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Domination parameters on Cayley digraphs of transformation semigroups with fixed sets

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Abstract: For a nonempty subset Y of a nonempty set X, denote by $Fix(X,Y)$ the semigroup of full transformations on the set X in which all elements in Y are fixed. The Cayley digraph $\mathrm{Cay}(Fix(X,Y),A)$ of $Fix(X,Y)$ with respect to a connection set $A \subseteq Fix(X,Y)$ is defined as a digraph whose vertex set is $Fix(X,Y)$ and two vertices α, β are adjacent in sense of drawing a directed edge (arc) from α to β if there exists $\mu \in A$ such that $\beta = \alpha\mu$. In this paper, we determine domination parameters of $\mathrm{Cay}(Fix(X,Y),A)$ where A is a subset of $Fix(X,Y)$ related to minimal idempotents and permutations in $Fix(X,Y)$.

Key words: Cayley digraphs of transformation semigroups, the (total/independent/connected/split) domination number

1. Introduction

Throughout the paper, all sets are considered to be finite. Let S be a semigroup and A be a nonempty subset of S. The Cayley digraph $\mathrm{Cay}(S,A)$ of a semigroup S with respect to a connection set A is a digraph with vertex set S and two vertices $x, y \in S$ are adjacent in sense of drawing a directed edge from x to y if there exists $a \in A$ in which $y = xa$. The Cayley graph was first introduced in 1878 by Arthur Cayley [5]. The concept was considered to construct graphs from finite groups and describe structural properties of abstract groups via graphs. Later, such concept was applied to visualize the structures of semigroups. One of interesting semigroups and also well known in semigroup theory is a transformation semigroup. In this paper, we focus on a special class of transformations in which certain elements are fixed. Such semigroup is known as a semigroup of transformations with fixed set. This semigroup was first defined by Honyam and Sanwong [11] in 2013. Let X be a nonempty set. Denote by $T(X)$ the semigroup of transformations from X into itself under the composition of maps. For a nonempty subset Y of X, the transformation semigroup $Fix(X,Y)$ with a fixed set Y is defined as follows:

$$Fix(X,Y) = \{\alpha \in T(X) : a\alpha = a \text{ for all } a \in Y\}.$$

Clearly, $Fix(X,Y)$ is a subsemigroup of $T(X)$. Indeed, if Y is a singleton set, then $Fix(X,Y)$ consists of all self-maps on X having the element of Y as their only common fixed element, which is well known in fixed point

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2010 *AMS Mathematics Subject Classification:* 05C20, 05C25, 05C69, 20M20 theory. In 2020, we [15] constructed the Cayley digraphs of $Fix(X,Y)$ and studied their connectedness and completeness. Hereafter, we apply our results in [15] to determine certain types of domination parameters of $\mathrm{Cay}(Fix(X,Y),A)$ where A is a special subset of $Fix(X,Y)$. Actually, the concept of domination parameters is a classical topic in graph theory. There are numerous research papers about domination parameters of graphs and digraphs. Further, the number of papers published on such topics is steadily growing. Some kinds of domination parameters are still investigated such as domination number, total domination number, independent domination number, connected domination number, and split domination number. For instance, in 2007, Arumugam et al. [3] provided results on total and connected domination in digraphs. Later in 2010, Blidia and Ould-Rabah [4] presented bounds on the domination number in oriented graphs. In 2013, Desormeaux et al. [8] introduced bounds on the connected domination number of a graph. In 2014, Amooshahi and Taeri [1] studied the domination and total domination numbers of Cayley sum graphs over a finite cyclic group. Later in 2015, Anupama et al. [2] purposed more results on the connected domination number of a jump graph. In the same year, Chaluvaraju and Appajigowda [6] determined the split domination number of a prism graph. Furthermore, López and Muntaner-Batle [13] studied the domination and total domination numbers of generalized products of graphs. In 2016, Desormeaux and Henning [9] presented the lower bounds on the total domination number of a graph. In 2018, Hao and Qian [10] introduced bounds on the domination number of a digraph. Moreover, Nupo and Panma [14] investigated the independent domination number in Cayley digraphs of rectangular groups. Recently, in 2019, Sivagami and Chelvam [17] considered the domination number of the trace graph of matrices. In addition, Ye et al. [18] provided more results on the domination number of the Cartesian product of two directed cycles.

2. Preliminaries and notations

We first recall that all sets mentioned in this paper are assumed to be finite. Some useful preliminaries and relevant notations are described here. For more definitions, terminologies and basic backgrounds about semigroups and digraphs not mentioned in this paper, we will refer to [7], [12] and [16].

Let $D=(V,E)$ be a digraph with vertex set V and arc set E. Further, let U be a nonempty subset of V.

The set U is called a *dominating set* of D if for each $v \in V \setminus U$, there exists $u \in U$ such that $(u,v) \in E$. Moreover, we say that u dominates v or v is dominated by u. The minimum cardinality among dominating sets of D is called the *domination number* of D and denoted by $\gamma(D)$, that is,

$$\gamma(D) = \min\{|U| : U \text{ is a dominating set of } D\}.$$

The set U is called a *total dominating set* of D if for each $v \in V$, there exists $u \in U$ such that $(u,v) \in E$. The minimum cardinality among total dominating sets of D is called the *total domination number* of D and denoted by $\gamma_t(D)$, that is,

$$\gamma_t(D) = \min\{|U| : U \text{ is a total dominating set of } D\}.$$

The set U is called an *independent dominating set* of D if U is a dominating set of D and independent in D, that is, $(u,v),(v,u) \notin E$ for all $u,v \in U$ such that $u \neq v$. The minimum cardinality among independent dominating sets of D is called the *independent domination number* of D and denoted by $\gamma_i(D)$, that is,

$$\gamma_i(D) = \min\{|U| : U \text{ is an independent dominating set of } D\}.$$

A digraph $D = (V, E)$ is said to be *weakly connected* or simply called *connected* if for each $u, v \in V$ in which $u \neq v$ there is a weakly dipath (simply called dipath) joining between u and v. In other words, we say that D is connected if for every distinct $u, v \in V$, there exist $x_1, x_2, \ldots, x_n \in V$ such that $(x_i, x_{i+1}) \in E$ or $(x_{i+1}, x_i) \in E$ for all $i = 0, 1, 2, \ldots, n$ where $u = x_0$ and $v = x_{n+1}$. Otherwise, if D is not connected, then D is said to be *disconnected*.

A digraph $D' = (V', E')$ is said to be a *subdigraph* of $D = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. Moreover, for any nonempty subset W of V, a subdigraph of D induced by W called an *induced subdigraph*, which is denoted by $D[W]$, is a subdigraph of D satisfying the condition that if $u, v \in W$ and $(u, v) \in E$, then (u, v) is an arc of $D[W]$, as well.

The set U is called a *connected dominating set* of D if U is a dominating set of D and the subdigraph of D induced by U is connected. The minimum cardinality among connected dominating sets of D is called the *connected domination number* of D and denoted by $\gamma_c(D)$, that is,

$$\gamma_c(D) = \min\{|U| : U \text{ is a connected dominating set of } D\}.$$

The set $U \subsetneq V$ is called a *split dominating set* of D if U is a dominating set of D and the subdigraph of D induced by $V \backslash U$ is disconnected. The minimum cardinality among split dominating sets of D is called the *split domination number* of D and denoted by $\gamma_s(D)$, that is,

$$\gamma_s(D) = \min\{|U| : U \text{ is a split dominating set of } D\}.$$

For a digraph with vertex set V and arc set E, we define

$$N^+(v) = \{u \in V \backslash \{v\} : (v, u) \in E\} \text{ and } N^-(v) = \{u \in V \backslash \{v\} : (u, v) \in E\} \text{ for any } v \in V.$$

We now describe some basic preliminaries about the semigroup $Fix(X, Y)$. For further information not mentioned here, we refer to [11] and [15]. Let X be a set and Y be a nonempty subset of X. For convenience, let $Y = \{a_i : i \in I\}$ throughout the paper, unless otherwise stated. Thus, for each $\alpha \in Fix(X, Y)$, we have $a_i\alpha = a_i$ for all $i \in I$. Let $X\alpha = \{a_i : i \in I\} \cup \{b_j : j \in J\}$ where $X\alpha$ denotes the image set of α. (Notice that the index set J can be empty, that is, $X\alpha = \{a_i : i \in I\} = Y$.) Then α can be denoted as

$$\alpha = \begin{pmatrix} A_i & B_j \\ a_i & b_j \end{pmatrix}$$

where $A_i = a_i\alpha^{-1}$ for each $i \in I$ and $B_j = b_j\alpha^{-1}$ for each $j \in J$. Notice that $A_i \cap Y = \{a_i\}$ for each $i \in I$ and $B_j \subseteq X \backslash Y$ for each $j \in J$.

Now, we consider some special subsemigroups of $Fix(X, Y)$ to construct the Cayley digraph. Such subsemigroups are E_m, the set of all minimal idempotents of $Fix(X, Y)$, and H_{id_X}, the group of permutations in $Fix(X, Y)$. Actually, Honyam and Sanwong [11] proved that

$$E_m = \left\{ \begin{pmatrix} A_i \\ a_i \end{pmatrix} : \{A_i : i \in I\} \text{ is a partition of } X \text{ with } a_i \in A_i \right\}.$$

In fact, every element in E_m is a left zero element of $Fix(X, Y)$. Furthermore, H_{id_X} is the Green's $\mathcal{H}$-class of $Fix(X, Y)$ containing the identity map id_X.

We now introduce the following lemma, which is useful for studying the domination parameters of $\mathrm{Cay}(Fix(X, Y), A)$.

Lemma 2.1 *[15] Let A be a nonempty subset of $Fix(X,Y)$ and μ be a vertex of* $\mathrm{Cay}(Fix(X,Y),A)$. *If $\mu \in E_m$, then $N^+(\mu)=\emptyset$.*

It is clear that, if $X=Y$, then $Fix(X,Y)=\{id_X\}$. Hence, hereafter we consider the case $Y \subsetneq X$ and, for convenience, we denote by Γ the Cayley digraph $\mathrm{Cay}(Fix(X,Y),A)$ of $Fix(X,Y)$ with a nonempty connection set A.

3. Domination parameters of Γ related to minimal idempotents

In this section, we present relevant results about domination parameters of Γ where the connection set is contained in the set E_m of minimal idempotents of $Fix(X,Y)$.

Lemma 3.1 *Let A be a connection set of Γ contained in E_m. Then the following statements hold.*

(i) *Every element in $Fix(X,Y)\backslash E_m$ is not dominated.*

(ii) *$Fix(X,Y)\backslash E_m \subseteq U$ for each dominating set U of Γ.*

Proof (i) Suppose that there exists $\alpha \in Fix(X,Y)\backslash E_m$ such that α is dominated by β for some $\beta \in Fix(X,Y)$, that is, $(\beta,\alpha)\in E(\Gamma)$. It follows that $\alpha=\beta\mu$ for some $\mu \in A \subseteq E_m$. Hence $X\alpha = X\beta\mu \subseteq X\mu \subseteq Y$. This implies that α is a minimal idempotent, which is a contradiction.

(ii) Let U be a dominating set of Γ, and let $\alpha \in Fix(X,Y)\backslash E_m$. Suppose that $\alpha \notin U$. By the property of the dominating set U, there exists $\beta \in U$ such that $(\beta,\alpha)\in E(\Gamma)$, which contradicts to (i). Consequently, α must belong to U, which completes the proof of our assertion. □

As the consequence of Lemma 3.1 (ii), we directly obtain a lower bound of a domination number of Γ, which is shown in the following proposition.

Proposition 3.2 *Let A be a connection set of Γ contained in E_m. Then*

$$\gamma(\Gamma) \geq |Fix(X,Y)| - |E_m| = |X|^{|X|-|Y|} - |Y|^{|X|-|Y|}.$$

However, some elements in E_m could be included into a dominating set of Γ up to a connection set A. Hereafter, for a connection set $A \subseteq E_m$, we define

$$\mathcal{A} = \{\mu \in E_m : (X\backslash Y)\mu \cap (\bigcup_{\alpha\in A}(X\backslash Y)\alpha) \neq \emptyset\} \text{ and}$$
$$\mathcal{B} = Fix(X,Y)\backslash \mathcal{A}.$$

Lemma 3.3 *Let A be a connection set of Γ contained in E_m. Then $\mathcal{B}$ is a dominating set of Γ.*

Proof Let $\beta \in Fix(X,Y)\backslash\mathcal{B}$. Then $\beta \in \mathcal{A}$, which follows that

$$\beta = \begin{pmatrix} A_i \\ a_i \end{pmatrix}$$

where $(X\backslash Y)\beta \cap (\bigcup\limits_{\alpha\in A}(X\backslash Y)\alpha) \neq \emptyset$. Thus, there exists $a_{i_0} \in (X\backslash Y)\beta$ in which $a_{i_0} \in (X\backslash Y)\lambda$ for some $\lambda \in A$. We get that there exist b_j and b_k in $X\backslash Y$ such that $b_j\lambda = a_{i_0} = b_k\beta$. We obtain that $b_k \in A_{i_0}$. Let $I' = I\backslash\{i_0\}$. Define $\varepsilon \in Fix(X,Y)$ by

$$\varepsilon = \begin{pmatrix} A_{i_0}\backslash\{b_k\} & A_{i'} & b_k \\ a_{i_0} & a_{i'} & b_j \end{pmatrix}.$$

Hence, $\varepsilon \notin \mathcal{A}$, which implies that $\varepsilon \in \mathcal{B}$. Moreover, we have $\varepsilon\lambda = \beta$, that is, $(\varepsilon,\beta) \in E(\Gamma)$. Therefore, $\mathcal{B}$ is a dominating set of Γ. □

We now present the following theorem, which is useful to determine the exact value of the domination number of Γ.

Theorem 3.4 *Let A be a connection set of Γ contained in E_m. Then*

$$\gamma(\Gamma) = |\mathcal{B}| = |X|^{|X|-|Y|} - |Y|^{|X|-|Y|} + |Y\backslash(\bigcup_{\alpha\in A}(X\backslash Y)\alpha)|^{|X|-|Y|}.$$

Proof By Lemma 3.3, we obtain that $\mathcal{B}$ is the dominating set of Γ, which leads to $\gamma(\Gamma) \leq |\mathcal{B}|$. We now suppose to the contrary that there exists a dominating set U of Γ in which $|U| < |\mathcal{B}|$. Thus, there exists at least one element $\beta \in \mathcal{B}\backslash U$. Hence $\beta \notin \mathcal{A}$. Since U is a dominating set of Γ, we get that there exists $\varepsilon \in U$ such that $(\varepsilon,\beta) \in E(\Gamma)$. That means $\beta = \varepsilon\mu$ for some $\mu \in A \subseteq E_m$. Thus, for each $x \in X$, we have $x\beta = x(\varepsilon\mu) = (x\varepsilon)\mu \in Y$. This implies that $\beta \in E_m$. Consider the element $\varepsilon \in U$, if $\varepsilon \in E_m$, then ε is a left zero element in $Fix(X,Y)$ and hence $\beta = \varepsilon\mu = \varepsilon \in U$, which is impossible. Thus, $\varepsilon \notin E_m$, that is, there exists $z \in X\backslash Y$ such that $z\varepsilon \in X\backslash Y$. Since $\beta,\mu \in E_m$ and $z\beta = z(\varepsilon\mu) = (z\varepsilon)\mu \in (X\backslash Y)\mu$, we obtain that

$$z\beta \in (X\backslash Y)\beta \cap (X\backslash Y)\mu \subseteq (X\backslash Y)\beta \cap (\bigcup_{\alpha\in A}(X\backslash Y)\alpha).$$

It follows that $\beta \in \mathcal{A}$, which is a contradiction. Consequently, there is no any dominating set with size less than $|\mathcal{B}|$. Note that

$$|\mathcal{A}| = |E_m| - |Y\backslash(\bigcup_{\alpha\in A}(X\backslash Y)\alpha)|^{|X|-|Y|}$$

where $|Y\backslash(\bigcup\limits_{\alpha\in A}(X\backslash Y)\alpha)|^{|X|-|Y|}$ is the number of elements $\lambda \in E_m$ whose intersections $(X\backslash Y)\lambda \cap (\bigcup\limits_{\alpha\in A}(X\backslash Y)\alpha)$ are empty. So we can conclude that

$$\gamma(\Gamma) = |\mathcal{B}| = |Fix(X,Y)| - |\mathcal{A}| = |X|^{|X|-|Y|} - |Y|^{|X|-|Y|} + |Y\backslash(\bigcup_{\alpha\in A}(X\backslash Y)\alpha)|^{|X|-|Y|}.$$

□

For the total domination number γ_t of Γ, the property of a total dominating set states that each element in $Fix(X,Y)$ has to be dominated. By Lemma 3.1 (i), the result of a total dominating set of Γ with respect to a connection set A, which is contained in E_m is provided in the following proposition.

Proposition 3.5 *For a connection set $A \subseteq E_m$, a total dominating set of Γ does not exist.*

In order to investigate the independent domination number of Γ with respect to a connection set $A \subseteq E_m$, the following lemma is needed.

Lemma 3.6 *If A is a connection set of Γ contained in E_m, then $\mathcal{B}$ is an independent dominating set of Γ.*

Proof Let A be a connection set of Γ such that $A \subseteq E_m$, and let $\alpha, \beta \in \mathcal{B}$ be distinct. There are three cases to consider as follows.

Case 1: Let $\alpha, \beta \notin E_m$. By Lemma 3.1 (i), we can conclude that $(\alpha, \beta), (\beta, \alpha) \notin E(\Gamma)$.

Case 2: Let $\alpha, \beta \in E_m$. It is shown in Lemma 2.1 that $N^+(\alpha) = \emptyset = N^+(\beta)$. Thus $(\alpha, \beta), (\beta, \alpha) \notin E(\Gamma)$.

Case 3: Suppose that either α or β belongs to E_m. Without loss of generality, we may assume that $\alpha \in E_m$ and $\beta \in Fix(X,Y)\backslash E_m$. Since $\alpha \in \mathcal{B}$, we have $\alpha \notin \mathcal{A}$, which implies that $(X\backslash Y)\alpha \cap (\bigcup\limits_{\delta \in A}(X\backslash Y)\delta) = \emptyset$. Indeed, $(\alpha, \beta) \notin E(\Gamma)$ since $N^+(\alpha) = \emptyset$ by Lemma 2.1. We now suppose to the contrary that $(\beta, \alpha) \in E(\Gamma)$. Then there exists $\mu \in A \subseteq E_m$ in which $\alpha = \beta\mu$. Since $\beta \notin E_m$, there exists $x \in X\backslash Y$ such that $x\beta \in X\backslash Y$. Thus $x\alpha = x(\beta\mu) = (x\beta)\mu \in (X\backslash Y)\mu$, which implies that $x\alpha \in (X\backslash Y)\alpha \cap (\bigcup\limits_{\delta \in A}(X\backslash Y)\delta)$, which is a contradiction. Therefore, $(\beta, \alpha) \notin E(\Gamma)$.

From the above three cases and by Lemma 3.3, we can conclude that $\mathcal{B}$ is an independent dominating set of Γ. □

Theorem 3.7 *If A is a connection set of Γ contained in E_m, then $\gamma_i(\Gamma) = \gamma(\Gamma)$.*

Proof Let A be a connection set of Γ such that $A \subseteq E_m$. We obtain by Lemma 3.6 that $\mathcal{B}$ is an independent dominating set of Γ. Consequently, $\gamma_i(\Gamma) \leq |\mathcal{B}| = \gamma(\Gamma)$ by Theorem 3.4. Moreover, since $\gamma(\Gamma) \leq \gamma_i(\Gamma)$ in general, immediately we have $\gamma_i(\Gamma) = \gamma(\Gamma)$, as required. □

Next, we consider the following lemma, which is useful for studying the connected domination number of Γ where a connection set is contained in E_m.

Lemma 3.8 *Let $A \subseteq E_m$ be a connection set of Γ. If $\mathcal{A} \subsetneq E_m$, then $N^-(\alpha) = \emptyset$ for each $\alpha \in E_m\backslash\mathcal{A}$.*

Proof Assume that $\mathcal{A} \subsetneq E_m$, and let $\alpha \in E_m\backslash\mathcal{A}$. Suppose to the contrary that there exists $\beta \in N^-(\alpha)$. Then $\beta \neq \alpha$ and $(\beta, \alpha) \in E(\Gamma)$, that is, $\alpha = \beta\mu$ for some $\mu \in A \subseteq E_m$. We first show that $(X\backslash Y)\alpha \cap (X\backslash Y)\mu \neq \emptyset$. If $x\beta \in Y$ for all $x \in X\backslash Y$, then $x\alpha = x\beta\mu = (x\beta)\mu = x\beta$, and by the definition of $Fix(X,Y)$, we have $y\alpha = y = y\beta$ for each $y \in Y$. This implies that $\alpha = \beta$, which is a contradiction. So, there exists $x \in X\backslash Y$ in which $x\beta \in X\backslash Y$. Consequently, $x\alpha = x\beta\mu = (x\beta)\mu \in (X\backslash Y)\alpha \cap (X\backslash Y)\mu$. It follows that $(X\backslash Y)\alpha \cap (\bigcup\limits_{\delta \in A}(X\backslash Y)\delta) \neq \emptyset$ and so $\alpha \in \mathcal{A}$, which contradicts to the assumption. Therefore, $N^-(\alpha) = \emptyset$. □

The following theorem provides some results on the connected domination number γ_c of Γ with respect to a connection set $A \subseteq E_m$.

Theorem 3.9 *Let A be a connection set of Γ contained in E_m. Then, the following statements hold.*

(i) *A connected dominating set of Γ exists if and only if Γ is connected.*

(ii) *If a connected dominating set of* Γ *exists, then* $\gamma_c(\Gamma) > \gamma(\Gamma)$.

Proof (i) Clearly, if Γ is a connected digraph, then a connected dominating set of Γ always exists since $Fix(X,Y)$ is a connected dominating set of Γ. Conversely, assume that Γ contains a connected dominating set, say U. To prove the connectedness of Γ, let $\alpha, \beta \in Fix(X,Y)$ be distinct vertices. Consider the following three cases.

Case 1: Let $\alpha, \beta \in U$. Since U is a connected dominating set, the subdigraph of Γ induced by U is connected. It follows that there exists a weakly dipath joining between α and β.

Case 2: Let $\alpha, \beta \notin U$. Then, there are $\lambda, \delta \in U$ such that λ and δ dominate α and β, respectively. That means $(\lambda, \alpha), (\delta, \beta) \in E(\Gamma)$. By the connectedness of the subdigraph of Γ induced by U, there exists a weakly dipath joining λ and δ. Thus, α and β are connected by a weakly dipath in Γ.

Case 3: Without loss of generality, assume that $\alpha \in U$ and $\beta \notin U$. By applying the concept of Case 2 and considering α instead of λ, we obtain that Γ contains a weakly dipath joining α and β including δ, which is the vertex that dominates β.

From the above three cases, we can conclude that Γ is a connected digraph.

(ii) Assume that U is a connected dominating set of Γ in which $|U| = \gamma_c(\Gamma)$. Clearly, U is a dominating set of Γ. By Lemma 3.1 (ii), we get that $Fix(X,Y)\backslash E_m \subseteq U$. By considering the definition of the set $\mathcal{A}$, we have that $Fix(X,Y)\backslash E_m \subseteq Fix(X,Y)\backslash \mathcal{A} = \mathcal{B}$, which is an independent set of Γ by Lemma 3.6. Hence, a subdigraph of Γ induced by $Fix(X,Y)\backslash E_m$ is disconnected. Since U is a connected dominating set of Γ, we conclude that U contains another element in E_m. If $\mathcal{A} \subsetneq E_m$, then $N^+(\mu) = \emptyset = N^-(\mu)$ for each $\mu \in E_m \backslash \mathcal{A}$ by Lemmas 2.1 and 3.8. That means the subdigraphs of Γ induced by such elements μ are loops. Thus, those elements μ must belong to U since U is a dominating set. However, the subdigraph of Γ induced by $Fix(X,Y)\backslash \mathcal{A}$ is also disconnected. Consequently,

$$\gamma_c(\Gamma) = |U| > |Fix(X,Y)\backslash \mathcal{A}| = |Fix(X,Y)| - |\mathcal{A}| = \gamma(\Gamma).$$

□

We now continue with the results of the split domination number $\gamma_s(\Gamma)$ of Γ with respect to a connection set $A \subseteq E_m$. Before that, we need the following lemmas.

Lemma 3.10 *Let* $A \subseteq E_m$ *be a connection set of* Γ. *If a split dominating set of* Γ *exists, then* $|Y| \geq 2$.

Proof Let U be a split dominating set of Γ and suppose that $|Y| = 1$. It is clear that $|E_m| = 1$, which yields $|A| = 1$. For convenience, let $A = \{\alpha\} = E_m$. By Lemma 3.1 (ii), we have $Fix(X,Y)\backslash E_m \subseteq U$. If $\alpha \in U$, then $U = Fix(X,Y)$, which is impossible by the definition of a split dominating set. This implies that $\alpha \notin U$, that is, $Fix(X,Y)\backslash U = \{\alpha\}$. It follows that $\Gamma[Fix(X,Y)\backslash U] = \Gamma[\{\alpha\}]$ is connected, which contradicts to the property of a split dominating set. Consequently, $|Y| \geq 2$. □

As a consequence of Lemma 3.10, we obtain that if $|Y| = 1$, then a split dominating set of Γ does not exist. Therefore, we will consider the case $|Y| \geq 2$ for investigating a split dominating set of Γ.

Lemma 3.11 *Let* $|Y| \geq 2$ *and* $A \subseteq E_m$ *be a connection set of* Γ. *Then,* $\mathcal{B}$ *is a split dominating set of* Γ *if one of the following statements hold.*

(i) $|A| = 1$ *and* $|X \backslash Y| \geq 2$.

(ii) $|A| \geq 2$.

Proof Assume that the condition holds. We now consider the following two cases.

Case 1: Let $|A| = 1$ and $|X \backslash Y| \geq 2$. Assume that $A = \{\alpha\} \subseteq E_m$. We can write

$$\alpha = \begin{pmatrix} A_i \\ a_i \end{pmatrix}.$$

We get that there exist b_j and b_k in $X \backslash Y$. Now, we define $\beta \in E_m$ such that $\beta \neq \alpha$ and $(X \backslash Y)\beta \cap (X \backslash Y)\alpha \neq \emptyset$.

If $b_j, b_k \in A_{i_0}$ for some $i_0 \in I$, then $b_j\alpha = a_{i_0} = b_k\alpha$ and we can choose a_{i_1} since $|Y| \geq 2$. Define $\beta \in E_m$ by

$$\beta = \begin{pmatrix} A_{i_0} \backslash \{b_j\} & A_{i_1} \cup \{b_j\} & A_{i'} \\ a_{i_0} & a_{i_1} & a_{i'} \end{pmatrix}$$

where $I' = I \backslash \{i_0, i_1\}$. We obtain that $b_k\alpha = a_{i_0} = b_k\beta \in (X \backslash Y)\beta \cap (X \backslash Y)\alpha$.

If $b_j \in A_{i_0}$ and $b_k \in A_{i_1}$ for some $i_0, i_1 \in I$, then we can define $\beta \in E_m$ by

$$\beta = \begin{pmatrix} (A_{i_0} \backslash \{b_j\}) \cup \{b_k\} & (A_{i_1} \backslash \{b_k\}) \cup \{b_j\} & A_{i'} \\ a_{i_0} & a_{i_1} & a_{i'} \end{pmatrix}$$

where $I' = I \backslash \{i_0, i_1\}$. We obtain that $b_j\alpha = a_{i_0} = b_k\beta \in (X \backslash Y)\beta \cap (X \backslash Y)\alpha$.

It follows that $\beta \in \mathcal{A}$. Actually, we can observe that $A \subseteq \mathcal{A}$ and so $|\mathcal{A}| \geq 2$. Further, we have by Lemma 3.3 that $\mathcal{B}$ is a dominating set of Γ, and $\Gamma[Fix(X,Y) \backslash \mathcal{B}] = \Gamma[\mathcal{A}]$ is the union of loops since $N^+(\mu) = \emptyset$ for all $\mu \in E_m$ by Lemma 2.1. Therefore, $\mathcal{B}$ is a split dominating set of Γ.

Case 2: Let $|A| \geq 2$. Hence $|\mathcal{A}| \geq 2$. By the similar argument stated in Case 1, we also obtain that $\mathcal{B}$ is a split dominating set of Γ. □

Theorem 3.12 *Let* $|Y| \geq 2$ *and* $A \subseteq E_m$ *be a connection set of* Γ. *Then the following statements hold.*

(i) *A split dominating set of* Γ *exists if and only if either* ($|A| = 1$ *and* $|X \backslash Y| \geq 2$) *or* $|A| \geq 2$.

(ii) *If a split dominating set of* Γ *exists, then* $\gamma_s(\Gamma) = \gamma(\Gamma)$.

Proof (i) Assume that a split dominating set of Γ exists, say U. Suppose to the contrary that $|A| = 1$ and $|X \backslash Y| = 1$. For convenience, let $A = \{\alpha\} \subseteq E_m$. Clearly, $Fix(X,Y) = E_m \cup \{id_X\}$ and $E(\Gamma) = \{(\mu,\mu) : \mu \in E_m\} \cup \{(id_X, \alpha)\}$. For each $\mu \in E_m \backslash \{\alpha\}$, we can observe that there is no $\beta \in Fix(X,Y) \backslash \{\mu\}$ such that β can dominate μ. Hence $E_m \backslash \{\alpha\} \subseteq U$ since U is a dominating set of Γ. Moreover, we have $\{id_X\} = Fix(X,Y) \backslash E_m \subseteq U$ by Lemma 3.1 (ii). Now, we have $Fix(X,Y) \backslash \{\alpha\} \subseteq U$. This implies that $U = Fix(X,Y) \backslash \{\alpha\}$ and so $\Gamma[Fix(X,Y) \backslash U] = \Gamma[\{\alpha\}]$ is connected, which contradicts to the property of a split dominating set U. The converse is clear by Lemma 3.11.

(ii) Assume that a split dominating set of Γ exists. By (i) and Lemma 3.11, we obtain that $\mathcal{B}$ is a split dominating set of Γ. Therefore, $\gamma_s(\Gamma) \leq |\mathcal{B}| = \gamma(\Gamma)$ by Lemma 3.4. In general, $\gamma(\Gamma) \leq \gamma_s(\Gamma)$, which implies that $\gamma_s(\Gamma) = \gamma(\Gamma)$, as required. □

4. Domination parameters of Γ related to permutations

In this section, we study domination parameters on a Cayley digraph Γ of $Fix(X,Y)$ whose connection set A is contained in the group H_{id_X} of permutations in $Fix(X,Y)$. We start with the following theorem, which presents the characterization of Γ.

Theorem 4.1 *The Cayley digraph Γ is the disjoint union of subdigraphs induced by E_m, H_{id_X} and $Fix(X,Y)\backslash(E_m \cup H_{id_X})$, respectively if and only if $A \subseteq H_{id_X}$.*

Proof Let A be a connection set of Γ contained in H_{id_X}. Further, let $(\alpha,\beta) \in E(\Gamma)$ be arbitrary. Then, $\beta = \alpha\mu$ for some $\mu \in A$. We consider the following three cases.

Case 1: Let $\alpha \in E_m$. Clearly, $\beta = \alpha \in E_m$. This implies that (α,β) is a loop in $\Gamma[E_m]$.

Case 2: Let $\alpha \in H_{id_X}$. Since $\mu \in A \subseteq H_{id_X}$ and H_{id_X} is a group, we have $\beta = \alpha\mu \in H_{id_X}$. Thus (α,β) is an edge in $\Gamma[H_{id_X}]$.

Case 3: Let $\alpha \notin E_m \cup H_{id_X}$. Then there exist $x, z \in X$ where $x \neq z$ such that $x\alpha = z\alpha$. Thus,

$$x\beta = x(\alpha\mu) = (x\alpha)\mu = (z\alpha)\mu = z(\alpha\mu) = z\beta.$$

This implies that $\beta \notin H_{id_X}$. Furthermore, since $\alpha \notin E_m$, there exists $w \in X\backslash Y$ in which $w\alpha \in X\backslash Y$. Hence $w\beta = w(\alpha\mu) = (w\alpha)\mu \in X\backslash Y$ as μ is a permutation on X. It follows that $\beta \notin E_m$. Consequently, $\beta \notin E_m \cup H_{id_X}$, which implies that (α,β) is an edge in $\Gamma[Fix(X,Y)\backslash(E_m \cup H_{id_X})]$.

From the above three cases, we can conclude that Γ is the disjoint union of subdigraphs induced by E_m, H_{id_X} and $Fix(X,Y)\backslash(E_m \cup H_{id_X})$.

Conversely, assume that the condition holds. Suppose to the contrary that there exists $\alpha \in A\backslash H_{id_X}$. Clearly, $(id_X, \alpha) \in E(\Gamma)$. We now observe that $\alpha \notin V(\Gamma[H_{id_X}])$ while $id_X \in V(\Gamma[H_{id_X}])$. This contradicts to the assumption of the disjoint union property of Γ. Therefore, $A \subseteq H_{id_X}$, as required. □

We now present certain prominent property of a dominating set of Γ as follows.

Lemma 4.2 *Let A be a connection set of Γ contained in H_{id_X}. If U is a dominating set of Γ, then $E_m \subseteq U$.*

Proof Let U be a dominating set of Γ, and let $\alpha \in E_m$. Suppose that $\alpha \notin U$. Since U is a dominating set of Γ, there exists $\beta \in U$ such that $(\beta,\alpha) \in E(\Gamma)$. Thus $\alpha = \beta\mu$ for some $\mu \in A \subseteq H_{id_X}$. As α is a left zero element of $Fix(X,Y)$, it follows that $\beta = \alpha\mu^{-1} = \alpha$, which is a contradiction. Therefore, $\alpha \in U$, which implies that $E_m \subseteq U$. □

As the fact that the identity $id_X \in Fix(X,Y)$ always induces a loop attached to every element in $Fix(X,Y)$ whenever $id_X \in A$. Thus, in order to present the bounds for the domination number of Γ, we consider the connection set A in which $id_X \notin A$ as follows.

Theorem 4.3 *Let A be a connection set of Γ contained in $H_{id_X}\backslash\{id_X\}$. Then,*

$$\frac{|A||Y|^{|X|-|Y|} + |X|^{|X|-|Y|}}{|A|+1} \leq \gamma(\Gamma) \leq \min\left\{|X|^{|X|-|Y|} - |A|, |X|^{|X|-|Y|} - \frac{(|X|-|Y|)!}{2}\right\}. \qquad (\star)$$

Proof Let $\alpha \in E_m$. If $(\beta, \alpha) \in E(\Gamma[E_m])$ for some $\beta \in E_m$, then $\alpha = \beta\mu$ for some $\mu \in A$. We have $\alpha = \beta$ since β is a left zero element. This implies that $N^-(\alpha) \cap E_m = \emptyset$. Hence $\gamma(\Gamma[E_m]) = |E_m|$. By Theorem 4.1, we see that Γ is the disjoint union of $\Gamma[E_m]$ and $\Gamma[Fix(X,Y)\backslash E_m]$. We conclude that $\gamma(\Gamma) = \gamma(\Gamma[E_m]) + \gamma(\Gamma[Fix(X,Y)\backslash E_m]) = |E_m| + \gamma(\Gamma[Fix(X,Y)\backslash E_m])$. Thus we need to consider the domination number of $\Gamma[Fix(X,Y)\backslash E_m]$.

We first investigate the lower bound of $\gamma(\Gamma[Fix(X,Y)\backslash E_m])$. Actually, each vertex of $\Gamma[Fix(X,Y)\backslash E_m]$ can dominate at most itself and other $|A|$ vertices. Hence

$$\gamma(\Gamma[Fix(X,Y)\backslash E_m]) \geq \frac{|Fix(X,Y)\backslash E_m|}{|A|+1} = \frac{|Fix(X,Y)| - |E_m|}{|A|+1} = \frac{|X|^{|X|-|Y|} - |Y|^{|X|-|Y|}}{|A|+1}.$$

For the upper bound of $\gamma(\Gamma[Fix(X,Y)\backslash E_m])$, we consider the fact that if $\alpha \in H_{id_X}$ is a given vertex of $\Gamma[Fix(X,Y)\backslash E_m]$, then all vertices in the neighborhood $N(\alpha) = \{\beta \in Fix(X,Y) : (\alpha,\beta) \in E(\Gamma[Fix(X,Y)\backslash E_m])\}$ of α will be dominated by α. It follows that $Fix(X,Y)\backslash(E_m \cup N(\alpha))$ is a dominating set of $\Gamma[Fix(X,Y)\backslash E_m]$. Thus $\gamma(\Gamma[Fix(X,Y)\backslash E_m]) \leq |Fix(X,Y)\backslash(E_m \cup N(\alpha))|$. Since $\alpha\mu \neq \alpha\eta$ for every distinct $\mu, \eta \in A \subseteq H_{id_X}\backslash\{id_X\}$, we have $|N(\alpha)| = |A|$. As $\alpha \in H_{id_X}$ and $A \subseteq H_{id_X}$, we get that $N(\alpha) \subseteq H_{id_X}$, which implies that $N(\alpha) \cap E_m = \emptyset$. We conclude that

$$\gamma(\Gamma[Fix(X,Y)\backslash E_m]) \leq |Fix(X,Y)\backslash(E_m \cup N(\alpha))| = |Fix(X,Y)| - |E_m| - |N(\alpha)| = |X|^{|X|-|Y|} - |Y|^{|X|-|Y|} - |A|.$$

Furthermore, let $\delta \in A$ be fixed. For each $\lambda \in H_{id_X}$, we observe that λ can dominate $\lambda\delta$, and $\lambda_1\delta \neq \lambda_2\delta$ for $\lambda_1 \neq \lambda_2$ in H_{id_X}. Hence those vertices λ together with all vertices in $Fix(X,Y)\backslash(E_m \cup H_{id_X})$ form a dominating set of $\Gamma[Fix(X,Y)\backslash E_m]$ with cardinality

$$\begin{aligned} N &= \left\lceil \frac{|H_{id_X}|}{2} \right\rceil + |Fix(X,Y)| - |H_{id_X}| - |E_m| \\ &= \left\lceil \frac{(|X|-|Y|)!}{2} \right\rceil + |X|^{|X|-|Y|} - (|X|-|Y|)! - |Y|^{|X|-|Y|}. \end{aligned}$$

Since $\emptyset \neq A \subseteq H_{id_X}\backslash\{id_X\}$, we obtain that $|X| - |Y| \geq 2$, which implies that $(|X|-|Y|)!$ is even. It follows that

$$\begin{aligned} N &= \frac{(|X|-|Y|)!}{2} + |X|^{|X|-|Y|} - (|X|-|Y|)! - |Y|^{|X|-|Y|} \\ &= |X|^{|X|-|Y|} - \frac{(|X|-|Y|)!}{2} - |Y|^{|X|-|Y|}. \end{aligned}$$

Consequently, $\gamma(\Gamma[Fix(X,Y)\backslash E_m]) \leq N$. Therefore,

$$\gamma(\Gamma[Fix(X,Y)\backslash E_m]) \leq \min\left\{|X|^{|X|-|Y|} - |Y|^{|X|-|Y|} - |A|, N\right\}.$$

By adding the value $|E_m| = |Y|^{|X|-|Y|}$ into both sides of inequalities for $\gamma(\Gamma[Fix(X,Y)\backslash E_m])$, the required bounds for $\gamma(\Gamma)$ stated as $(\star)$ are obtained. □

We now consider the Cayley digraph of $Fix(X,Y)$ related to a special connection set, that is, $A = H_{id_X}$ for illustrating the result of the domination number more clearly. In order to study certain structural properties of such digraphs, we need the following prescription.

For $\alpha \in Fix(X,Y)$, the symbol π_α denotes the partition of X induced by the map α, which is well known as a set of kernel classes of α, namely,

$$\pi_\alpha = \{x\alpha^{-1} : x \in X\alpha\}.$$

Then, we have $|\pi_\alpha| = |X\alpha|$.

Note that Γ is said to be *complete* if $(\alpha,\beta) \in E(\Gamma)$ for any $\alpha,\beta \in Fix(X,Y)$. In order to present the following proposition, we will mention some notations as follows. Let $\mathcal{R}$ be one of Green's relations on a semigroup. It is well known that $\mathcal{R}$ is an equivalence relation on a semigroup. In [11], Honyam and Sanwong characterized the Green's relation $\mathcal{R}$ on $Fix(X,Y)$ as follows. For each $\alpha,\beta \in Fix(X,Y)$,

$$\alpha\mathcal{R}\beta \text{ if and only if } \pi_\alpha = \pi_\beta.$$

Therefore, for each $\alpha \in Fix(X,Y)$, the Green's $\mathcal{R}$-class of $Fix(X,Y)$ containing α is

$$R_\alpha = \{\beta \in Fix(X,Y) : \pi_\beta = \pi_\alpha\}.$$

Proposition 4.4 *Let $A = H_{id_X}$ be a connection set of Γ. Then the following statements hold.*

(i) *For $\alpha,\beta \in Fix(X,Y)$, $(\alpha,\beta) \in E(\Gamma)$ if and only if $\pi_\alpha = \pi_\beta$.*

(ii) *Γ is the disjoint union of $\Gamma[R_\alpha]$ where $\alpha \in Fix(X,Y)$.*

(iii) *For each $\alpha \in Fix(X,Y)$, an induced subdigraph $\Gamma[R_\alpha]$ is complete.*

Proof (i) Let $\alpha,\beta \in Fix(X,Y)$. Assume that $(\alpha,\beta) \in E(\Gamma)$. Then $\beta = \alpha\mu$ for some $\mu \in A$. Moreover, μ^{-1} exists in A, which leads to $\alpha = \beta\mu^{-1}$, that is, $(\beta,\alpha) \in E(\Gamma)$. Now, let $P \in \pi_\alpha$ and $x,y \in P$. Thus $x\alpha = y\alpha$, which implies that $x\beta = x(\alpha\mu) = (x\alpha)\mu = (y\alpha)\mu = y(\alpha\mu) = y\beta$. Furthermore, if $x\beta = y\beta$, then $x\alpha = x(\beta\mu^{-1}) = (x\beta)\mu^{-1} = (y\beta)\mu^{-1} = y(\beta\mu^{-1}) = y\alpha$. We obtain that $P \in \pi_\beta$. Thus, $\pi_\alpha \subseteq \pi_\beta$. The other containment can be proved, similarly. Consequently, $\pi_\alpha = \pi_\beta$.

Conversely, suppose that $\pi_\alpha = \pi_\beta = \{A_i : i \in I\} \cup \{B_j : j \in J\}$ where $a_i\alpha^{-1} = A_i = a_i\beta^{-1}$ for each $i \in I$ and $b_j\alpha^{-1} = B_j = c_j\beta^{-1}$ for each $j \in J$. Then, we can write

$$\alpha = \begin{pmatrix} A_i & B_j \\ a_i & b_j \end{pmatrix} \text{ and } \beta = \begin{pmatrix} A_i & B_j \\ a_i & c_j \end{pmatrix}.$$

Let $X\backslash X\alpha = \{x_k : k \in K\}$ and $X\backslash X\beta = \{x'_k : k \in K\}$ for some index set K. Since $|X\backslash X\alpha| = K = |X\backslash X\beta|$, there exists a bijection $\varphi : X\backslash X\alpha \to X\backslash X\beta$. Define $\mu \in Fix(X,Y)$ by

$$\mu = \begin{pmatrix} a_i & b_j & x_k \\ a_i & c_j & x_k\varphi \end{pmatrix}.$$

It follows that $\mu \in H_{id_X}$ and $\beta = \alpha\mu$. This implies that $(\alpha,\beta) \in E(\Gamma)$, immediately.

(ii) It is well known that the set of all $\mathcal{R}$-classes forms a partition of $Fix(X,Y)$. Clearly, each directed edge of Γ joins only vertices in the same $\mathcal{R}$-class by (i). Hence the result follows.

(iii) Let $\alpha \in Fix(X,Y)$. Clearly, for each $\lambda,\delta \in R_\alpha$, we have that $\pi_\lambda = \pi_\alpha = \pi_\delta$ and we conclude by (i) that $(\lambda,\delta) \in E(\Gamma)$. Hence $(\lambda,\delta) \in E(\Gamma[R_\alpha])$, as well. Therefore, an induced subdigraph $\Gamma[R_\alpha]$ of Γ is complete. □

In combinatorics, the Stirling number $S(n,k)$ of the second kind is a well-known parameter for counting the number of ways to partition a set of n objects into k nonempty subsets. The explicit formula of $S(n,k)$ is expressed as

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n.$$

Surprisingly, whenever the connection set A of Γ equals H_{id_X}, the domination number $\gamma(\Gamma)$ is concerned with the Stirling number of the second kind as shown in the following theorem.

Theorem 4.5 *Let $A = H_{id_X}$ be a connection set of Γ. Then*

$$\gamma(\Gamma) = \sum_{k=|Y|}^{|X|} S(|X|,k).$$

Proof We can conclude by Proposition 4.4 (ii) and (iii) that Γ is the disjoint union of complete subdigraphs induced by R_α mentioned in the above proposition where $\alpha \in Fix(X,Y)$. As the fact that the number of ways to partition the set X into k nonempty subsets is equal to $S(|X|,k)$ and the possibilities of k are $|Y|, |Y|+1, |Y|+2, \ldots, |X|$, it follows that

$$\gamma(\Gamma) = \sum_{k=|Y|}^{|X|} S(|X|,k)$$

with the dominating set containing one vertex from each complete subdigraph of Γ. □

We continue with the result of a connected dominating set of Γ as follows.

Theorem 4.6 *If $A \subseteq H_{id_X}$ is a connection set of Γ, then a connected dominating set of Γ does not exist.*

Proof Let A be a connection set of Γ contained in H_{id_X}. Since we consider the case $X \neq Y$ in this paper, clearly $|Fix(X,Y)| \geq 2$ with at least one minimal idempotent, say μ. By Lemma 4.2, we obtain that μ must be contained in every dominating set of Γ. Since $A \subseteq H_{id_X}$, we get that μ induces a loop attached to itself and $N^-(\mu) = \emptyset = N^+(\mu)$. Hence an induced subdigraph of Γ containing μ as its vertex must be disconnected. Therefore, a connected dominating set of Γ does not exist, certainly. □

Now, we characterize an existence of a split dominating set of Γ with respect to a connection set $A = H_{id_X}$ as follows.

Lemma 4.7 *Let $A = H_{id_X}$ be a connection set of Γ. Then a split dominating set of Γ exists if and only if $|X \backslash Y| \geq 2$.*

Proof Let U be a split dominating set of Γ. Suppose that $|X\backslash Y| = 1$. It follows that $A = H_{id_X} = \{id_X\}$. Then Γ is the disjoint union of loops, which implies that $U = Fix(X,Y)$. This contradicts to the definition of a split dominating set. Hence $|X\backslash Y| \geq 2$.

Conversely, assume that $|X\backslash Y| \geq 2$. Let $b_{j_0}, b_{j_1} \in X\backslash Y$. Consider $\alpha, \beta, \lambda \in Fix(X,Y)$ as follows:

$$\alpha = \begin{pmatrix} a_i & X\backslash Y \\ a_i & b_{j_0} \end{pmatrix},\ \beta = \begin{pmatrix} a_i & X\backslash Y \\ a_i & b_{j_1} \end{pmatrix} \text{ and } \lambda = \begin{pmatrix} a_i & b_{j'} & b_{j_0} & b_{j_1} \\ a_i & b_{j'} & b_{j_1} & b_{j_0} \end{pmatrix}$$

where $J' = J\backslash\{j_0, j_1\}$. We observe that $\beta \in R_\alpha$ and $\lambda \in R_{id_X}$. By Proposition 4.4 (ii) and (iii), we obtain that Γ is the disjoint of complete subdigraphs induced by the Green's $\mathcal{R}$-class. By choosing one vertex from each complete subdigraph, and let U be the set of such vertices, we obtain that $\Gamma[Fix(X,Y)\backslash U]$ is disconnected. Hence U is a split dominating set of Γ. $\square$

Finally, we present the sufficient condition to imply that the parameters $\gamma(\Gamma), \gamma_t(\Gamma), \gamma_i(\Gamma)$ and $\gamma_s(\Gamma)$ are equal.

Theorem 4.8 *Let $A = H_{id_X}$ be a connection set of Γ. If $|X\backslash Y| \geq 2$, then $\gamma(\Gamma) = \gamma_t(\Gamma) = \gamma_i(\Gamma) = \gamma_s(\Gamma)$.*

Proof Let $|X\backslash Y| \geq 2$. By Lemma 4.7, we have that a split dominating set of Γ exists. By Proposition 4.4 (ii) and (iii), we obtain that Γ is the disjoint union of complete subdigraphs. By choosing one vertex from each complete subdigraph, and by letting U be the set of such vertices, we obtain that U is a split dominating set and an independent dominating set of Γ. Hence $\gamma_s(\Gamma) \leq |U|$ and $\gamma_i(\Gamma) \leq |U|$. Actually, $|U| = \sum_{k=|Y|}^{|X|} S(|X|, k) = \gamma(\Gamma)$ by Theorem 4.5. So we conclude that $\gamma_i(\Gamma) = \gamma(\Gamma) = \gamma_s(\Gamma)$. Moreover, since $id_X \in A$, we obtain that $(\alpha, \alpha) \in E(\Gamma)$ for all $\alpha \in U$. Therefore, U forms a total dominating set of Γ and thus $\gamma_t(\Gamma) \leq |U|$. Consequently, $\gamma(\Gamma) = \gamma_t(\Gamma) = \gamma_i(\Gamma) = \gamma_s(\Gamma)$, as required. $\square$

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