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C_{11}-modules via left exact preradicals

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Abstract: In this article, we study modules with the condition that every image of a submodule under a left exact preradical has a complement which is a direct summand. This new class of modules properly contains the class of C_{11}-modules (and hence also CS-modules). Amongst other structural properties, we deal with direct sums and decompositions with respect to the left exact preradicals of this new class of modules. It is obtained a decomposition such that the image of the module itself is a direct summand for the left exact radical, which enjoys the new condition.

Key words: Left exact preradical, complement submodule, Goldie torsion submodule, CS-module, C_{11}-module

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary right modules. Let R be a ring and let M be an R-module. A submodule N of M is *essential* (or *large*) in M if for every $0 \neq K$ submodule of M, we have $N \cap K \neq 0$. Given a submodule C of M, by a *complement* (submodule) of C in M, we mean a submodule D of M, maximal with respect to the property $C \cap D = 0$. A submodule, which is a complement of a submodule in M is called a *complement* in M. Let N be a submodule of M. A complement (submodule) K in M is called the *closure of* N *in* M provided that N is essentially contained in K. Note that, a closure of a submodule need not be unique. However, if the module is nonsingular then every submodule has a unique-closure (see, [6, 14]).

Recall that a module is said to be CS (or *extending*) or said to satisfy the C_1 condition if every submodule is essential in a direct summand. Equivalently, every complement is a direct summand (see, [4, 14]). Extending modules and their generalizations play an important role in modules and rings. To this end, several generalizations of CS notion have been worked out extensively by many authors (see, for example [1, 5, 8–10, 12–14]). This kind of investigations are traced back to the theory of C_{11}-modules as well as C_{11}-rings. A module M is called C_{11}-module (or satisfies C_{11}) if every submodule has a complement in M which is a direct summand of M [9, 10]. In this trend, as the first attempt, Tercan [13] defined ES-module notion as a generalization of CLS-modules (so, CS-modules) in terms of left exact preradicals, for a ring R [13]. M is called an ES-module provided that every exact submodule is a direct summand of M. Since left exact preradicals are main tools in this work, it would be better to give some information about them. Recall that a functor r from the category of right R-modules to itself is called a *left exact preradical* if it has the following

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properties:

(i) $r(M)$ is a submodule of M for every right R-module M,

(ii) $r(N) = N \cap r(M)$ for every submodule N of a right R-module M, and

(iii) $\varphi(r(M)) \subseteq r(M')$ for every homomorphism $\varphi : M \to M'$, for right R-modules M, M'.

Let r be a left exact preradical in the category of R-modules. Amongst foregoing properties, $r(M_1 \oplus M_2) = r(M_1) \oplus r(M_2)$ holds true for all right R-modules M_1, M_2. Furthermore, r is called a *radical* if $r(M/r(M)) = 0$ for every right R-module M. It is clear that the singular submodule and socle are left exact preradicals, and the second singular submodule (or Goldie torsion submodule) is a radical. For an excellent treatment of left exact preradicals, the reader is referred to [11].

In this paper, first of all we mention some basic information on ES-modules and some related modules in literature. Then, we define rC_{11}*-modules* and investigate their structural properties. In particular, we think of direct sums and direct decompositions of such modules for a left exact preradical in the category of right R-modules. We reduce our consideration for a left exact radical whenever we need to have additional properties. Since any result including a left exact preradical in the category of right R-modules constructs a framework, our results can be applied directly to a right module with its fundamental submodules like socle, Goldie torsion submodule, etc.

We use r to signify a left exact preradical in the category of right R-modules. Moreover, let M be a right R-module. Then $N \leq M$, $SocM$ and $Z_2(M)$ will denote N is a submodule of M, socle of M and second singular submodule (or Goldie torsion submodule) of M, respectively. For any other terminology or unexplained definitions, we refer to [4, 6, 11, 14, 15].

2. Some remarks on ES-modules

In this section, we deal with basic observations on ES-modules and related concepts. Let r be a left exact radical in the category of right R-modules and let M be any R-module. Let us call M r_c*-module* if every exact submodule of M is a complement in M. In other words, for every submodule N of M, $r(M/N) = 0$ implies that N is a complement in M. For example, if $r = Soc$ then r_c-module and C-module definitions coincide (see [5]). Then, we have the following straightforward observation.

Lemma 2.1 *If M_R is a r_c-module with CS property then M is an ES-module.*

Proof Let N be any exact submodule of M. By hypothesis, N is a complement and hence a direct summand of M. □

Modified proof of [5, Proposition 3.11] gives the subsequent general result on r_c-modules.

Proposition 2.2 *r_c-modules are closed under quotients.*

Proof Let M be a r_c-module and N a submodule of M. Let us show that M/N is a r_c-module. For this aim, assume that there is an exact submodule K/N in M/N, which is not complement in M/N where $N \leq K \leq M$. Then, $r((M/N)/(K/N)) \cong r(M/K) = 0$, and there is a submodule L/N in M/N such that K/N is essential in L/N where $K \leq L \leq M$. Since M is an r_c-module and $r(M/K) = 0$, K is a complement in M. In consideration of K/N is essential in L/N, K is an essential submodule of L, a contradiction. It follows that M/N is a r_c-module. □

Corollary 2.3 *Suppose that M_1, M_2 are r_c-modules with CS property. If M is a direct sum $M_1 \oplus M_2$ of M_1, M_2 such that M_1 is M_2-injective then M is an ES-module.*

Proof By Lemma 2.1, M_1 and M_2 are ES-modules. Now [13, Theorem 6] yields that M is an ES-module. □

It has come to our attention that the following general results in [13, Lemma 4, Theorem 9] has been missed out by the some other authors (see [1, 5]). It seems that there is no direct way to achieve the latter paper. To this end, it is better to mention these results in [13] without their proofs for preserving the completeness of future works.

Lemma 2.4 *([13, Lemma 4]) Any direct summand of an ES-module is an ES-module.*

Theorem 2.5 *([13, Theorem 9]) Let R be a ring and let r be the left exact radical for a stable hereditary torsion theory for the category of right R-modules. Then, a right R-module M is an ES-module if and only if $M = r(M) \oplus M'$ for some submodule M' of M and both $r(M)$ and M' are ES-modules.*

By the aforementioned results, as special cases [5, Proposition 3.9 and Theorem 3.12] can be obtained. We give the following easy example which shows that CS property does not imply the condition worked as d-extending in [5].

Example 2.6 *Let D be a commutative ring with $SocR = 0$ and let M be a left faithful simple D-module. Let R be the trivial extension ring of D with M, i.e.*

$$R = \begin{bmatrix} D & \diagdown & M \\ 0 & & D \end{bmatrix} = \left\{ \begin{bmatrix} d & m \\ 0 & d \end{bmatrix} : d \in D,\ m \in M \right\}.$$

Then, R is a commutative ring. Since M is a left faithful D-module, R is an indecomposable uniform R-module. Hence R is a right CS-module. Now, let $r = Soc$. So $SocR = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$. Define $\varphi : R \to D$ by $\varphi\left(\begin{bmatrix} d & m \\ 0 & d \end{bmatrix}\right) = d$. It is easy to check that φ is an epimorphism with $ker\varphi = SocR$. It follows that $R/SocR \cong D$. Thus $Soc(R/SocR) = 0$. Since R is indecomposable, $SocR$ is not a direct summand of R.

In a pattern by Example 2.6, we may get several same type examples.

3. rC_{11}-Modules

We introduce and investigate the rC_{11}-modules. To do this, we restrict our consideration on the definition of C_{11}-modules to a special type of submodules namely the class of submodules which consists of images of all submodules under a left exact preradical r in the category of right R-modules.

Definition 3.1 *A module M satisfies rC_{11} (or rC_{11}-module) if for each submodule N of M, there exists a direct summand K of M such that K is a complement of $r(N)$ in M.*

Lemma 3.2 *Let N be a submodule of M and let K be a direct summand of M. Then, K is a complement of N in M if and only if $K \cap N = 0$ and $K \oplus N$ is essential in M.*

Proof Immediate by definitions. □

Combining Definition 3.1 together with the previous lemma, we have the following useful characterization of rC_{11}-modules for a left exact preradical in the category of right R-modules.

Proposition 3.3 *The following conditions are equivalent.*

(i) M satisfies rC_{11}.

(ii) For any submodule N of M there exists a direct summand K of M such that $r(N) \cap K = 0$ and $r(N) \oplus K$ is essential in M.

Proof It follows from Lemma 3.2. □

It is clear from Proposition 3.3 that any C_{11}-module satisfies rC_{11}. In particular CS-modules (and hence uniform or injective modules) satisfy rC_{11}. It is well-known that any indecomposable module with C_{11} is uniform (see [10]). In contrast there are indecomposable rC_{11}-modules which are not uniform as the following example illustrates. This example also makes it clear that the class of C_{11}-modules is properly contained in the class of rC_{11}-modules.

Example 3.4 *(i) The Specker group $\prod_{i=1}^{\infty} \mathbb{Z}$ does not satisfy C_{11} but it satisfies rC_{11}. Let $r = Soc$ and let $M_{\mathbb{Z}} = \prod_{i=1}^{\infty} \mathbb{Z}$. Then M does not satisfy C_{11} [10, Lemma 3.4]. Note that $M_{\mathbb{Z}}$ is nonsingular from [6, Proposition 1.12]. Hence [6, Corollary 1.26] yields that $SocM_{\mathbb{Z}} = 0$. So $M_{\mathbb{Z}}$ is a rC_{11}-module.*
(ii) Let R be a principal ideal domain. If R is not a complete discrete valuation ring, then there exists an indecomposable torsion-free R-module M of rank 2 [7, Theorem 19]. For M, $SocM = 0$. Hence M satisfies rC_{11} with respect to $r = Soc$. However, M_R has uniform dimension 2. It follows that M does not satisfy C_{11}.
(iii) Let $\mathbb{R}$ be the real field and S the polynomial ring $\mathbb{R}[x,y,z]$. Then the ring $R = S/Ss$, where $s = x^2 + y^2 + z^2 - 1$, is a commutative Noetherian domain. Moreover, the free R-module $M = R \oplus R \oplus R$ contains a direct summand K which does not satisfy C_{11} [9]. Note that K_R is indecomposable with uniform dimension 2. Since $SocM = 0$, $Soc(K_R) = 0$. It follows that K_R is a rC_{11}-module with respect to $r = Soc$.

In a similar vein to Example 3.4(iii) we may have abundance of examples as follows. If $n \geq 3$ is any odd integer, S is the polynomial ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$ in the indeterminates $x_1, x_2, \ldots, x_n$ over $\mathbb{R}$, $s = x_1^2 + x_2^2 + \cdots + x_n^2 - 1$, and R is the commutative Noetherian domain S/Ss, then the free module $M = \bigoplus_{i=1}^{n} R$ has an indecomposable direct summand K with uniform dimension $n-1$ and $Soc(K_R) = 0$ (see also [14]).

Recall that, in contrast to CS-modules, any direct sum of modules with C_{11} is also a C_{11}-module [10, Theorem 2.4]. Natural question arises whether a direct sum of modules with rC_{11} is a rC_{11}-module. However, the left exact preradicals bring a framework which forces to take into account different types of submodules have the common property. Of course if $r(M) = 0$ then trivially M satisfies rC_{11}. So, we have the following fact.

Theorem 3.5 *Any direct sum of rC_{11}-modules with essential image under a left exact preradical satisfies rC_{11}.*

Proof Let $M_\lambda(\lambda \in \Lambda)$ be a non-empty collection of modules, each satisfying rC_{11} and having essential $r(M_\lambda)$. Let $\lambda \in \Lambda$. Let N be a submodule of M_λ. Note that $r(N) = N \cap r(M_\lambda)$ is essential in N. By rC_{11}, there exists a direct summand K of M_λ such that $r(N) \cap K = 0$ and $r(N) \oplus K$ is essential in M_λ. Now, we have that $r(N) \oplus K \leq N \oplus K \leq M_\lambda$. Since $r(N) \oplus K$ is essential in M_λ, $N \oplus K$ is essential in M_λ. It follows that each $M_\lambda(\lambda \in \Lambda)$ satisfies C_{11}. Then, by [10, Theorem 2.4], $\underset{\lambda \in \Lambda}{\oplus} M_\lambda$ satisfies rC_{11}.

□

One might expect that whether submodules of a rC_{11}-module need to be rC_{11}-module. However, any module, which does not satisfy rC_{11} (see, Example 3.9) is contained in a rC_{11}-module, namely its injective hull. Our next result and its corollary gather up certain classes of submodules of a rC_{11}-module, which satisfy rC_{11} property.

Proposition 3.6 *Let M be a rC_{11}-module and X a submodule of M. If the intersection of X with any direct summand of M is a direct summand of X, then X is a rC_{11}-module.*

Proof Let X be a submodule of M and let Y be a submodule of X. Now $r(Y)$ is a submodule of M. Then, there exists a direct summand D of M such that $r(Y) \cap D = 0$ and $r(Y) \oplus D$ is essential in M. By assumption, $X \cap D$ is a direct summand of X. Note that $r(Y) \cap (X \cap D) = 0$ and $X \cap (r(Y) \oplus D)$ is essential in X. By the modular law, $X \cap (r(Y) \oplus D) = r(Y) \oplus (X \cap D)$. It follows that X is a rC_{11}-module. □

Corollary 3.7 *Let M_R be a rC_{11}-module. If N is a submodule of M such that $f(N) \subseteq N$ where $f^2 = f \in End(M_R)$, then N is a rC_{11}-module.*

Proof Let N be a submodule of M such that $f(N) \subseteq N$ where $f^2 = f \in End(M_R)$. Let K be a direct summand of M. Consider $\pi : M \to K$ the canonical projection. Then, $\pi(r(N)) \subseteq r(N) \cap K$ is a direct summand of N. Hence N is a rC_{11}-module, by Proposition 3.6. □

It is obvious that Corollary 3.7 holds, in particular whenever we replace projection invariant submodule with fully invariant submodule in M.

Lemma 3.8 *Let M be a module which satisfies rC_{11}. Then $M = M_1 \oplus M_2$ where M_1 and M_2 are submodules such that $r(M_1)$ is essential in M_1 and $r(M_2) = 0$.*

Proof By Proposition 3.3, there exist submodules M_1, M_2 of M such that $M = M_1 \oplus M_2$, $r(M) \cap M_2 = 0$, and $r(M) \oplus M_2$ is an essential submodule of M. Since r is left exact, it follows that $r(M_2) = M_2 \cap r(M) = 0$. Let $\pi : M \to M_1$ denote the canonical projection. Then, $\pi(r(M)) \subseteq r(M_1)$. For any $0 \neq m \in M_1$, there exists $t \in R$ such that $0 \neq mt \in r(M) \oplus M_2$, and, hence, $0 \neq mt = \pi(mt) \in \pi(r(M)) \subseteq r(M_1)$. It follows that $r(M_1)$ is an essential submodule of M_1.

□

The converse of Lemma 3.8 is not true in general. On using $r = Soc$ and $r = Z_2$, we provide two examples, which are as follows:

Example 3.9 *(i) Let R be the trivial extension of the ring $\mathbb{Z}$ with the finite direct sum of $\mathbb{Z}$-module, $\overset{n}{\underset{i=1}{\oplus}}\mathbb{Z}$ where $n \geq 1$, i.e. $R = \begin{bmatrix} \mathbb{Z} & \overset{n}{\underset{i=1}{\oplus}}\mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix} = \left\{ \left[\begin{smallmatrix} n & m \\ 0 & n \end{smallmatrix}\right] : n \in \mathbb{Z},\ m \in \overset{n}{\underset{i=1}{\oplus}}\mathbb{Z} \right\}$. Now let $M_1 = R$ and $M_2 = R/I$ where $I = Soc(R) = \begin{bmatrix} 0 & \overset{n}{\underset{i=1}{\oplus}}\mathbb{Z} \\ 0 & 0 \end{bmatrix}$. Let $M = M_1 \oplus M_2$ and $r = Soc$. Then $Soc(M_1)$ is essential in M_1 and $Soc(M_2) = 0$. However it is easy to see that M is not rC_{11}-module.*

(ii) [3, Example 1.6]. Let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{bmatrix}$ be the trivial extension of $\mathbb{Z}$ and the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Let $I = \left\{ \begin{bmatrix} 4n & 0 \\ 0 & 4n \end{bmatrix} : n \in \mathbb{Z} \right\}$ and $J = \left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} : x \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \right\}$. Set $M = M_1 \oplus M_2$ where $M_1 = R/I$ and $M_2 = R/J$. Note that $M_1 = Z_2(M_1)$ and M_1 is indecomposable. Furthermore, $Z_2(M_2) = 0$ and M_2 is uniform. Since M is not a C_{11}-module, M_1 is not a C_{11}-module (see [3, Example 1.6]). It follows that there exists a submodule Y in M_1 such that there is no any direct summand of M_1 which is a complement of Y in M_1. Observe that $Y = Z_2(Y)$ which gives that M_1 is not a rC_{11}-module. Thus M is not a rC_{11}-module.

Observe that Lemma 3.8 provides a direct summand of a rC_{11}-module, which enjoys the property. However, by Example 3.4(iii) with $r = Z$, the property rC_{11} is not inherited by direct summands.

Even for a special case like $r(M)$ is a direct summand of M, it is not clear when $r(M)$ has rC_{11} property. Obviously, not all preradicals are of relevance in our aim, since $r(M)$ will be zero in many cases. For instance, the only preradicals of interest are those, which are subgenerated by some submodule K of M and the related class of radical modules is just the class of $\sigma[K]$ in which case any $r(M)$ is of the form $Tr(\sigma[K], M)$. For more details, see [15].

The next objective is to obtain when $r(M)$ has rC_{11} for an R-module M with rC_{11}. For our purpose let us consider the following property for a left exact preradical r in the category of right R-modules which is interesting in its own right. Let M be a right R-module.

(Y) For each submodule, N and each direct summand D of M, $r(N) \oplus D$ has a complement, which is a direct summand of M.

It can be seen easily that the following implications hold.

$$C_{11} \Rightarrow (Y) \Rightarrow rC_{11}.$$

Furthermore the conditions *(Y)* and rC_{11} are equivalent for indecomposable modules. Therefore, Example 3.9 also shows that the class of C_{11}-modules are properly contained in the class of modules which satisfy the property *(Y)*. However, we could not settle whether rC_{11} implies *(Y)* at this time. Perhaps it would be helpful to provide an example which has non-zero socle and satisfy the property *(Y)*. Let R be the ring as in [6, Example 3.2], i.e.

$R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{bmatrix}$ be the split null extension ring. Let $r = Soc$. Since $\mathbb{Z}_2$ is a faithful left $\mathbb{Z}_2$-module, $SocR_R = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$. Note that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{bmatrix}$ are the only direct summands of R which has zero intersection with socle. However, $SocR_R$ is simple and not essential in R. It follows that $SocR_R \oplus \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{bmatrix}$, which is essential in R. Hence, $SocR_R \oplus \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{bmatrix}$ has a complement $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So R_R satisfies *(Y)*.

We are in a position to prove that a module, which satisfies property *(Y)* can be decomposed into two C_{11}-modules in terms of a left exact preradical r in such a way that one piece has zero image and the other has essential image under r. First, we need to have the following basic lemma on closure submodules in a module.

Lemma 3.10 *Let N be a submodule of a module M such that N has a unique closure K in M. Then, K is the sum of all submodules L of M containing N and such that N is essential in L.*

Proof It is straightforward. □

Theorem 3.11 *Let R be a ring, r a left exact preradical for the category of right R-modules, and M a right R-module such that $r(M)$ has a unique closure in M. If M has the property (Y) then $M = M_1 \oplus M_2$ is a direct sum of rC_{11}-modules M_1 and M_2 such that $r(M_1)$ is essential in M_1 and $r(M_2) = 0$. In this case, M has rC_{11}.*

Proof Suppose M has (Y). By Lemma 3.8, $M = M_1 \oplus M_2$ with $r(M_1)$ is essential in M_1 and $r(M_2) = 0$. Note that $r(M) = r(M_1) \oplus r(M_2) = r(M_1)$, so M_1 is the (unique) closure of $r(M)$ in M. Let $\pi : M \to M_1$ denote the canonical projection. It is clear that M_2 has rC_{11}.

Let N be any submodule of M_1. By assumption, there exist submodules K, K' of M such that $M = K \oplus K'$, $(r(N) \oplus M_2) \cap K = 0$, and $r(N) \oplus M_2 \oplus K$ is essential in M. Since $K \cap M_2 = 0$, it follows that $K \cong \pi(K)$. Note that because r is left exact, $r(\pi(K)) = \pi(K) \cap r(M_1)$ is essential in $\pi(K)$. Hence, $r(K)$ is essential in K and, in addition, $r(M) = r(K) \oplus r(K')$ is essential in $K \oplus r(K')$. By Lemma 3.10, $K \oplus r(K') \subseteq M_1$ and, in particular, $K \subseteq M_1$. Now, $M_1 = K \oplus (M_1 \cap K')$, and $r(N) \oplus K = (r(N) \oplus M_2 \oplus K) \cap M_1$, by the modular law. It follows that $r(N) \oplus K$ is essential in M_1. By Proposition 3.3, M_1 satisfies rC_{11}. The second part follows from Theorem 3.5. □

Since a direct summand of a module M is a complement in M and any complement in M has itself own closure in M, Theorem 3.11 applies in the case where $r(M)$ is a complement of M and, in particular, when $r(M)$ is a direct summand of M. Thus, Theorem 3.11 gives the following consequence, which is a fundamental result in the theory of C_{11}-modules (see [10, Theorem 2.9]).

Corollary 3.12 *A nonsingular module M satisfies C_{11} if and only if $M = M_1 \oplus M_2$ where M_1 is a module satisfying C_{11} and having essential socle and M_2 is a module satisfying C_{11} and having zero socle.*

Proof The sufficiency is clear by [10, Theorem 2.4]. Conversely, suppose that M satisfies C_{11}. It can be checked that M satisfies (Y). Since $r = Soc$ is a left exact preradical in the category of right R-modules, Theorem 3.11 yields the result. □

Following [11, p.152], a hereditary torsion theory is called *stable* if the class of torsion modules is closed under injective envelopes. From [11, Proposition 7.3, p.153], the Goldie torsion theory is stable. Thus, the following result provides a useful decomposition into rC_{11}-modules.

Corollary 3.13 *Let R be a ring and r the left exact radical for a stable hereditary torsion theory for the category of right R-modules. If M satisfies (Y), then $M = r(M) \oplus K$ for some submodule K and both $r(M)$ and K satisfy rC_{11}.*

Proof Suppose M satisfies *(Y)*. By Lemma 3.8, $M = M_1 \oplus M_2$ such that $r(M_1)$ is essential in M_1 and $r(M_2) = 0$. By hypothesis, $r(M_1) = M_1$. Moreover, $r(M) = r(M_1) \oplus r(M_2) = M_1$, and hence $M = r(M) \oplus K$ where $K = M_2$. Now, the result follows from Theorem 3.11. □

Corollary 3.13 has the following special case, which is the very important characterization of modules with C_{11} property.

Corollary 3.14 *A module M satisfies C_{11} if and only if $M = Z_2(M) \oplus K$ for some (nonsingular) submodule K of M and both $Z_2(M)$ and K satisfy C_{11}.*

Proof The sufficiency is clear by [10, Theorem 2.4]. The necessity follows from Theorem 3.11 because M satisfies (Y), $r = Z_2$ is a left exact radical and the Goldie torsion theory is stable. □

In the rest of this paper, we focus on when direct summands of a rC_{11}-module are also rC_{11}-modules.

Proposition 3.15 *Let $M = M_1 \oplus M_2$. Then M_1 satisfies rC_{11} if and only if for every submodule N of M_1, there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap r(N) = 0$, and $K \oplus r(N)$ is an essential submodule of M.*

Proof Suppose M_1 satisfies rC_{11}. Let N be any submodule of M_1. By Proposition 3.3, there exists a direct summand L of M_1 such that $r(N) \cap L = 0$ and $r(N) \oplus L$ is essential in M_1. It is clear that $(L \oplus M_2) \cap r(N) = 0$ and $(L \oplus M_2) \oplus r(N)$ is essential in M. Conversely, suppose that M_1 has the stated property. Let H be any submodule of M_1. By hypothesis, there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap r(H) = 0$, and $K \oplus r(H)$ is an essential submodule of M. Now, $K = K \cap (M_1 \oplus M_2) = (K \cap M_1) \oplus M_2$ so that $K \cap M_1$ is a direct summand of M, and hence also of M_1, $r(H) \cap (K \cap M_1) = 0$, and $r(H) \oplus (K \cap M_1) = M_1 \cap (r(H) \oplus K)$, which is an essential submodule of M_1. It follows that M_1 satisfies rC_{11}. □

The next result applies in the case that M is a rC_{11}-module satisfying condition C_3. Recall that a module M has C_3 provided that if M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M (see, [4, 14]).

Theorem 3.16 *Let $M = M_1 \oplus M_2$ be a rC_{11}-module such that for every direct summand K of M with $K \cap M_2 = 0$, $K \oplus M_2$ is a direct summand of M. Then M_1 is a rC_{11}-module.*

Proof Let N be any submodule of M_1. By hypothesis, there exists a direct summand K of M such that $(r(N) \oplus M_2) \cap K = 0$ and $r(N) \oplus M_2 \oplus K$ is an essential submodule of M by Proposition 3.3. Moreover, $M_2 \oplus K$ is a direct summand of M. Now, the result follows from Proposition 3.15. □

Corollary 3.17 *Let M be a module, which satisfies rC_{11} and C_3. Then every direct summand of M satisfies rC_{11} and C_3.*

Proof C_3 property is inherited by direct summands (see, for example [14]). Then, the result follows by the Theorem 3.16. □

Proposition 3.18 *Let M be a rC_{11}-module and K a direct summand of M such that M/K is K-injective. Then K satisfies rC_{11}.*

Proof There exists a submodule K' of M such that $M = K \oplus K'$, and by hypothesis, K' is K-injective. Let L be a direct summand of M such that $L \cap K' = 0$. By Lemma 7.5 in [4], there exists a submodule H of M such that $H \cap K' = 0$, $M = H \oplus K'$, and $L \subseteq H$. Now, L is a direct summand of H, and hence $L \oplus K'$ is a direct summand of $M = H \oplus K'$. By Theorem 3.16, K satisfies rC_{11}. □

Corollary 3.19 *Let M be a module, which satisfies rC_{11}. Let N be a direct summand of M such that M/N is an injective module. Then N satisfies rC_{11}.*

Proof Since M/N is N-injective, N satisfies rC_{11} by Proposition 3.18.

□

Corollary 3.20 *Let $M = M_1 \oplus M_2$ be a direct sum of a submodule M_1 and an injective submodule M_2. If M satisfies rC_{11} then M_1 satisfies rC_{11}.*

Proof If M satisfies rC_{11} then M_1 satisfies rC_{11} by Proposition 3.18. □

Notice that conditions rC_{11} and (Y) are equivalent for indecomposable modules. The author thinks that rC_{11} does not imply (Y). But he does not have any counter example at this time. It turns out that the following problem is reasonable for future work.

Open Problem

Investigate the class of modules such that the conditions rC_{11} and (Y) are equivalent for a left exact preradical r in the category of right modules.

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