla}f(x) \rangle = \langle a, a \rangle$ for all $x \in M$. Then in this case, M is a null hypersurface if and only if a is a constant null vector. □

If T is defined by an equation of type (3.2) and a is a constant null vector then by Lemma 3.1, M is a null hypersurface and the vector field $\xi := a$ spanned the radical distribution, and $\forall X \in \Gamma(TM)$, $\nabla_X \xi = 0$, thus M is a totally geodesic null hypersurface of $Q_t^{n+2}(c)$.
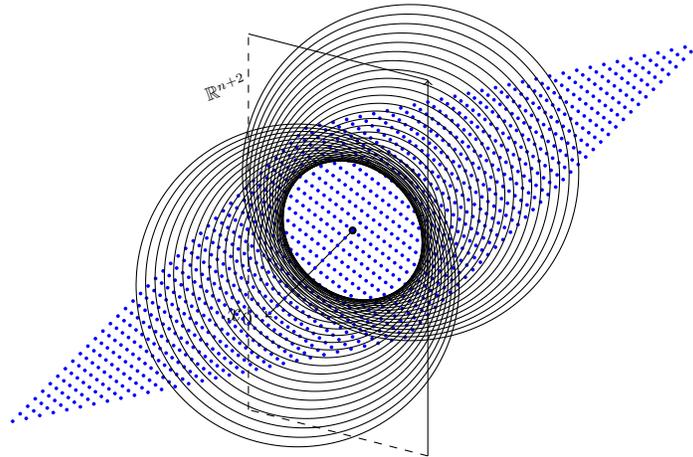
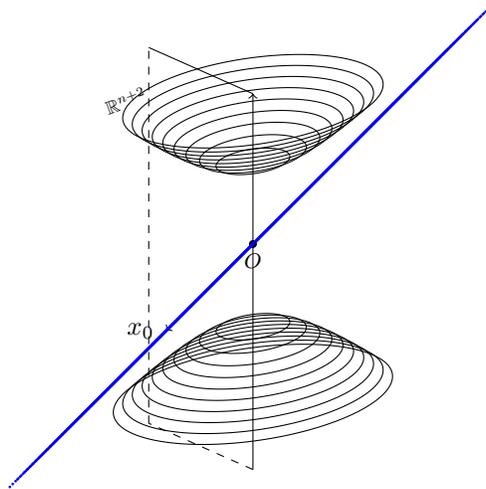**Figure 1**. Section of $\mathbb{S}_1^{n+2}$ by the hyperplane $T : -\sqrt{n+2}x_0 + x_1 + \cdots + x_{n+2} = 0$.

Figure 2. Section of $\mathbb{H}_1^{n+2}$ by the hyperplane $S : -\sqrt{n}x_0 - x_1 + \cdots + x_{n+2} = 0$.

Example 3.2 *The section of $\mathbb{S}_1^{n+2}$ (resp. $\mathbb{H}_1^{n+2}$) by the hyperplane $T : -\sqrt{n+2}x_0 + x_1 + \cdots + x_{n+2} = 0$, see Figure 1, (resp. $S : -\sqrt{n}x_0 - x_1 + \cdots + x_{n+2} = 0$, see Figure 2) is a totally geodesic null hypersurface.*

If T is defined by an equation of type (3.1) and $a \in Q_t^{n+2}(c)$ then by Lemma 3.1, M is a null hypersurface. We consider on M the normalizing pair

$$
\begin{aligned}
N &= \frac{1}{2}\left(\frac{1 + ca_0^2}{(a_0 - x_0)^2} - c \right)(a - x) - \frac{1}{a_0 - x_0}\partial_{x_0} - \frac{x_0}{a_0 - x_0}ca, \\
\xi &= a - x.
\end{aligned}
$$

The shape operators are then given by

$$
\overset{\star}{A}_\xi(X) = PX, \quad A_N(X) = \frac{1}{2}\left(\frac{1 + ca_0^2}{(a_0 - x_0)^2} - c \right)PX \quad \forall X \in \Gamma(TM).
$$

Thus, the screen is totally umbilical and M is a totally umbilical null hypersurface.

3.2. Proof of Theorem 1.2

Lemma 3.3 *Let (M, g, N) be a normalized null hypersurface of a semi-Riemannian space form $Q_t^{n+2}(c)$. If A_N is symmetric and commute with $\overset{\star}{A}_\xi$ or M is totally geodesic then, there exists a (re-)normalization $\widetilde{N}$ such that $\tau^{\widetilde{N}} \equiv 0$.*

Proof Since A_N is symmetric and commute with $\overset{\star}{A}_\xi$ or M is totally geodesic, Equation (2.16) shows that the $1-$form τ^N is closed and by the Poincaré lemma, it is locally exact. Thus, there exists a smooth function f defined at least locally and such that $\tau^N = -df$. Set $\varphi = e^f$ and consider the (re-)normalization $\widetilde{N} = \varphi N$ (which conserve the screen distribution), one has by using Lemma 2.1 in [6],

$$\tau^{\widetilde{N}} = \tau^N + d\ln|\varphi| = \tau^N + df = 0.$$

$\square$

Proof [Proof of Theorem 1.2] If the screen distribution is totally umbilical then A_N it is symmetric and commute with $\overset{\star}{A}_\xi$. Hence, A_N is symmetric and commute with $\overset{\star}{A}_\xi$ or M is totally geodesic and by Lemma 3.3, there exists a normalization N for which $\tau^N \equiv 0$. Let ρ be the umbilicity factor of $\overset{\star}{A}_\xi$ (with $\rho = 0$ in the case where M is totally geodesic) and $(\rho\eta)^\sharp$ the gradient of the $1-$form $\rho\eta$ with respect to any non degenerate metric on M (for example the associated metric g_η). Now, consider the (re-)normalization $\widetilde{N} = e^{(\rho\eta)^\sharp} N$, one has $\tau^{\widetilde{N}} = \rho\eta = \rho e^{-(\rho\eta)^\sharp} \widetilde{\eta}$. By using Lemma 2.1, one obtains

$$\xi\left(\rho e^{(-\rho\eta)^\sharp}\right) = 0 \text{ and } PU\left(\rho e^{-(\rho\eta)^\sharp}\right) = 0 \quad \forall U \in \Gamma(TM).$$

Then, $\rho e^{-(\rho\eta)^\sharp}$ is constant on M. Now, setting $a = e^{-(\rho\eta)^\sharp}\xi + \rho e^{-(\rho\eta)^\sharp} x$ a direct calculation gives $\overset{o}{\nabla}_U a = 0$ (where $\overset{o}{\nabla}$ is the flat connection of $\mathbb{R}_q^{n+3}$). Then, a is a constant vector on M which can be extended to a constant vector on $\mathbb{R}_q^{n+3}$. Finally,

$$\forall x \in M, \langle a, x \rangle = \rho e^{-(\rho\eta)^\sharp} c.$$

That proves the direct sense of the two items and the converse was shown in subsection 3.1. $\square$

4. Almost isoparametric and isoparametric normalized null hypersurfaces

A nondegenerate hypersurface M in a real space-form $Q(c)$ of constant sectional curvature c is said to be isoparametric if it has constant principal curvatures. An isoparametric hypersurface M in $\mathbb{R}^n$ can have at most two different principal curvatures, and M must be an open subset of a hyperplane, hypersphere or a spherical cylinder $\mathbb{S}^k \times \mathbb{R}^{n-k-1}$. This was shown by Levi–Civita [26] for $n = 3$ and by Segre [31] for arbitrary n. Similarly, Cartan [12] proved that an isoparametric hypersurface M in a hyperbolic space $\mathbb{H}^n$ can have at most two different principal curvatures, and M must be either totally umbilical or else be an open subset of a standard product $\mathbb{S}^k \times \mathbb{H}^{n-k-1}$ in $\mathbb{H}^n$.

Definition 4.1   **a)** *A normalized null hypersurface* (M, g, N) *is said to be isoparametric if the screen principal curvatures (eigenfunctions of* $\overset{\star}{A}_\xi$*) are constants.*

  b) *A normalized null hypersurface* (M, g, N) *is said to be almost isoparametric if the screen distribution* $\mathscr{S}(N)$ *is integrable and all the screen principal curvatures are constants on each leaf of* $\mathscr{S}(N)$*.*

Example 4.2 *Every totally geodesic null hypersurface is isoparametric.*

Let us recall the following result which gives Cartan's identity for null hypersurfaces.

Theorem 4.3 *[10] Let* (M, g, N) *be a lightlike hypersurface of an* $(n+2)-$ *dimensional Lorentzian space-form* $Q_t^{n+2}(c)$ *with* $\tau = 0$*. Assume that* $E_0 = \xi, E_1, ..., E_n$ *are eigenfunctions of* $\overset{\star}{A}_\xi$ *satisfying* $\overset{\star}{A}_\xi E_i = \lambda_i E_i$ *(*$i \geq 1$*) and* λ_i *is constant along* $\mathscr{S}(N)$*. Then for every* $i \geq 1$*,*

$$\sum_{j=1, \lambda_j \neq \lambda_i} \frac{c + \lambda_j g(A_N E_i, E_i) + \lambda_i g(A_N E_j, E_j)}{\lambda_i - \lambda_j} = 0.$$

Lemma 4.4 *Let* $x : (M, g, N) \to Q_1^{n+2}(c)$ *be an almost isoparametric normalized null hypersurface isommetrically immersed into a Lorentzian space-form with constant sectional curvature* $c \leq 0$*. If there exists a conformal screen (re-)normalization with vanishing* $1-$ *form* τ*, then* M *has at most two different screen principal curvatures. In particular when* $c = 0$*,* M *has at most one nonzero screen principal curvature and when* $c < 0$*,* M *has exactly two or no nonzero screen principal curvatures.*

Proof   Let $x : (M, g, N) \to Q_1^{n+2}(c)$ be a almost isoparametric normalized null hypersurface with conformal screen distribution $(A_N = \varphi \overset{\star}{A}_\xi)$ and $\tau = 0$. Let $\lambda_1, ..., \lambda_p$ be all distinct screen principal curvatures of the sharp operator $\overset{\star}{A}_\xi$, with algebraic multiplicities $\nu_1, ..., \nu_p$. By the previous theorem for any $i = 1, ..., p$ Cartan's identity can be written as

$$\sum_{j=1, j \neq i} \nu_j \frac{c + 2\varphi \lambda_j \lambda_i}{\lambda_i - \lambda_j} = 0. \tag{4.1}$$

Without loss of generalities, we may assume $\lambda_1 < \lambda_2 < \cdots < \lambda_p$, and $\lambda_p \geq 0$. Choose the largest nonnegative λ_i such that $2\varphi \lambda_i \lambda_{i-1} \leq c$. Then

$$\frac{c + 2\varphi \lambda_j \lambda_i}{\lambda_i - \lambda_j} \leq 0$$

for any $j \neq i$. Hence $2\varphi \lambda_i \lambda_j = c$ if $i \neq j$. Therefore $p \leq 2$. □

5. Normalized null hypersurfaces $x : (M, g, N) \to Q_1^{n+2}(c),\ c \neq 0$ **satisfying** $\Delta^\eta x = \lambda x + b,\ \lambda \in \mathbb{R}, b \in \mathbb{R}^{n+3}$

Let $x : (M, g, N) \to Q_1^{n+2}(c) \subset \mathbb{R}_q^{n+3}$ be a normalized null hypersurface immersed into a nonflat Lorentzian space-form $Q_1^{n+2}(c)$ and let $a \in \mathbb{R}_q^{n+3}$ be a fixed vector. We can write

$$a = a^\top + \langle a^x, \xi \rangle N + c \langle a, x \rangle x,$$

where a^x is the orthogonal projection of a on $TQ_1^{n+2}(c)$ and $a^\top$ is the projection of a^x on TM with respect to decomposition (2.1). Then, $\langle x, a \rangle \in \mathcal{C}^\infty(M)$ and $\forall U \in \Gamma(TM)$ using the fact that $\langle a^x, \xi \rangle = \langle a, \xi \rangle$ one has

$$g_\eta(\nabla^\eta \langle x, a \rangle, U) = U \cdot \langle x, a \rangle = \langle U, a \rangle = g_\eta(Pa^\top + \langle \xi, a \rangle \xi, U).$$

Thus,

$$\nabla^\eta \langle x, a \rangle = Pa^\top + \langle \xi, a \rangle \xi = a - \langle N, a \rangle \xi - \langle \xi, a \rangle N + \langle \xi, a \rangle \xi - \langle \xi, a \rangle x, \tag{5.1}$$

and for all $U \in \Gamma(TM)$, by using (2.6), (2.7), we get

$$
\begin{aligned}
\nabla_U \nabla^\eta \langle x, a \rangle &= \langle A_N(U) - \overset{\star}{A}_\xi(U) - 2\tau(U)\xi, a \rangle \xi + \langle N, a \rangle \overset{\star}{A}_\xi(U) \\
&\quad + (A_N(U) - \overset{\star}{A}_\xi(U))\langle \xi, a \rangle - c\langle x, a \rangle U.
\end{aligned} \tag{5.2}
$$

It follows that

$$
\begin{aligned}
\Delta^\eta \langle x, a \rangle &= trace(\nabla \nabla^\eta \langle x, a \rangle) \\
&= \langle A_N \xi, a \rangle + (S_1 - \overset{\star}{S}_1 - 2\tau(\xi))\langle \xi, a \rangle + \overset{\star}{S}_1 \langle N, a \rangle - c(n+1)\langle x, a \rangle.
\end{aligned} \tag{5.3}
$$

We extend Δ^η on $\otimes^{n+3}\mathcal{C}^\infty(M)$ by

$$\Delta^\eta(f_0, ..., f_{n+2}) = (\Delta^\eta f_0, ..., \Delta^\eta f_{n+2}), \tag{5.4}$$

for all $(f_0, ..., f_{n+2}) \in \otimes^{n+3}\mathcal{C}^\infty(M)$. Let $(e_0, ..., e_{n+2})$ be the standard orthonormal basis of $\mathbb{R}_q^{n+3}$. Then by using (5.3) and (5.4),

$$
\begin{aligned}
\Delta^\eta x &= (\epsilon_0 \Delta^\eta \langle x, e_0 \rangle, ..., \epsilon_{n+1}\Delta^\eta \langle x, e_{n+1} \rangle_1) \\
&= A_N \xi + (S_1 - \overset{\star}{S}_1 - 2\tau(\xi))\xi + \overset{\star}{S}_1 N - c(n+1)x,
\end{aligned} \tag{5.5}
$$

where $\epsilon_A = \langle e_A, e_A \rangle = \pm 1$. We say that a normalized null hypersurface (M, g, N) is pseudo-harmonic (of first kind) provided $\Delta^\eta x = 0$.

Lemma 5.1 *Pseudo-harmonic normalized null hypersurface does not exist in de Sitter and antide Sitter spaces.*

Proof As $A_N \xi \in \mathscr{S}(N)$ and ξ, N and x are linearly independent, Equation (5.5) show that $\Delta^\eta x$ cannot vanish. $\qquad\square$

Proof [Proof of Theorem 1.3] Suppose that x satisfies $\Delta^\eta x = \lambda x + b$ then,

$$A_N(\xi) + (S_1 - \overset{\star}{S}_1 - 2\tau(\xi))\xi + \overset{\star}{S}_1 N - c(n+1)x = \lambda x + b. \tag{5.6}$$

Taking covariant derivative of (5.6) by ξ and using Gauss–Weingarten equations one obtains

$$
\begin{aligned}
\lambda \xi = &\overset{\star}{\nabla}_\xi A_N \xi - \overset{\star}{S}_1 A_N \xi + (\xi \cdot \overset{\star}{S}_1 + \tau(\xi) \overset{\star}{S}_1)N - c\overset{\star}{S}_1 x \\
&+ \Big[C(\xi, A_N(\xi)) + \xi \cdot S_1 - \xi \cdot \overset{\star}{S}_1 - 2\xi \cdot \tau(\xi) \\
&- S_1 \tau(\xi) + \overset{\star}{S}_1 \tau(\xi) + 2\tau(\xi)^2 - c(n+1) \Big]\xi.
\end{aligned} \tag{5.7}
$$

Hence,

$$
\begin{cases}
\overset{\star}{\nabla}_\xi A_N(\xi) = \overset{\star}{S}_1 A_N(\xi) \\
\xi \cdot \overset{\star}{S}_1 + \tau(\xi)\, \overset{\star}{S}_1 = 0 \overset{(2.20)}{\Longrightarrow} tr\left(\overset{\star}{A}_\xi^{\,2}\right) = 0 \\
\overset{\star}{S}_1 = 0 \\
C(\xi, A_N(\xi)) + \xi \cdot S_1 - \xi \cdot \overset{\star}{S}_1 - 2\xi \cdot \tau(\xi) - S_1 \tau(\xi) + \overset{\star}{S}_1 \tau(\xi) + 2\tau(\xi)^2 - c(n+1) = \lambda
\end{cases}
.
$$

Thus, M is maximal and by Lemma 2.5, M is totally geodesic and by Theorem 1.2, M is an open piece of a section of $Q_1^{n+2}(c)$ with hyperplane of $\mathbb{R}^{n+3}$ passing through the origin. $\qquad \square$

6. The second-order differential operators L_r

We assume now that the normalization N is closed with $1-$form τ^N such that $\tau^N|_{\mathscr{S}(N)} \equiv 0$. Then $A_N \xi = 0$ and

$$
C(\xi, X) = 0, \ \forall Y \in \Gamma(\mathscr{S}(N)). \tag{6.1}
$$

Let $(X_0 = \xi, X_1, ..., X_n)$ be a $g_\eta-$orthonormal basis of $\Gamma(TM)$ with $\mathrm{span}\{X_1, ..., X_n\} = \mathscr{S}(N)$. Let f be a smooth function on M. By a direct calculating,

$$
\left(\nabla_\xi \overset{\star}{T}_r\right)\xi = (-1)^r \xi\left(\overset{\star}{S}_r\right)\xi. \tag{6.2}
$$

By definition,

$$
\begin{aligned}
div^\nabla(\overset{\star}{T}_r \nabla^\eta f) &= trace(\nabla \overset{\star}{T}_r \nabla^\eta f) \\
&= \sum_{i=1}^n \left\{ \left\langle \left(\nabla_{X_i} \overset{\star}{T}_r\right)\nabla^\eta f, X_i \right\rangle + \left\langle \overset{\star}{T}_r \nabla_{X_i}\nabla^\eta f, X_i \right\rangle \right\} \\
&\quad + \eta(\nabla_\xi \overset{\star}{T}_r \nabla^\eta f).
\end{aligned} \tag{6.3}
$$

For each i,

$$
\begin{aligned}
\left\langle \left(\nabla_{X_i} \overset{\star}{T}_r\right)\nabla^\eta f, X_i \right\rangle &= \left\langle \left(\nabla_{X_i} \overset{\star}{T}_r\right) X_i, \nabla^\eta f \right\rangle - \eta(\overset{\star}{T}_r \nabla^\eta f)B(X_i, X_i) \\
&\quad + \eta(\nabla^\eta f)B(\overset{\star}{T}_r X_i, X_i) \\
&= \left\langle \left(\nabla_{X_i} \overset{\star}{T}_r\right) X_i, \nabla^\eta f \right\rangle - (-1)^r \overset{\star}{S}_r \eta(\nabla^\eta f)B(X_i, X_i) \\
&\quad + \eta(\nabla^\eta f)B(\overset{\star}{T}_r X_i, X_i) \\
&= \left\langle \left(\nabla_{X_i} \overset{\star}{T}_r\right) X_i, \nabla^\eta f \right\rangle + \eta(\nabla^\eta f)\left\langle \overset{\star}{A}_\xi^{\,2} \circ \overset{\star}{T}_r X_i, X_i \right\rangle.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
div^\nabla(\overset{\star}{T}_r \nabla^\eta f) &= \left\langle \nabla^\eta f, div^\nabla(\overset{\star}{T}_r) \right\rangle + \eta(\nabla^\eta f)tr(\overset{\star}{A}^{\,2} \circ \overset{\star}{T}_{r-1}) + tr(\overset{\star}{T}_r \circ \nabla_\eta^2 f) \\
&\quad - \eta(\overset{\star}{T}_r \nabla_\xi \nabla^\eta f) + \eta(\nabla_\xi \overset{\star}{T}_r \nabla^\eta f)
\end{aligned} \tag{6.4}
$$

By using (2.6) and (6.1),

$$\eta(\nabla_\xi \overset{*}{T}_r \nabla^\eta f) = (-1)^r \eta(\nabla_\xi \overset{*}{S}_r \nabla^\eta f) + C(\xi, \overset{*}{T}_{r-1} \circ \overset{*}{A}_\xi \nabla^\eta f)$$

$$\eta(\nabla_\xi \overset{*}{T}_r \nabla^\eta f) = (-1)^r \eta(\nabla^\eta f)\xi(\overset{*}{S}_r) + (-1)^r \overset{*}{S}_r \eta(\nabla_\xi \nabla^\eta f), \tag{6.5}$$

and,

$$\eta(\overset{*}{T}_r \nabla_\xi \nabla^\eta f) = (-1)^r \overset{*}{S}_r \eta(\nabla_\xi \nabla^\eta f). \tag{6.6}$$

Replace (6.5) and (6.6) in (6.4) we obtain

$$\begin{aligned} div^\nabla(\overset{*}{T}_r \nabla^\eta f) &= \left\langle \nabla^\eta f, div^\nabla(\overset{*}{T}_r) \right\rangle + tr(\overset{*}{T}_r \circ \nabla_\eta^2 f) \\ &+ \eta(\nabla^\eta f)\left((-1)^r \xi(\overset{*}{S}_r) + tr(\overset{*}{A}^2 \circ \overset{*}{T}_{r-1})\right). \end{aligned} \tag{6.7}$$

Thanks to [2] and since the ambient has constant sectional curvature, the divergence $div^\nabla(\overset{*}{T}_r)$ is $TM^\perp-$valued and $(-1)^r \xi(\overset{*}{S}_r) + tr(\overset{*}{A}_\xi^2 \circ \overset{*}{T}_{r-1}) = \tau^N(\xi)tr(\overset{*}{A}_\xi \circ \overset{*}{T}_{r-1})$. Hence (6.7) becomes

$$div^\nabla(\overset{*}{T}_r \nabla^\eta f) = tr(\overset{*}{T}_r \circ \nabla_\eta^2 f) + \eta(\nabla^\eta f)\tau^N(\xi)tr(\overset{*}{A}_\xi \circ \overset{*}{T}_{r-1}). \tag{6.8}$$

Let r be an integer with $0 \le r \le n$. We define the second-order differential operator $L_r : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ for a normalized null hypersurface in a Lorentzian space-form by

$$L_r f = tr(\overset{*}{T}_r \circ \nabla_\eta^2 f) + \eta(\nabla^\eta f)\tau^N(\xi)tr(\overset{*}{A}_\xi \circ \overset{*}{T}_{r-1}), \quad \forall f \in \mathcal{C}^\infty(M). \tag{6.9}$$

It is easy to check that L_o is nothing but the first kind pseudo-Laplacian operator Δ^η and that L_r satisfies for $f, g \in \mathcal{C}^\infty(M)$,

$$L_r(fg) = fL_r g + gL_r f + 2\left\langle \overset{*}{T}_r \nabla^\eta f, \nabla^\eta g \right\rangle + 2\eta(\overset{*}{T}_r \nabla^\eta f)\eta(\nabla^\eta g). \tag{6.10}$$

7. Normalized null hypersurfaces $x : (M, g, N) \to Q_1^{n+2}(c)$ satisfying $L_r x = \mathcal{U}x + b$

Definition 7.1 *A L_r-normalized null hypersurface of $Q_1^{n+2}(c)$ is a normalized null hypersurface $x : (M, g, N) \to Q_1^{n+2}(c)$ satisfying the following linear condition on the second order differential equation*

$$L_r x = \mathcal{U}x + b, \tag{7.1}$$

for some field of screen constant matrices $\mathcal{U} \in \mathbb{R}^{(n+3)\times(n+3)}$ and field of screen constant vectors $b \in \mathbb{R}^{n+3}$.

Our goal in this section is to classify L_r-normalized null hypersurfaces $x : (M, g, N) \to Q_1^{n+2}(c)$. In [28], the authors showed that nondegenerate hypersurface of $Q_1^{n+2}(c)$ satisfying the above Equation (7.1) with $\mathcal{U} \in \mathbb{R}^{(n+3)\times(n+3)}$, $b \in \mathbb{R}^{n+3}$, constant matrix and constant vector respectively, are $(r+1)-$maximal hypersurfaces, totally umbilical hypersurfaces or isoparametric hypersurfaces with at most two different principal curvatures. In our case (of null hypersurfaces), we will assume that the matrix $\mathcal{U}$ and the vector b are constant

along the leaves of the screen distribution, since in nondegenerate case, the screen distribution is nothing but the tangent space globally. Furthermore, a difficulty to solve this problem in the degenerate case is the choice of normalization. In [8], to find null hypersurfaces in the Lorentz-Minkowski space $\mathbb{R}_1^{n+2}$ satisfying the equation (7.1), the authors considered a specific set of normalizations that they called UCC-normalization with vanishing $1-$form τ. But, the existence of a UCC-normalization with vanishing $1-$form τ is improbable for several null hypersurfaces of $Q_1^{n+2}(c)$. For instance, a totally geodesic null hypersurface of $Q_1^{n+2}(c)$ does not admit such a normalization since the screen distribution cannot be totally geodesic (see [20]). So, we will first consider closed normalizations such that $\tau^N|_{\mathscr{S}(N)} = 0$.

Let $a \in \mathbb{R}^{n+3}$ be a fixed vector. Using definition of the second-order operator and (5.2) one has

$$
\begin{aligned}
L_r \langle x, a \rangle &= trace(\overset{\star}{T}_r \circ \nabla_\eta^2 \langle x, a \rangle) \\
&= tr(\overset{\star}{T}_r \circ \langle A_N - \overset{\star}{A}_\xi - 2\tau^N(\xi)\xi, a\rangle\xi) + \langle N, a\rangle tr(\overset{\star}{T}_r \circ \overset{\star}{A}_\xi) \\
&\quad + \langle \xi, a\rangle tr\left(\overset{\star}{T}_r \circ (A_N - \overset{\star}{A}_\xi)\right) - c\langle x, a\rangle tr(\overset{\star}{T}_r).
\end{aligned}
$$

By using Proposition 2.3 one obtains

$$
\begin{aligned}
L_r \langle x, a \rangle &= (-1)^r (r+1)\, \overset{\star}{S}_{r+1} \langle N, a\rangle + (-1)^r\, \overset{\star}{S}_r \langle A_N\xi - 2\tau^N(\xi)\xi, a\rangle \\
&\quad + tr\left(\overset{\star}{T}_r \circ (A_N - \overset{\star}{A}_\xi)\right) \langle \xi, a\rangle - (-1)^r c(n+1-r)\, \overset{\star}{S}_r \langle x, a\rangle. \quad (7.2)
\end{aligned}
$$

After extending L_r on $\otimes^{n+3}\mathcal{C}^\infty(M)$, one has

$$
\begin{aligned}
L_r x &= (-1)^r (r+1)\, \overset{\star}{S}_{r+1} N + (-1)^r \left(A_N\xi - 2\tau^N(\xi)\xi\right) \overset{\star}{S}_r \\
&\quad + tr\left(\overset{\star}{T}_r \circ (A_N - \overset{\star}{A}_\xi)\right) \xi - (-1)^r c(n+1-r)\, \overset{\star}{S}_r x.
\end{aligned}
$$

If one assumes that the normalization is such that $A_N - \overset{\star}{A}_\xi = \varphi_1 P + \varphi_2 \overset{\star}{A}_\xi$, from the last equation one obtains

$$
L_r x = f_1 N + f_2 \xi + f_3 x, \quad (7.3)
$$

with $f_1 = (-1)^r (r+1)\, \overset{\star}{S}_{r+1}$, $f_2 = (-1)^r \left[(n-r)\varphi_1 \overset{\star}{S}_r + (r+1)\varphi_2 \overset{\star}{S}_{r+1}\right]$,

$f_3 = (-1)^{r+1}(n-r+1)\, \overset{\star}{S}_r c$.

Our goal is then to find closed L_r-normalized null hypersurfaces $x : (M, g, N) \to Q_1^{n+2}(c)$ with $\tau^N|_{\mathscr{S}(N)} = 0$.

7.1. Examples

7.1.1. Screen totally umbilical or totally geodesic normalized null hypersurfaces $x : (M, g, N) \to Q_1^{n+2}(c)$

We have already shown that screen totally umbilical or totally geodesic normalized null hypersurfaces $x : (M, g, N) \to Q_1^{n+2}(c)$ are sections of $Q_1^{n+2}(c)$ by hyperplanes of $\mathbb{R}_q^{n+3}$.

Let $a \in \mathbb{R}_q^{n+3}$ be a constant vector such that $\langle a, a \rangle = c$ and M the null hypersurface, section of $Q_1^{n+2}(c)$ by the hyperplane passing through a. Thus,

$$M = \{x \in Q_1^{n+2}(c); f(x) = \langle a, x \rangle = c\}.$$

We consider (locally) on M the Normalizing pair

$$N = \frac{1}{2}\left(\frac{1+ca_0^2}{a_0 - x_0} - c(a_0 - x_0)\right)(a - x) - \partial x_0 - x_0 ca,$$

$$\xi = \frac{1}{a_0 - x_0}a - x.$$

By a direct calculating,

$$\overset{\star}{A}_{\xi} = P, \quad A_N = \frac{1}{2}\left(\frac{1+ca_0^2}{(a_0 - x_0)^2} - c\right)P, \quad \tau = 0, \ \eta = x^\star dx_0,$$

and it follows that

$$\overset{\star}{S}_r = \binom{n}{r}\left(\frac{1}{a_0 - x_0}\right)^r, \quad A_N - \overset{\star}{A}_{\xi} = \frac{1}{2}\left(ca_0^2 - 1 - (a_0 - x_0)^2 c\right)\overset{\star}{A}_{\xi}.$$

Therefore, Equation (7.3) leads to

$$(r+1)^{-1}\binom{n}{r+1}^{-1}(a_0 - x_0)^{-r}L_r x = \left[\frac{ca_0^2}{a_0 - x_0} + \frac{(a_0 - x_0)c}{n - r}\right]x$$

$$+ \left[(a_0 - x_0)c + \frac{ca_0^2}{a_0 - x_0} - c\right]a + \partial x_0,$$

which is solution of Equation (7.1).

7.1.2. Standard degenerate product in $\mathbb{S}_1^{n+2}$

Let $f : \mathbb{R}_1^{n+3} \supset \mathbb{S}_1^{n+2} \to \mathbb{R}$, $x \mapsto -x_0^2 + x_1^2 + \cdots + x_m^2$, $1 \le m \le n+1$, be a smooth function. Set $D = diag[1, ..., 1, 0, ..., 0]$, one then has $f(x) = \langle Dx, x \rangle$. For all $x \in \mathbb{S}_1^{n+2}$, $df(x) = -2x_0 dx_0 + 2x_1 dx_1 + \cdots + 2x_m dx_m$. setting $M = f^{-1}(0)$, and taking the restriction of f on the open subset $\mathbb{S}_1^{n+2} \smallsetminus \{0\} \times \mathbb{S}^{n-m+2}$ to remove singular points and make M a hypersurface of $\mathbb{S}_1^{n+2}$ one has

$$M = \Lambda_0^m \times \mathbb{S}^{n-m+1}.$$

It follows that $df(x)(X_x) = \langle Dx, X_x \rangle$, thus Dx is normal to M and since Dx is null vector. One concludes that M is a null hypersurface of $\mathbb{S}_1^{n+2}(c)$. Now, we consider the globally defined normalizing pair $\{N, \xi\}$ given by

$$N = \frac{1}{\sqrt{2}}\left(x_0 + \frac{1}{x_0}\right)Dx - \sqrt{2}\partial x_0 - \sqrt{2}x_0 x, \quad \xi = \frac{1}{x_0\sqrt{2}}Dx.$$

By direct calculating, one gets

$$
\begin{cases}
\overset{\star}{A}_\xi(X) = \frac{X_0}{x_0^2}Dx - \frac{1}{x_0\sqrt{2}}DX = -\frac{1}{x_0\sqrt{2}}DPX \\
A_N(X) = x_0\sqrt{2}X + \frac{1}{\sqrt{2}}\left(\frac{X_0}{x_0^2} - X_0\right)Dx - \frac{1}{\sqrt{2}}\left(\frac{1}{x_0} + x_0\right)DX \\
\quad = x_0\sqrt{2}PX - \frac{1}{\sqrt{2}}\left(\frac{1}{x_0} + x_0\right)DPX \\
\tau^N \equiv 0 \\
\eta = \sqrt{2}dx_0
\end{cases} .
$$

Using the (local) basis

$$X_0 = \xi,\ X_1 = -x_m\partial x_1 + x_1\partial x_m, \cdots, X_{m-1} = -x_m\partial x_{m-1} + x_{m-1}\partial x_m,$$

$$X_m = -x_{n+2}\partial x_{m+1} + x_{m+1}\partial x_{n+2}, ..., X_n = -x_{n+2}\partial x_{n+1} + x_{n+1}\partial x_{n+2},$$

the principal curvatures are

$$\overset{\star}{k}_0 = 0, \quad \overset{\star}{k}_1 = \cdots = \overset{\star}{k}_{m-1} = -\frac{1}{x_0\sqrt{2}}, \quad \overset{\star}{k}_m = \cdots = \overset{\star}{k}_n = 0,$$

$$k_0 = 0, k_1 = \cdots = k_{m-1} = -\frac{1}{\sqrt{2}}\left(x_0 - \frac{1}{x_0}\right), k_m = \cdots = k_n = \sqrt{2}x_0.$$

Then, the $r-$th mean curvatures of the shape operator $\overset{\star}{A}_\xi$ are given by

$$
\overset{\star}{H}_r = \binom{n+1}{r}^{-1}\overset{\star}{S}_r = \begin{cases} \binom{n+1}{r}^{-1}\binom{m-1}{r}\left(-\frac{1}{x_0\sqrt{2}}\right)^r & \text{if } 0 \le r \le m-1 \\ 0 & \text{if } m \le r \le n+1 \end{cases} .
$$

A direct calculating gives $A_N - \overset{\star}{A}_\xi = \sqrt{2}x_0 P + x_0^2 \overset{\star}{A}_\xi$ and

$$
\begin{aligned}
tr\left(\overset{\star}{T}_r \circ (A_N - \overset{\star}{A}_\xi)\right) &= \sqrt{2}x_0 tr\left(\overset{\star}{T}_r \circ P\right) - x_0^2 tr\left(\overset{\star}{T}_r \circ \overset{\star}{A}_\xi\right) \\
&= (-1)^r(n-r)x_0\sqrt{2}\,\overset{\star}{S}_r - (-1)^r(r+1)x_0^2\,\overset{\star}{S}_{r+1} .
\end{aligned}
$$

- If $m \le r \le n+1$, then (7.3) becomes

$$L_r x = 0,$$

  which is a solution of Equation (7.1) with $\mathcal{U} = 0$ and $b = 0$.

- If $0 \le r \le m-2$, then (7.3) becomes

$$
\begin{aligned}
L_r x &= \frac{1}{2}\left(\frac{1}{x_0\sqrt{2}}\right)^r\left[(2n - m - r + 1)\binom{m-1}{r} - 1 - \frac{1}{x_0^2}\right]Dx \\
&\quad + \binom{m-1}{r}\left(\frac{1}{x_0\sqrt{2}}\right)^r\left[(m - r - 2)\frac{1}{x_0}\partial x_0 - (n - m + 2)x\right],
\end{aligned}
$$

  which is a solution of Equation (7.1).

-

$$L_{m-1}x = -\left(\frac{1}{x_0\sqrt{2}}\right)^{m-1}[(n - m + 1)D - 1]x,$$

  which is a solution of Equation (7.1).

7.1.3. Standard degenerate product in $\mathbb{H}_1^{n+2}$

Let $f : \mathbb{R}_2^{n+3} \supset \mathbb{H}_1^{n+2} \to \mathbb{R}$, $x \mapsto -x_0^2 + x_2^2 \cdots + x_{m+1}^2$, $1 \leq m \leq n+1$, be a smooth function. Set $D = diag[1, 0, 1..., 1, 0, ..., 0]$, one then has $f(x) = \langle Dx, x \rangle$. For all $x \in \mathbb{H}_1^{n+2}(c)$, $df(x) = -2x_0 dx_0 + 2x_2 dx_2 + \cdots + 2x_{m+1} dx_{m+1}$. We set $M = f^{-1}(0)$, and take the restriction of f on the open subset $\mathbb{H}_1^{n+2} \smallsetminus \{0\} \times \mathbb{H}^{n-m+2}$ to remove singular points and make that M be a hypersurface of $\mathbb{H}_1^{n+2}$. It follows that

$$M = \Lambda_0^m \times \mathbb{H}^{n-m+1},$$

and $df(x)(X_x) = \langle Dx, X_x \rangle$. Thus, Dx is normal to M and since Dx is null vector. One concludes that M is a null hypersorface of $\mathbb{H}_1^{n+2}$. Now, we consider the global defined normalizing pair $\{N, \xi\}$ given by

$$N = \frac{1}{\sqrt{2}}\left(\frac{1}{x_0} - x_0\right)Dx - \sqrt{2}\partial x_0 + \sqrt{2}x_0 x, \quad \xi = \frac{1}{x_0\sqrt{2}}Dx.$$

By direct calculating, one gets

$$\begin{cases} \overset{\star}{A}_\xi = -\frac{1}{x_0\sqrt{2}}D \circ P \\ A_N = -\frac{1}{\sqrt{2}}\left(\frac{1}{x_0} - x_0\right)D \circ P - \sqrt{2}x_0 P \\ \tau^N \equiv 0 \\ \eta = \sqrt{2}dx_0 \end{cases}.$$

Using the (local) basis

$$X_0 = \xi, \ X_1 = -x_{m+1}\partial x_2 + x_2\partial x_{m+1}, \cdots X_{m-1} = -x_{m+1}\partial x_m + x_m\partial x_{m+1},$$

$$X_m = -x_{n+2}\partial x_2 + x_2\partial x_{n+2},$$

$$X_{m+1} = -x_{n+2}\partial x_{m+2} + x_{m+2}\partial x_{n+2} \cdots X_n = -x_{n+2}\partial x_{n+1} + x_{n+1}\partial x_{n+2},$$

the principal curvatures are

$$\overset{\star}{k}_0 = 0, \quad \overset{\star}{k}_1 = \cdots = \overset{\star}{k}_{m-1} = -\frac{1}{x_0\sqrt{2}}, \quad \overset{\star}{k}_m = \cdots = \overset{\star}{k}_n = 0,$$

$$k_0 = 0, k_1 = \cdots = k_{m-1} = -\frac{1}{\sqrt{2}}\left(x_0 + \frac{1}{x_0}\right), k_m = \cdots = k_n = -\sqrt{2}x_0.$$

Then, the $r-$th mean curvatures of the shape operator $\overset{\star}{A}_\xi$ are given by

$$\overset{\star}{H}_r = \binom{n+1}{r}^{-1}\overset{\star}{S}_r = \begin{cases} \binom{n+1}{r}^{-1}\binom{m-1}{r}\left(-\frac{1}{x_0\sqrt{2}}\right)^r & \text{if } 0 \leq r \leq m-1 \\ 0 & \text{if } m \leq r \leq n+1 \end{cases}.$$

A direct calculating gives $A_N - \overset{\star}{A}_\xi = -\sqrt{2}x_0 P - x_0^2 \overset{\star}{A}_\xi$ and

$$\begin{aligned} tr\left(\overset{\star}{T}_r \circ (A_N - \overset{\star}{A}_\xi)\right) &= -\sqrt{2}x_0 tr\left(\overset{\star}{T}_r \circ P\right) - x_0^2 tr\left(\overset{\star}{T}_r \circ \overset{\star}{A}_\xi\right) \\ &= -(-1)^r(n-r)x_0\sqrt{2}\,\overset{\star}{S}_r - (-1)^r(r+1)x_0^2\,\overset{\star}{S}_{r+1}. \end{aligned}$$

- If $m \le r \le n+1$, then (7.3) becomes

$$L_r x = 0,$$

  which is solution of Equation (7.1) with $\mathcal{U} = 0$ and $b = 0$.

- If $0 \le r \le m-2$, then (7.3) becomes

$$L_r x = -\binom{m-1}{r}\left(\frac{1}{x_0\sqrt{2}}\right)^r \left[\left(\frac{m-r-1}{2x_0^2} + n - m + 1\right)D + (n+m-2r)I\right]x$$
$$+ (m-r-2)\binom{m-1}{r}\left(\frac{1}{x_0\sqrt{2}}\right)^r \frac{1}{x_0}\partial x_0,$$

  which is solution of Equation (7.1).

- Finally,(())

$$L_{m-1}x = -\left(\frac{1}{x_0\sqrt{2}}\right)^{m-1}\left[(n-m+1)D + (n-m+2)I\right]x,$$

  which is solution of Equation (7.1).

7.2. Classification of null hypersurfaces $x : (M, g, N) \to Q_1^{n+2}(c)$, satisfying $L_r x = \mathcal{U}x$

In the same manner we defined $L_r f$ in Section 6, let us introduce $L_{r|\mathscr{S}}f$ to be the trace of the restriction of the endomorphism $\overset{\star}{T}_r \circ \nabla_\eta^2 f : \Gamma(TM) \to \Gamma(TM)$ on $\Gamma(\mathscr{S}(N))$. From now on, X, Y, Z are sections of $\mathscr{S}(N)$ and we assume that the normalization is such that

$$A_N - \overset{\star}{A}_\xi = \varphi_1 P + \varphi_2 \overset{\star}{A}_\xi, \tag{7.4}$$

with $\varphi_1, \varphi_2 \in \mathcal{C}^\infty(M)$. Let us compute $L_{r|\mathscr{S}}(x)$, $L_{r|\mathscr{S}}(N)$, $L_{r|\mathscr{S}}(L_r x)$ and $L_{r|\mathscr{S}}(\xi)$.

- $L_{r|\mathscr{S}}(x)$

$$\langle \overset{\star}{\nabla} \langle a, x\rangle, X\rangle = X \cdot \langle a, x\rangle = \langle Pa^\top, X\rangle$$

$$\Rightarrow \overset{\star}{\nabla} \langle a, x\rangle = Pa^\top = a - \langle a, N\rangle\xi - \langle a, \xi\rangle N - \langle a, x\rangle cx.$$

  Take covariant derivative,

$$\overset{\circ}{\nabla}_X \overset{\star}{\nabla} \langle a, x\rangle = \langle a, N\rangle \overset{\star}{A}_\xi(X) + \langle a, \xi\rangle A_N(X) - \langle a, x\rangle cX.$$

  From here using the assumption (7.4) and formula (2.13) one obtains

$$\overset{\star}{\nabla}_X \overset{\star}{\nabla} \langle a, x\rangle = \langle a, N + (1 + \varphi_2)\xi\rangle \overset{\star}{A}_\xi(X) + \langle a, \xi\rangle\varphi_1 X - \langle a, x\rangle cX$$

  From here, using the definition of $L_{r|\mathscr{S}}$ and Proposition 2.3,

$$L_{r|\mathscr{S}}(x) = (-1)^r(r+1)\overset{\star}{S}_{r+1}\left(N + (1 + \varphi_2)\xi\right)$$
$$+ (-1)^r(n-r)\overset{\star}{S}_r \xi - (-1)^r(n-r)\overset{\star}{S}_r cx. \tag{7.5}$$

- $L_{r|\mathscr{S}}(N)$

$$\langle \overset{\star}{\nabla} \langle a, N \rangle, X \rangle = X \cdot \langle a, N \rangle = -\langle A_N(a^\top), x \rangle.$$

Hence,

$$\overset{\star}{\nabla} \langle a, N \rangle = -(1 + \varphi_2) \overset{\star}{A}_\xi (a^\top) - \varphi_1 P a^\top. \tag{7.6}$$

Take covariant derivative, use the assumption (7.4) and formula (2.13),

$$\begin{aligned}
\overset{\star}{\nabla}_X \overset{\star}{\nabla} \langle a, N \rangle = &- \left(\varphi_1^2 X + 2\varphi_1(1 + \varphi_2) \overset{\star}{A}_\xi (X) + (1 + \varphi_2)^2 \overset{\star}{A}_\xi (X)^2 \right) \langle a, \xi \rangle \\
&+ \left((1 + \varphi_2) \overset{\star}{A}_\xi (X) + \varphi_1 X \right) c \langle a, x \rangle - (1 + \varphi_2) \left(\nabla_{a^\top} \overset{\star}{A}_\xi \right) X \\
&- \varphi_1 \overset{\star}{A}_\xi (X) \langle a, N \rangle + (1 + \varphi_2) C(X, \overset{\star}{A}_\xi (a^\top)) \xi.
\end{aligned}$$

Hence by using Proposition 2.3,

$$\begin{aligned}
L_{r|\mathscr{S}}(N) = &(-1)^{r+1} \left[(n - r) \varphi_1^2 \overset{\star}{S}_r + 2\varphi_1(1 + \varphi_2)(r + 1) \overset{\star}{S}_{r+1} \right] \\
&+ (1 + \varphi_2)^2 \left[\overset{\star}{S}_1 \overset{\star}{S}_{r+1} - (r + 2) \overset{\star}{S}_{r+2} \right] \xi \\
&+ (-1)^r \left[(1 + \varphi_2)(r + 1) \overset{\star}{S}_{r+1} + (n - r) \varphi_1 \overset{\star}{S}_r \right] cx \\
&+ (-1)^{r+1} \overset{\star}{\nabla} \overset{\star}{S}_{r+1} + (-1)^{r+1}(r + 1)\varphi_1 \overset{\star}{S}_{r+1} N.
\end{aligned} \tag{7.7}$$

- $L_{r|\mathscr{S}}(L_r x)$ applying $L_{r|\mathscr{S}}$ to equality (7.3) and using (6.10) and (7.6),

$$\begin{aligned}
L_{r|\mathscr{S}}(L_r x) = &f_1 L_{r|\mathscr{S}}(N) + L_{r|\mathscr{S}}(f_1) N + f_2 L_{r|\mathscr{S}}(\xi) \\
&+ L_{r|\mathscr{S}}(f_2) \xi + f_3 L_{r|\mathscr{S}}(x) + L_{r|\mathscr{S}}(f_3) x \\
&- 2 \left(A_N \circ \overset{\star}{T}_r \right) \left(\overset{\star}{\nabla} f_1 \right) + 2 \overset{\star}{T}_r \left(\overset{\star}{\nabla} f_3 \right) \\
&- 2 \left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_r \right) \left(\overset{\star}{\nabla} f_2 \right).
\end{aligned} \tag{7.8}$$

- $L_{r|\mathscr{S}}(\xi)$ by the same manner,

$$\begin{aligned}
L_{r|\mathscr{S}}(\xi) = &(-1)^{r+1} \left[(r + 1)\varphi_1 \overset{\star}{S}_{r+1} - (1 + \varphi_2) \left(\overset{\star}{S}_1 \overset{\star}{S}_{r+1} - (r + 2) \overset{\star}{S}_{r+2} \right) \right] \xi \\
&+ (-1)^r \overset{\star}{\nabla} \overset{\star}{S}_{r+1} + (-1)^r (r + 1) \overset{\star}{S}_{r+1} cx.
\end{aligned} \tag{7.9}$$

Let us make the assumption that the immersion $x : (M, g, N) \to Q(c)_1^{n+2}$ satisfies Equation (7.1). Then

we have $L_{r|\mathscr{S}}(L_r x) = \mathcal{U} L_{r|\mathscr{S}}(x)$, that jointly with (7.8) and (7.3), yields

$$
\begin{aligned}
& f_1 L_{r|\mathscr{S}}(N) + L_{r|\mathscr{S}}(f_1)N + f_2 L_{r|\mathscr{S}}(\xi) + L_{r|\mathscr{S}}(f_2)\xi \\
& + f_3 L_{r|\mathscr{S}}(x) + L_{r|\mathscr{S}}(f_3)x \\
& - 2\left(A_N \circ \overset{\star}{T}_r\right)\left(\overset{\star}{\nabla} f_1\right) - 2\left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_r\right)\left(\overset{\star}{\nabla} f_2\right) + 2\,\overset{\star}{T}_r\left(\overset{\star}{\nabla} f_3\right) \\
& = (-1)^r(r+1)\,\overset{\star}{S}_{r+1}\left(\mathcal{U}N + (1+\varphi_2)\mathcal{U}\xi\right) \\
& + (-1)^r(n-r)\varphi_1\,\overset{\star}{S}_r\,\mathcal{U}\xi - (-1)^r(n-r)\,\overset{\star}{S}_r\,c\mathcal{U}x.
\end{aligned}
\tag{7.10}
$$

On the other hand, from (7.1), and using again (7.3), we have

$$
\mathcal{U}x = -Pb^\top + (f_1 - \langle b,\xi\rangle)N + (f_2 - \langle b,N\rangle)\xi + (f_3 - \langle b,x\rangle c)x
\tag{7.11}
$$

from here and (7.10), we get

$$
\begin{aligned}
& f_1 L_{r|\mathscr{S}}(N) + L_{r|\mathscr{S}}(f_1)N + f_2 L_{r|\mathscr{S}}(\xi) + L_{r|\mathscr{S}}(f_2)\xi + f_3 L_{r|\mathscr{S}}(x) \\
& + L_{r|\mathscr{S}}(f_3)x - 2\left(A_N \circ \overset{\star}{T}_r\right)\left(\overset{\star}{\nabla} f_1\right) - 2\left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_r\right)\left(\overset{\star}{\nabla} f_2\right) + 2\,\overset{\star}{T}_r\left(\overset{\star}{\nabla} f_3\right) \\
& - (-1)^r(n-r)\,\overset{\star}{S}_r\,cPb^\top + (-1)^r(n-r)\,\overset{\star}{S}_r\,c\varphi_1 c(f_1 - \langle b,\xi\rangle)N \\
& + (-1)^r(n-r)\,\overset{\star}{S}_r\,c(f_2 - \langle b,N\rangle)\xi + (-1)^r(n-r)\,\overset{\star}{S}_r\,c(f_3 - \langle b,x\rangle c)x \\
& = (-1)^r(r+1)\,\overset{\star}{S}_{r+1}\left(\mathcal{U}N + (-1)^r\left[(r+1)(1+\varphi_2)\mathcal{U}\,\overset{\star}{S}_{r+1}\right.\right. \\
& \left.\left. + (n-r)\varphi_1\,\overset{\star}{S}_r\,\mathcal{U}\xi - (-1)^r(n-r)\,\overset{\star}{S}_r\right]\xi.
\end{aligned}
\tag{7.12}
$$

Since $\mathcal{U}$ and b are constants on the leaves of the screen distribution, taking covariant on (7.3) and using (7.1) one has

$$
\begin{aligned}
\mathcal{U}X = {}& -f_1 A_N(X) - f_2\,\overset{\star}{A}_\xi(X) + f_3 X \\
& + \langle \overset{\star}{\nabla} f_1, X\rangle N + \langle \overset{\star}{\nabla} f_2, X\rangle\xi + \langle \overset{\star}{\nabla} f_3, X\rangle x.
\end{aligned}
\tag{7.13}
$$

$$
\langle \mathcal{U}X, Y\rangle = -f_1\langle A_N(X), Y\rangle - f_2\langle \overset{\star}{A}_\xi(X), Y\rangle + f_3\langle X, Y\rangle.
$$

Therefore $\mathcal{U}$ is selfadjoint if and only if the following conditions hold.

$$
\begin{cases}
\langle \mathcal{U}X,\xi\rangle = \langle X,\mathcal{U}\xi\rangle, & \langle \mathcal{U}X,N\rangle = \langle X,\mathcal{U}N\rangle \quad \langle \mathcal{U}X,x\rangle = \langle X,\mathcal{U}x\rangle \\
\langle \mathcal{U}N,\xi\rangle = \langle N,\mathcal{U}\xi\rangle & \langle \mathcal{U}N,x\rangle = \langle N,\mathcal{U}x\rangle \quad \langle \mathcal{U}\xi,x\rangle = \langle \xi,\mathcal{U}x\rangle
\end{cases}
$$

From here and by using (7.11) and (7.13) one see that

$$
\overset{\star}{\nabla}\langle b,x\rangle = c\,\overset{\star}{\nabla} f_3 = (-1)^r(n+1-r)\,\overset{\star}{\nabla}\overset{\star}{S}_r,
\tag{7.14}
$$

and so $\langle b, x \rangle - (-1)^r(n+1-r) \overset{\star}{\nabla}\overset{\star}{S}_r$ is constant on $\mathscr{S}(N)$. A direct consequence is that

$$(-1)^r(n+1-r)L_{r|\mathscr{S}}\left(\overset{\star}{S}_r\right) = L_{r|\mathscr{S}}(\langle b, x \rangle) \tag{7.15}$$

$$= (-1)^r \left((n-r)\varphi_1 \overset{\star}{S}_r + (1+\varphi_2)(r+1) \overset{\star}{S}_{r+1}\right) \langle b, \xi \rangle$$

$$+ (-1)^r(r+1) \overset{\star}{S}_{r+1} \langle b, N \rangle - (-1)^r(n-r) \overset{\star}{S}_r c\langle b, x \rangle.$$

With (7.12) one has

$$(-1)^r(r+1) \overset{\star}{S}_{r+1} \langle (\mathcal{U}N, X) + (-1)^r \left[(r+1)(1+\varphi_2) \overset{\star}{S}_{r+1} + (n-r)\varphi_1 \overset{\star}{S}_r\right] \langle \mathcal{U}\xi, X \rangle$$

$$= (-1)^r(1+\varphi_2)f_1\langle\overset{\star}{\nabla}\overset{\star}{S}_{r+1}, X\rangle - (-1)^r f_2\langle\overset{\star}{\nabla}\overset{\star}{S}_r, X\rangle - 2\varphi_1\langle\overset{\star}{T}_r\left(\overset{\star}{\nabla} f_1, X\right)$$

$$- 2(1+\varphi_2)\langle\left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_r\right)\left(\overset{\star}{\nabla} f_1\right), X\rangle - 2\langle\left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_r\right)\left(\overset{\star}{\nabla} f_2\right), X\rangle$$

$$+ 2\langle\overset{\star}{T}_r\left(\overset{\star}{\nabla} f_3\right), X - (-1)^r(n-r) \overset{\star}{S}_r c\langle Pb^\top, X\rangle. \tag{7.16}$$

From here, using the fact that $\langle \mathcal{U}X, N \rangle = \langle \overset{\star}{\nabla} f_2, X \rangle$ and $\langle \mathcal{U}X, \xi \rangle = \langle \overset{\star}{\nabla} f_1, X \rangle$ one gets

$$\left[(r+1)^2 \varphi_2 \overset{\star}{S}_{r+1} + (r+1)(n-r)\varphi_1 \overset{\star}{S}_r + (r+1)^2(1+\varphi_2) \overset{\star}{S}_{r+1}\right.$$

$$\left. - (1+\varphi_2)(r+1) \overset{\star}{S}_{r+1}\right] \overset{\star}{\nabla}\overset{\star}{S}_{r+1} + 2(-1)^r(r+1)\varphi_1 \overset{\star}{T}_r\left(\overset{\star}{\nabla}\overset{\star}{S}_{r+1}\right)$$

$$+ 4(-1)^r(r+1)(1+\varphi_2)\left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_r\right)\left(\overset{\star}{\nabla}\overset{\star}{S}_{r+1}\right) \tag{7.17}$$

$$= \left[-(r+1)(n-r)\varphi_1 \overset{\star}{S}_{r+1} - (n-r)\varphi_1 \overset{\star}{S}_r - (r+1)\varphi_2 \overset{\star}{S}_{r+1}\right.$$

$$\left. - (n-r)(n+1-r) \overset{\star}{S}_r c\right] \overset{\star}{\nabla}\overset{\star}{S}_r - 2(-1)^r(n-r)\varphi_1 \left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_r\right)\left(\overset{\star}{\nabla}\overset{\star}{S}_r\right)$$

$$- 2(-1)^r(n+1-r)c \overset{\star}{T}_r\left(\overset{\star}{\nabla}\overset{\star}{S}_r\right).$$

Lemma 7.2 *Let* $x : (M, g, N) \to Q_1^{n+2}(c)$ *be a normalized null hypersurface endowed with a closed normalization* N *such that* $A_N - \overset{\star}{A}_\xi = \varphi_1 P + \varphi_2 \overset{\star}{A}_\xi$ *with* $\varphi_1 \neq 0$ *or* $\varphi_2 \neq -1/2$ *and* $\tau \equiv 0$. *Assume that* M *satisfies Equation (7.1) for some* $r = 1, ..., n$. *If the* $r-th$ *mean curvature* $\overset{\star}{H}_r$ *is screen constant then, the* $(r+1)-th$ *mean curvature* $\overset{\star}{H}_{r+1}$ *is also screen constant.*

Proof Let us assume that $\overset{\star}{H}_r$ is screen constant. Let L be a leaf of $\mathscr{S}(N)$ and consider the open set

$$\mathscr{U}_{r+1} = \{p \in L; \; \overset{\star}{\nabla}\overset{\star}{H}_{r+1}^2 (p) = 0\}.$$

We need to show that $\mathscr{U}_{r+1}$ is empty. If $\mathscr{U}_{r+1}$ is not empty then, from (7.12),

$$\left[r\varphi_1 + \overset{\star}{S}_r + (r+1)(r+2)(1+2\varphi_2) \overset{\star}{S}_{r+1}\right] \overset{\star}{\nabla}\overset{\star}{S}_{r+1}$$

$$= 2(-1)^{r+1}(r+1)\left[\varphi_1 \overset{\star}{T}_r\left(\overset{\star}{\nabla}\overset{\star}{S}_{r+1}\right) + (1+2\varphi_2) \overset{\star}{T}_{r+1}\left(\overset{\star}{\nabla}\overset{\star}{S}_{r+1}\right)\right].$$

Consider $\{\overset{\star}{E}_0= \xi, \overset{\star}{E}_1, \ldots, \overset{\star}{E}_n\}$ a local pseudo-orthonornal basis of principal direction of $\overset{\star}{A}_\xi$. One can write $\overset{\star}{\nabla}\overset{\star}{S}_{r+1}= \sum_{i=1}^n \left\langle \overset{\star}{\nabla}\overset{\star}{S}_{r+1}, \overset{\star}{E}_i \right\rangle \overset{\star}{E}_i$, and by using Proposition 2.3,

$$\sum_{i=1}^n \left[r\varphi_1+ \overset{\star}{S}_r +(r+1)(r+2)(1+2\varphi_2) \overset{\star}{S}_{r+1} \right] \langle \overset{\star}{\nabla}\overset{\star}{S}_{r+1}, \overset{\star}{E}_i \rangle \overset{\star}{E}_i$$

$$= 2(r+1) \sum_{i=1}^n \left[(1+2\varphi_2) \overset{\star}{S}_{r+1}^{i} -\varphi_1 \overset{\star}{S}_r^{i} \right] \langle \overset{\star}{\nabla}\overset{\star}{S}_{r+1}, \overset{\star}{E}_i \rangle \overset{\star}{E}_i . \text{ on } \mathscr{U}_{r+1},$$

for every $i = 1, \ldots, n$. Therefore, for every i such that $\langle \overset{\star}{\nabla}\overset{\star}{S}_{r+1}, \overset{\star}{E}_i \rangle \neq 0$, one gets

$$\left[r\varphi_1 \overset{\star}{S}_r +(r+1)(r+2)(1+2\varphi_2) \overset{\star}{S}_{r+1} \right] = 2(r+1) \left[(1+2\varphi_2) \overset{\star}{S}_{r+1}^{i} -\varphi_1 \overset{\star}{S}_r^{i} \right] \text{ on } \mathscr{U}_{r+1}. \tag{7.18}$$

We claim that $\langle \overset{\star}{\nabla}\overset{\star}{S}_{r+1}, \overset{\star}{E}_i \rangle = 0$ for some i. Otherwise, taking summation in (7.18) and using Proposition 2.3,

$$(r+1)(1+2\varphi_2)(2n+r-4) \overset{\star}{S}_{r+1}= (2(r+1)(n-r) - r\varphi_1) \overset{\star}{S}_r, \tag{7.19}$$

and then, $\overset{\star}{H}_{r+1}$ is constant on $\mathscr{U}_{r+1}$, which is a contradiction. Now rearranging the local pseudo-orthonormal basis if necessary or even taking another pseudo-orthonormal basis of principal directions, we may assume that there exists some $m \in \{1, \ldots, n-1\}$ such that

$$\begin{cases} \left\langle \overset{\star}{\nabla}\overset{\star}{S}_{r+1}, \overset{\star}{E}_i \right\rangle \neq 0 & \text{for } i = 1, \ldots, m, \text{ and } \overset{\star}{k}_1< \cdots <\overset{\star}{k}_m \\ \left\langle \overset{\star}{\nabla}\overset{\star}{S}_{r+1}, \overset{\star}{E}_i \right\rangle = 0 & \text{for } i = m+1, \ldots, 1 \end{cases}. \tag{7.20}$$

Using the same method in the proof of the Lemma 7.1 in [8], one proves that for every subset $J \subset \{1, \ldots, m\}$,

$$2(r+1)(1+2\varphi_2) \overset{\star}{S}_{r+1}^{J} -2(r+1)\varphi_1 \overset{\star}{S}_r^{J}= \left[r\varphi_1 \overset{\star}{S}_r +(r+1)(r+2)(1+2\varphi_2) \overset{\star}{S}_{r+1} \right] \text{ on } \mathscr{U}_{r+1}. \tag{7.21}$$

From (7.13) and (7.20) one has

$$\mathcal{U} \overset{\star}{E}_i= C_i \overset{\star}{E}_i \text{ for } i = 1, \ldots, m,$$

where
$C_i = (-1)^{r+1} \left[\left((r+1) + (r+1)(1+2\varphi_2 \overset{\star}{k}_i) \right) \overset{\star}{S}_{r+1} + \left((n-r)\varphi_1 \overset{\star}{k}_i +(n-r+1)c \right) \overset{\star}{S}_r \right]$ is screen constant as eigenvalue of the screen constant matrix $\mathcal{U}$. From (7.21) for the set $J = \{1, \ldots, m\}$ we get

$$\left[r\varphi_1 \overset{\star}{S}_r +(r+1)(r+2)(1+2\varphi_2) \overset{\star}{S}_{r+1} \right] \left[(r+1)(1+2\varphi_2) \overset{\star}{S}_{r+1} +(n-r)\varphi_1 \overset{\star}{S}_r \right]^{r+1}$$

$$= 2(r+1)(1+2\varphi_2) \sum_{m<i_1<\cdots<i_{r+1}<n} \eta_{i_1} \cdots \eta_{i_{r+1}} \tag{7.22}$$

$$- 2(r+1)\varphi_1 \left[(1+2\varphi_2)(r+1) \overset{\star}{S}_{r+1} +(n-r)\varphi_1 \overset{\star}{S}_r \right] \sum_{m<i_1<\cdots<i_r<n} \eta_{i_1} \cdots \eta_{i_r},$$

where $\eta_i = C_i - (r+1) \overset{\star}{S}_{r+1} + (n-r+1)c \overset{\star}{S}_r$. Looking (7.22) as an equation in $\overset{\star}{S}_{r+1}$, its both sides are trivially zero if and only if $\varphi_1 = 0$ and $\varphi_2 = -1/2$. Then for $\varphi_1 \neq 0$ or $\varphi_2 \neq -1/2$, $\overset{\star}{H}_{r+1}$ is locally constant on $\mathscr{U}_{r+1}$, which is a contradiction. This completes the proof. □

Proof [Proof of Theorem 1.1] Let us assume that the immersion $x : (M, g, N) \to Q_1^{n+2}(c)$ satisfies the condition $L_r x = \mathcal{U} x$, for some screen constant matrix $\mathcal{U} \in \mathbb{R}^{(n+2)\times(n+2)}$. Since $b = 0$, Equation (7.14) implies that $\overset{\star}{H}_r$ is screen constant and by Lemma 7.2, $\overset{\star}{H}_{r+1}$ is also screen constant. Let us assume that $\overset{\star}{H}_{r+1}$ is a nonzero constant (otherwise, there is nothing to prove). Then, from (7.10) one obtains

$$f_1 L_{r|\mathscr{S}}(N) + f_2 L_{r|\mathscr{S}}(\xi) + f_3 L_{r|\mathscr{S}}(x) + (-1)^r (n-r) \overset{\star}{S}_r cf_3 x$$

$$+ (-1)^r (n-r) \overset{\star}{S}_r c\varphi_1 cf_1 N + (-1)^r (n-r) \overset{\star}{S}_r cf_2 \xi$$

$$= (-1)^r (r+1) \overset{\star}{S}_{r+1} \mathcal{U} N + (-1)^r \left[(r+1)(1+\varphi_2) \overset{\star}{S}_{r+1} + (n-r)\varphi_1 \overset{\star}{S}_r \right] \mathcal{U}\xi,$$

what we write as

$$\alpha_1 \xi + \alpha_2 x + \alpha_3 N = (r+1) \overset{\star}{S}_{r+1} \mathcal{U} N + \left[(r+1)(1+\varphi_2) \overset{\star}{S}_{r+1} + (n-r)\varphi_1 \overset{\star}{S}_r \right] \mathcal{U}\xi.$$

From here, taking covariant derivative and use the assumption (7.4) one gets

$$(-1)^{r+1}(r+1)\varphi_1 \overset{\star}{S}_{r+1} \mathcal{U} X + \alpha_4 \mathcal{U} \overset{\star}{A}_\xi(X) = -(\alpha_1 + \alpha_3(1+\varphi_2)) \overset{\star}{A}_\xi(X)$$

$$+ (\alpha_2 - \alpha_3\varphi_1)X + \langle \overset{\star}{\nabla} \alpha_1, X \rangle \xi, \tag{7.23}$$

for every $X \in \Gamma(\mathscr{S}(N))$. Also, (7.13) and (7.11) give

$$\mathcal{U} X = -f_1 A_N(X) - f_2 \overset{\star}{A}_\xi(X) + f_3 X, \tag{7.24}$$

and

$$\mathcal{U} x = f_1 N + f_2 \xi + f_3 x, \tag{7.25}$$

Applying $\overset{\star}{A}_\xi$ on (7.24) and using the assumption (7.4),

$$\mathcal{U} \overset{\star}{A}_\xi(X) = (f_3 - f_1\varphi_1) \overset{\star}{A}_\xi(X) - (f_2 + (1+\varphi_2)f_1) \overset{\star}{A}_\xi{}^2(X).$$

Jointly this with (7.23) and (7.24) it follow that α_1 is screen constant and

$$\alpha_5 X + \alpha_6 \overset{\star}{A}_\xi(X) + \alpha_7 \overset{\star}{A}_\xi{}^2(X) = 0,$$

where,

$$
\begin{cases}
\alpha_2 &= (r+1)\,\overset{\star}{S}_{r+1}\left[(r+1)(1+2\varphi_2)\,\overset{\star}{S}_{r+1} + 2(n-r)\varphi_1\,\overset{\star}{S}_r\right]c \\
\alpha_3 &= (r+1)^2\,\overset{\star}{S}_{r+1}^{\,2} - (r+1)\,\overset{\star}{S}_r\overset{\star}{S}_{r-1}\,c \\
\alpha_4 &= (-1)^{r+1}\left[2(r+1)(1+\varphi_2)\,\overset{\star}{S}_{r+1} + (n-r)\varphi_1\,\overset{\star}{S}_r\right] \\
\alpha_5 &= -(r+1)\,\overset{\star}{S}_{r+1}\left[(r+1)(1+2\varphi_2)\,\overset{\star}{S}_{r+1} + (n-r)\varphi_1\,\overset{\star}{S}_r\right] \\
\alpha_6 &= (r+1)^2(1+\varphi_2)\varphi_1\,\overset{\star}{S}_{r+1}^{\,2} + (\alpha_1 + \alpha_3(1+\varphi_2)) \\
&\quad + (r+1)\,\overset{\star}{S}_{r+1}\,\varphi_1\left[(n-r)\varphi_1\,\overset{\star}{S}_r + (r+1)\varphi_2\,\overset{\star}{S}_{r+1}\right] \\
&\quad -(-1)^r\alpha_4\left[(n+1-r)\,\overset{\star}{S}_r\,c + (r+1)\varphi_1\,\overset{\star}{S}_{r+1}\right] \\
\alpha_7 &= (-1)^{r+1}\alpha_4\left[(r+1)(1+\varphi_2)\,\overset{\star}{S}_{r+1} + (n-r)\varphi_1\,\overset{\star}{S}_r + (r+1)\varphi_2\,\overset{\star}{S}_{r+1}\right]
\end{cases}
,
$$

$\alpha_5, \alpha_6, \alpha_7$ are screen constant. Then M has at most two nonzero principal curvatures and is a totally umbilical or an almost isoparametric normalized null hypersurface of $Q(c)_1^{n+2}$ with exactly two nonzero principal curvatures. $\square$

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