



Statistically multiplicative convergence on locally solid Riesz algebras

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Abstract: In this paper, we introduce the statistically multiplicative convergent sequences in locally solid Riesz algebras with respect to algebra multiplication and solid topology. We study this concept and the notion of st_m -bounded sequences, and also, we prove some relations between this convergence and the other convergences such as the order convergence and the statistical convergence in topological spaces. Also, we give some results related to semiprime f -algebras.

Key words: Statistically multiplicative convergence, statistically multiplicative bounded sequence, statistical convergence, Riesz algebra, Riesz spaces, locally solid Riesz spaces

1. Preliminary and introductory facts

The statistical convergence is a natural and efficient tool in the theory of functional analysis. It is a generalization of the ordinary convergence of a real sequence; see for example [9, 14, 16]. The idea of statistical convergence was firstly introduced by Zygmund [21] in the first edition of his monograph in 1935. Fast [12] and Steinhaus [19] independently improved this idea in the same year 1951. Fridy [13] defined the concept of statistical Cauchyness and convergence for real sequences. He also dealt with some Tauberian theorems. Recently, several generalizations and applications of this concept have been investigated by several authors in series of recent papers (c.f. [3, 8, 10, 11]).

The concept of statistical boundedness has firstly appeared in the paper of Fridy and Orhan [14]. Contrary to statistical convergence, it has not drawn that much attention of researchers. Bhardwaj and Gupta [7] presented some generalizations of statistical boundedness by introducing counterparts of the notions of statistical convergence of order α and λ -statistical convergence. The natural density of subsets of \mathbb{N} plays a crucial role in the definition of statistical convergence. For a subset A of natural numbers if the following limit exists

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : k \in A\}|$$

, then this unique limit is called *the density of A* and mostly abbreviated by $\delta(A)$, where $|\{k \leq n : k \in A\}|$ is the number of members of A not exceeding n . Also, a sequence $x = (x_k)$ statistically converges to L provided that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0 \quad (1)$$

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for each $\varepsilon > 0$. It is written by $S - \lim x_k = L$. If $L = 0$ then x is a statistically null sequence. A sequence $x = (x_k)$ is said to be statistically bounded if there exists a number $B \geq 0$ such that $\delta(\{k : |x_k| > B\}) = 0$. Throughout this paper, the vertical bar of sets will stand for the cardinality of the given sets.

Riesz space is another concept of functional analysis and it is ordered vector space that has many applications in measure theory, operator theory, and applications in economics, see for example [1, 2, 20]. Riesz space was introduced by F. Riesz in [18]. A real-valued vector space E with an order relation " \leq " on E (i.e. it is an antisymmetric, reflexive, and transitive relation) is called *ordered vector space* whenever, for every $x, y \in E$, we have

- (1) $x \leq y$ implies $x + z \leq y + z$ for all $z \in E$,
- (2) $x \leq y$ implies $\lambda x \leq \lambda y$ for every $0 \leq \lambda \in \mathbb{R}$.

An ordered vector space E is called *Riesz space* or *vector lattice* if, for any two vectors $x, y \in E$, the infimum and the supremum

$$x \wedge y = \inf\{x, y\} \quad \text{and} \quad x \vee y = \sup\{x, y\}$$

exist in E , respectively. A vector lattice is called *order complete* if every nonempty bounded above subset has a supremum (or, equivalently, whenever every nonempty bounded below subset has an infimum). For an element x in a vector lattice E , the *positive part*, the *negative part*, and *module* of x , respectively

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0 \quad \text{and} \quad |x| := x \vee (-x).$$

In the present paper, the vertical bar $|\cdot|$ of elements of the vector lattices will stand for the module of the given elements. It is clear that the positive and negative parts of a vector are positive. Ercan gave a characterization of statistical convergence on Riesz spaces in [10]. Recall that the order convergence is crucial for the vector lattices. A net $(x_\alpha)_{\alpha \in A}$ in a vector lattice E is called *order convergent* to $x \in E$ whenever the following conditions hold:

- (1) there exists a decreasing net $(y_\beta)_{\beta \in B} \downarrow$ such that $\inf y_\beta = 0$ in E ;
- (2) for any β , there is α_β with $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$.

Now we turn our attention to the solid topology on vector lattices. By a *linear topology* τ on a vector space E , we mean a topology τ on E that makes the addition and the scalar multiplication are continuous. A *topological vector space* (E, τ) is a vector space E equipped with linear topology τ . Neighborhoods of zero on topological vector spaces will often be referred to as zero neighborhoods. Every zero neighborhood V in topological vector spaces is *absorbing*, that is, for every element x , there exists a positive real $\lambda > 0$ such that $\lambda x \in V$. A linear topology τ on a vector space E has a base \mathcal{N} for the zero neighborhoods satisfying the following:

- (1) Each $V \in \mathcal{N}$ is balanced, i.e., $\lambda V \subseteq V$ for all scalars $|\lambda| \leq 1$,
- (2) For each $V_1, V_2 \in \mathcal{N}$, there is $V \in \mathcal{N}$ such that $V \subseteq V_1 \cap V_2$,
- (3) For every $V \in \mathcal{N}$, there exists $U \in \mathcal{N}$ with $U + U \subseteq V$,
- (4) For any scalar λ and each $V \in \mathcal{N}$, the set λV is also in \mathcal{N} .

In this article, unless otherwise stated, when we mention a zero neighborhood, it means that it always belongs to a base that holds the above properties. A subset A of a vector lattice E is called *solid* if, for each $x \in A$ and $y \in E$ with $|y| \leq |x|$ it holds $y \in A$. Let E be vector lattice and τ be a linear topology on it. Then the pair (E, τ) is said to be *locally solid vector lattice* (or, *locally solid Riesz space*) if τ has a base which consists of solid sets; for much more details on these notions, see [1, 2, 20].

The statistical convergence in a locally solid Riesz space was introduced and studied by Aydin in [3] with respect to unbounded order convergence on locally solid Riesz spaces. Let (x_n) be a sequence in a locally solid vector lattice E is said to be *statistically unbounded τ -convergent* to $x \in E$ if, for every zero neighborhood U , it satisfies

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - x| \wedge u \notin U\}| = 0 \tag{2}$$

for all $u \in E_+$. We abbreviate this convergence as $x_n \xrightarrow{st-u_\tau} x$, or shortly, we say that (x_n) *st- u_τ -converges* to x .

In this paper, we aim to introduce the concept of statistical convergence on locally solid Riesz algebras. So, let give some basic notations and terminologies of Riesz algebras that will be used. A Riesz space E is called a *Riesz algebra* (or *l-algebra*, for short), if E is an associative algebra whose positive cone $E_+ = \{x \in E : x \geq 0\}$ is closed under the algebra multiplication, i.e.,

$$x, y \in E_+ \Rightarrow x \cdot y \in E_+.$$

We routinely use the following fact: $y \leq x$ implies $u \cdot y \leq u \cdot x$ for arbitrary elements x, y and for every positive element u in l -algebras. We refer the reader for an exposition on the following notions to [2, 6, 15, 17, 20].

Definition 1.1 *An l-algebra E is said to be*

- (1) *d-algebra whenever $u \cdot (x \wedge y) = (u \cdot x) \wedge (u \cdot y)$ and $(x \wedge y) \cdot u = (x \cdot u) \wedge (y \cdot u)$ holds for all $u, x, y \in E_+$;*
- (2) *almost f-algebra if $x \wedge y = 0$ implies $x \cdot y = 0$ for all $x, y \in E_+$;*
- (3) *f-algebra if $x \wedge y = 0$ implies $(u \cdot x) \wedge y = (x \cdot u) \wedge y = 0$ for all $u, x, y \in E_+$;*
- (4) *semiprime whenever the only nilpotent element in E is zero, where an element x in E is called nilpotent whenever $x^n = 0$ for some natural number $n \in \mathbb{N}$;*
- (5) *unital if E has a multiplicative unit;*
- (6) *have the factorization property if, for given $x \in E$, there exist $y, z \in E$ such that $x = y \cdot z$.*

The important observation on Definition 1.1 is that every f -algebra is both d - and almost f -algebra. A vector lattice E is called *Archimedean* whenever $\frac{1}{n}x \downarrow 0$ holds in E for each $x \in E_+$. Every Archimedean f -algebra is commutative; see [20, Thm.140.10].

Assume E is an Archimedean f -algebra with the multiplicative unit vector e . Then, by applying [20, Thm.142.1.(v)], it can be seen that e is a positive vector. In arbitrary l -algebras the unit element (if existing) need not be positive; see for example [15, Exam.1.2(iii)]. In this article, unless otherwise stated, all vector lattices are assumed to be real and Archimedean. Different types of convergences in Riesz algebras have attracted the attention of several authors in a series of recent papers; see for example [4–6].

Example 1.2 Let E be a vector lattice. Consider $Orth(E) := \{\pi \in L_b(E) : x \perp y \text{ implies } \pi x \perp y\}$ the set of orthomorphisms on E , i.e., $|x| \wedge |y| = 0$ in E implies $|\pi x| \wedge |y| = 0$. Then, for an order complete E , $Orth(E)$ is an order complete unital f -algebra; see [17, Thm.15.4].

2. Definitions and basic properties of the st_m -convergence

In this section, we introduce the st_m -convergence and we give some basic results. Let us start with the asymptotic density that is of great importance for the statistical convergences. Let K be a subset of the set \mathbb{N} of natural numbers. Define a new set $K_n = \{k \in K : k \leq n\}$. If the limit of

$$\delta(K) := \lim_{n \rightarrow \infty} |K_n|/n$$

exists then $\delta(K)$ is called the *asymptotic density* of the set K . Recall that $\delta(\mathbb{N} - A) = 1 - \mu(A)$ for $A \subseteq \mathbb{N}$. Let X be a topological space and (x_n) be a sequence in X . Recall that (x_n) is said to be *statistically convergent in topology* to $x \in X$ whenever, for each neighborhood U of x , we have $\delta(\{n \in \mathbb{N} : x_n \notin U\}) = 0$; see for example [9, 16].

Definition 2.1 Let E be l -algebra and (E, τ) be a locally solid vector lattice. Then the pair (E, τ) is called locally solid l -algebra.

We continue with the following notion, which is crucial for the present paper.

Definition 2.2 Let (E, τ) be a locally solid l -algebra and (x_n) be a sequence in E . Then (x_n) is said to be statistically multiplicative convergent to $x \in E$ if for every zero neighborhood U

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : u \cdot |x_k - x| \notin U\}| = 0 \tag{3}$$

holds for all $u \in E_+$.

We abbreviate this convergence as $x_n \xrightarrow{st_m} x$, or we say that (x_n) st_m -converges to x , for short. Let's take $K = \{n \in \mathbb{N} : u \cdot |x_k - x| \notin U\}$ for arbitrary zero neighborhood U and for each positive vector u in a locally solid l -algebra E . Then, briefly, (x_n) is st_m -convergent to $x \in E$ whenever $\mu(K) = 0$ for every $u \in E_+$. Moreover, the limit in (3) can be also defined for the right multiplicative or both sides. But, to simplify the presentation, we suppose that Riesz algebras under consideration are commutative. The first observation is in the following remark.

Remark 2.3 Let (E, τ) be a positive unital locally solid l -algebra. Then, the statistically multiplicative convergence implies the statistical convergence in topology. Indeed, let $e \geq 0$ be the positive multiplicative unit element of E . Then, it follows from the positivity of e and the equality $e \cdot |x_n - x| = |x_n - x|$ that one can get the desired result.

We continue with several basic and useful results that are motivated by their analogies from vector lattice theory, and also, they are similar to the classical results for so many kinds of statistical convergences such as [3, Thm.2.2.] and similar to [1, Thm.2.17.], so their proofs are omitted.

Theorem 2.4 Let $x_n \xrightarrow{st_m} x$ and $y_n \xrightarrow{st_m} y$ be two sequences in a locally solid l -algebra (E, τ) . Then, we have

(i) $x_n \vee y_n \xrightarrow{st_m} x \vee y$;

(ii) $x_n \xrightarrow{st_m} x$ iff $(x_n - x) \xrightarrow{st_m} 0$ iff $|x_n - x| \xrightarrow{st_m} 0$;

(iii) $rx_n + sy_n \xrightarrow{st_m} rx + sy$ for any $r, s \in \mathbb{R}$;

(iv) $x_{n_k} \xrightarrow{st_m} x$ for any subsequence (x_{n_k}) of x_n .

In particular, we have

$$x_n^+ \xrightarrow{st_m} x^+, \quad x_n^- \xrightarrow{st_m} x^- \quad \text{and} \quad |x_n| \xrightarrow{st_m} |x|.$$

Proposition 2.5 Let (E, τ) be a Hausdorff locally solid l -algebra. Then limit of the st_m -convergence is uniquely determined in E .

The following result shows that the st_m -convergence satisfies normality in topological vector spaces.

Proposition 2.6 Let (x_n) and (y_n) be two sequences in a locally solid l -algebra (E, τ) . If $x_n \xrightarrow{st_m} 0$ and $0 \leq y_n \leq x_n$ for all $n \in \mathbb{N}$, then $y_n \xrightarrow{st_m} 0$.

Proof Let U be an arbitrary zero neighborhood in E . Then, we have $\mu(K) = 0$, where $K = \{n \in \mathbb{N} : u \cdot x_k \notin U\}$ for each $u \in E_+$ because (x_n) is positive and $x_n \xrightarrow{st_m} 0$. By using the solidness of U , we get $u \cdot y_k \in U$ for each $u \in E_+$ because of $u \cdot y_k \leq u \cdot x_k \in U$ for all $u \in E_+$. Therefore, we have $\mu(\{n \in \mathbb{N} : u \cdot y_k \notin U\}) = 0$, i.e., we have $y_n \xrightarrow{st_m} 0$ because the set $\{n \in \mathbb{N} : u \cdot y_k \notin U\}$ is a subset of $\{n \in \mathbb{N} : u \cdot x_k \notin U\}$ and U is arbitrary zero neighborhood. \square

Definition 2.7 Let (E, τ) be a locally solid l -algebra. A sequence (x_n) in E is said to be st_m -Cauchy if $(x_n - x_m)_{(m,n) \in \mathbb{N} \times \mathbb{N}} \xrightarrow{st_m} 0$. Also, E is said to be st_m -complete if every st_m -Cauchy sequence in E is st_m -convergent.

Proposition 2.8 Every st_m -convergent sequence in a locally solid l -algebra is st_m -Cauchy.

Proof Suppose that a sequence (x_n) is st_m -convergent to a point $x \in E$. Take an arbitrary zero neighborhood U . Then, there is another zero neighborhood V such that $V + V \subseteq U$. Thus, we have $\mu(K) = 1$, where $K = \{n \in \mathbb{N} : u \cdot |x_n - x| \in V\}$ for every $u \in E_+$. Thus, for each $u \in E_+$, we can obtain the inequality

$$u \cdot |x_n - x_m| \leq u \cdot |x_n - x| + u \cdot |x_m - x| \in V + V \subseteq U.$$

for all $n, m \in K$. Hence we get $K \subseteq \{n \in \mathbb{N} : u \cdot |x_n - x_m| \in U\}$, and so, $\mu(\{n \in \mathbb{N} : u \cdot |x_n - x_m| \in U\}) = 1$ holds for every $u \in E_+$. Therefore, we get the desired result that (x_n) is st_m -Cauchy \square

Definition 2.9 A sequence (x_n) in a locally solid l -algebra is called statistically multiplicative bounded (or st_m -bounded, for short) whenever, for each zero neighborhood U , there exists a positive element $w \in E_+$ such that

$$\mu(\{n \in \mathbb{N} : w \cdot |x_n| \notin U\}) = 0.$$

It is clear that statistically multiplicative bounded sequence coincides with the notion the statistically bounded for real sequences (c.f. [14]). Recall that a linear topology τ on an ordered vector space is called *locally full* whenever it has a base for the τ -neighborhoods at zero consisting of full neighborhoods (c.f. [1, Exer.1. p.72])

Theorem 2.10 An st_m -convergent sequence (x_n) in a locally solid l -algebra is statistically multiplicative bounded.

Proof Suppose (E, τ) is a locally solid f -algebra. Then, since the topology τ is a locally solid topology on E , it follows from [6, Thm.4.] and [6, Lem.2.1] that the topology τ is locally full in E . Now, we can apply it. Let \mathcal{N}_0 be the base of zero τ -neighborhoods. Thus, we can define a new zero neighborhood

$$W_{u,U} := \{x \in E : u \cdot |x| \in U\} \tag{4}$$

for any element $u \in E_+$ and any zero neighborhood $U \in \mathcal{N}_0$. Then the class $\mathcal{B} := \{W_{u,U} : u \in E_+, U \in \mathcal{T}\}$ is a base of zero neighborhood of locally solid topology τ_m on E ; see [6, Thm.17.].

Now, assume that a sequence $x_n \xrightarrow{st_m} x$ in E . Let U be an arbitrary zero τ -neighborhood in \mathcal{N}_0 . Then there exists another $V \in \mathcal{N}_0$ such that $V + V \subseteq U$. By using the st_m -convergence of (x_n) , we have $\mu(K) = 0$, where $K = \{n \in \mathbb{N} : u \cdot |x_n - x| \notin V\}$ for all $u \in E_+$. Let's take a fixed positive element $u \in E_+$. Then we have $\mu(K_W) = 0$ for the set $K_W = \{n \in \mathbb{N} : (x_n - x) \notin W_{u,V}\}$. On the other hand, since every linear topology is absorbing, there exists a positive scalar $\lambda > 0$ such that $\lambda x \in W_{u,V}$. Now, take another scalar $\gamma > 0$ such that $\gamma \leq 1$ and $\gamma \leq \lambda$. Hence, we have $\gamma x \in W_{u,V}$ because $W_{u,V}$ is solid set and $\gamma|x| \leq \lambda|x| \in W_{u,V}$.

Next, by using $V + V \subseteq U$, one can see that $W_{u,V} + W_{u,V} \subseteq W_{u,U}$ for all $u \in E_+$. So, we have

$$\gamma x_n = \gamma(x_n - x) + \gamma x \in W_{u,V} + W_{u,V} \subseteq W_{u,U}.$$

for every $n \in \mathbb{N} - K_W$, and so, we get $\mu(\{n \in \mathbb{N} : \gamma x_n \notin W_{u,U}\}) = 0$. If we choose the constant element w in Definition 2.9 as $w := \gamma u$ for the given u then we have $w \in E_+$ and $\mu(\{n \in \mathbb{N} : w \cdot |x_n| \notin U\}) = 0$. Therefore, we get the desired, the st_m -boundedness of (x_n) result because U is arbitrary zero neighborhood. \square

3. Main results in Riesz algebras

In this section, we give some results of the st_m -convergence that are related to the multiplication on Riesz algebras. We begin with the following useful result for which we will use the basic properties in Theorem 2.4 and [6, Prop.2.2.].

Theorem 3.1 Let (E, τ) be a locally solid l -algebra. Then, the following facts are equivalent:

- (i) the st_m -convergence has a unique limit;

(ii) if $y_n \leq x_n$ for all $n \in \mathbb{N}$, $x_n \xrightarrow{st_m} x$ and $y_n \xrightarrow{st_m} y$ then $y \leq x$;

(iii) the positive cone E_+ is closed under the st_m -convergence in E .

Proof (i) \iff (ii) Suppose that $x_n \xrightarrow{st_m} x$, $y_n \xrightarrow{st_m} y$, and $y_n \leq x_n$ for all $n \in \mathbb{N}$. Then we get $(x_n - y_n) \xrightarrow{st_m} (x - y)$, and also, we have

$$x_n - y_n = (x_n - y_n)^+ \xrightarrow{st_m} (x - y)^+ \geq 0$$

because of $y_n \leq x_n$ for each $n \in \mathbb{N}$. Since the st_m -convergence has a unique limit, we have $x - y = (x - y)^+ \geq 0$, i.e., we get $x \geq y$.

For the converse, we assume, on the contrary, that the st_m -convergence does not have a unique limit. Suppose that there exists a sequence (x_n) in E and $x, y \in E$, $x \neq y$, such that $x_n \xrightarrow{st_m} x$ and $x_n \xrightarrow{st_m} y$. Thus, we have $0 = x_n - x_n \xrightarrow{st_m} x - y$. As a result, we have proved that the zero sequence is st_m -convergent to $|x - y|$, i.e., we have $0 \xrightarrow{st_m} |x - y|$. Now, we consider two sequences $y_n = z_n = 0$ in E . Then, we can say $y_n \xrightarrow{st_m} y := 0$ and $z_n \xrightarrow{st_m} z := |x - y| \neq 0$. Thus, one can observe $y \not\leq z$, which disrupts the condition (ii). Therefore, we get the desired a contradiction that proves the uniqueness limit of the st_m -convergence.

(ii) \iff (iii) To show the closedness of the positive cone E_+ under the st_m -convergence, one can consider the following fact $0 =: y_n \leq x_n \xrightarrow{st_m} x$ implies $x \geq 0$.

Conversely, suppose $x_n \xrightarrow{st_m} x$, $y_n \xrightarrow{st_m} y$, and $y_n \leq x_n$ for all $n \in \mathbb{N}$. Then, we have $(x_n - y_n) \xrightarrow{st_m} (x - y)$. Since E_+ is closed under the st_m -convergence in E and $x_n - y_n \in E_+$ for each $n \in \mathbb{N}$, we get $x - y \in E_+$, i.e., we have $x \geq y$. \square

Modifying the proof of Proposition 2.6 in [3] and applying Theorem 3.1, we have the following result, which gives a relation between the st_m -convergence and the order convergence.

Proposition 3.2 *Every monotone st_m -convergent sequence is order convergent to its st_m -limit point in locally solid l -algebras in which the st_m -convergence has a unique limit.*

In the next result, we prove that the algebra multiplication is statistically multiplicative continuous, i.e. that for every net $(x_\alpha)_{\alpha \in A}$ in a locally solid l -algebra E , $x_\alpha \xrightarrow{c} x$ implies $y \cdot x_\alpha \xrightarrow{c} y \cdot x$ for all elements $x, y \in E$, where c is a linear convergence on E ; see [6, Def.5.3].

Theorem 3.3 *Let (x_n) be an st_m -convergent sequence to a point x in a locally solid l -algebra E . Then $y \cdot x_n \xrightarrow{st_m} y \cdot x$ for all $y \in E$.*

Proof Suppose $x_n \xrightarrow{st_m} x$ in E and $y \in E$. So, by applying Theorem 2.4, we can get $(x_n - x)^+ \xrightarrow{st_m} 0$ and $(x_n - x)^- \xrightarrow{st_m} 0$. Then, one can get $y^+ \cdot (x_n - x)^\pm \xrightarrow{st_m} 0$ and $y^- \cdot (x_n - x)^\pm \xrightarrow{st_m} 0$. Indeed, consider a zero neighborhood U . If we choose $w_u := u \cdot y^+ \in E_+$ for every $u \in E_+$, then we have

$$u \cdot |y^+ \cdot (x_n - x)^\pm| = u \cdot y^+ \cdot (x_n - x)^\pm = w_u \cdot (x_n - x)^\pm$$

because the negative and the positive part of elements in vector lattices are positive. Therefore, we have $\mu(\{n \in \mathbb{N} : u \cdot |y^+ \cdot (x_n - x)^\pm| \notin U\}) = 0$ for all $u \in E_+$ because of $\mu(\{n \in \mathbb{N} : w_u \cdot (x_n - x)^\pm \notin U\}) = 0$ for

all $w_u \in E_+$, i.e., we have $y^+ \cdot (x_n - x)^\pm \xrightarrow{st_m} 0$, where we use \pm for both cases the negative and the positive parts of $(x_n - x)$. Similarly, one can show $y^- \cdot (x_n - x)^\pm \xrightarrow{st_m} 0$.

Now, by applying the equality $x = x^+ - x^-$ for any element in vector lattices, we can consider the following equality

$$\begin{aligned} y \cdot (x_n - x) &= (y^+ - y^-) \cdot [(x_n - x)^+ - (x_n - x)^-] \\ &= y^+ \cdot (x_n - x)^+ - y^+ \cdot (x_n - x)^- \\ &\quad - y^- \cdot (x_n - x)^+ + y^- \cdot (x_n - x)^-. \end{aligned}$$

Thus, we can say $y \cdot (x_n - x) \xrightarrow{st_m} 0$ because of $y^+ \cdot (x_n - x)^+$, $y^+ \cdot (x_n - x)^-$, $y^- \cdot (x_n - x)^+$ and $y^- \cdot (x_n - x)^-$ are st_m -convergent to zero. Thus, we get the desired result, i.e., we have $y \cdot x_n \xrightarrow{st_m} y \cdot x$. \square

Remind that a net (x_α) in an f -algebra E is said to be multiplicative order convergent to $x \in E$ (shortly, (x_α) mo -converges to x) if $|x_\alpha - x| \cdot u \xrightarrow{o} x$ for all $u \in E_+$; see [4]. Instead of the order convergence, if taking any c -convergence then the mo -convergence is called mc -convergence; see [6]. It is known that the product of mo -convergent nets in f -algebras is mo -convergent (see [4]), and the product of mc -convergent nets in commutative l -algebras is mc -convergent (see [6]). Fortunately, we also have a positive answer for the st_m -convergence in Theorem 3.8. But, the product of statistically multiplicative bounded sequences does not need to be st_m -bounded in general because the product of zero neighborhood sets can be bigger than them.

Problem 3.4 *Let (x_n) and (y_n) are st_m -bounded sequences in a locally solid l -algebra with the factorization property. Is the product of $(x_n \cdot y_n)$ st_m -bounded sequence?*

Now, we give the following notion to give a partial answer of the product bounded sequences.

Definition 3.5 *Let E be a locally solid l -algebra. Then, E is said to have the neighborhood factorization property whenever, for every zero neighborhood U in E , there exist zero neighborhoods V, W with $V \cdot W \subseteq U$.*

It is clear that the real line has the neighborhood factorization property. For the following example, we consider [17, Exam.11.2].

Example 3.6 *Let E be the collection of all real-valued continuous functions f which are piecewise polynomials with the norm topology on $[0, 1]$. With respect to the pointwise operations, it is an Archimedean f -algebra with the unit element e i.e., $e(x) = 1$ for all $0 \leq x \leq 1$. Then E has the factorization property, and also, it has the neighborhood factorization property.*

Proposition 3.7 *If (x_n) and (y_n) are st_m -bounded sequences in a locally solid l -algebra with the neighborhood factorization property then the product sequence $(x_n \cdot y_n)$ is also st_m -bounded sequence.*

Proof Suppose (x_n) and (y_n) are st_m -bounded in a locally solid l -algebra E with the neighborhood factorization property, and U is a zero neighborhood in E . Then, there exist zero neighborhoods V, W such that $V \cdot W \subseteq U$. Since (x_n) and (y_n) are st_m -bounded, there exists positive elements $w_x, w_y \in E_+$ such that

$$\mu(\{n \in \mathbb{N} : w_x \cdot |x_n| \notin V\}) = 0$$

and

$$\mu(\{n \in \mathbb{N} : w_y \cdot |y_n| \notin W\}) = 0.$$

Now, by taking $w = w_x \cdot w_y$ and applying properties in [15, p.151], we can obtain the following inequality

$$w \cdot |x_n \cdot y_n| \leq (w_x \cdot |x_n|) \cdot (w_y \cdot |y_n|) \in V \cdot W \subseteq U$$

for each $n \in \mathbb{N}$. Therefore, for the given zero neighborhood U , we find a positive element w such that $\mu(\{n \in \mathbb{N} : w \cdot |x_n \cdot y_n| \notin U\}) = 0$, i.e., we get the desired result, the st_m -boundedness of $(x_n \cdot y_n)$. \square

In the next result, we prove that, under some conditions, the product of two st_m -convergent sequences is also st_m -convergent.

Theorem 3.8 *If $x_n \xrightarrow{st_m} x$ and $y_n \xrightarrow{st_m} y$ in a locally solid l -algebra E , then the product sequence $(x_n \cdot y_n)$ st_m -converges to $x \cdot y$ whenever at least one of these two sequences is order bounded.*

Proof Suppose $x_n \xrightarrow{st_m} x$ and $y_n \xrightarrow{st_m} y$ in E . Without loss of generality, suppose $|x_n| \leq w$ for all $n \in \mathbb{N}$ and some $w \in E_+$. Then, by applying the properties in [15, P.151], we have the following inequality

$$\begin{aligned} |x_n \cdot y_n - x \cdot y| &= |x_n \cdot y_n - x_n \cdot y + x_n \cdot y - x \cdot y| \\ &\leq |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x| \\ &\leq v \cdot |y_n - y| + |y| \cdot |x_n - x| \end{aligned}$$

for all $n \in \mathbb{N}$. Since $x_n \xrightarrow{st_m} x$ and $y_n \xrightarrow{st_m} y$, it follows from Theorem 2.4 that $|x_n - x|$ and $|y_n - y|$ are st_m -convergent to zero. Thus, by using Theorem 3.3, we get $v \cdot |y_n - y|, |y| \cdot |x_n - x| \xrightarrow{st_m} 0$. Lastly, by using Proposition 2.6 and Theorem 2.4, one can get $|x_n \cdot y_n - x \cdot y| \xrightarrow{st_m} 0$, i.e., $x_n \cdot y_n \xrightarrow{st_m} x \cdot y$. \square

Problem 3.9 *Is Theorem 3.8 true in the case that at least one of two sequences is st_m -bounded?*

4. The st_m -convergence on f -algebras

In this section, we prove some results on f -algebras. Firstly, we give an extension of Proposition 2.5 in the following result.

Theorem 4.1 *The limit of a st_m -convergent sequence is uniquely determined in locally solid semiprime f -algebras.*

Proof We show firstly that if a sequence (x_n) consists of nilpotents of E such that $x_n \xrightarrow{st_m} x$, then x is also a nilpotent element in E . To see this, take a fixed zero neighborhood U . Thus, we have $\mu(K) = 0$ for the set $K = \{n \in \mathbb{N} : u \cdot |x_k - x| \notin U\}$. Now, we fix a positive element $u \in E_+$. Then from [20, Thm.142.1(ii)] and [17, Prop.10.2(iii)], we have

$$u \cdot |x_n - x| = |u \cdot x_n - u \cdot x| = |u \cdot x| = u \cdot |x|. \tag{5}$$

Let's define a new constant sequence $y_n(u) := u \cdot |x|$ for the fixed $u \in E_+$ and for all $n \in \mathbb{N}$. Then it follows from (5) that

$$\{n \in \mathbb{N} : w \cdot |y_n(u) - 0| \notin U\} = \{n \in \mathbb{N} : w \cdot y_n(u) \notin U\}$$

for each $w \in E_+$. Thus, we get $y_n(u) \xrightarrow{stm} 0$ or $u \cdot |x| \xrightarrow{stm} 0$ because of $\mu(K) = 0$. As a result, we see that $u \cdot x = 0$ for every $u \in E_+$ because $u \cdot |x|$ is constant and converges to zero for all $u \in E_+$. Then, $y \cdot x = 0$ for all $y \in E$. Hence, from [15, p.157], we get that x is a nilpotent element in E .

Next, we show that st_m -limit is unique. Assume it is not true, i.e., there exists a sequence (z_n) in E such that $z_n \xrightarrow{stm} z_1$ and $z_n \rightarrow z_2$ in E . Consider a zero neighborhood U . Then, there exists another zero neighborhood V such that $V + V \subseteq U$. Then, we have $\mu(K_x) = \mu(K_y) = 1$, where $K_x = \{n \in \mathbb{N} : v \cdot |x_k - x| \in V\}$ and $K_y = \{n \in \mathbb{N} : v \cdot |x_k - y| \in V\}$ for all $v \in E_+$. Also, we have

$$|z_1 - z_2| \leq |z_n - z_1| + |z_n - z_2|. \tag{6}$$

By taking $v = |z_1 - z_2| \in E_+$ and considering (6), we have

$$|z_1 - z_2|^2 \leq |z_1 - z_2| \cdot |z_n - z_1| + |z_1 - z_2| \cdot |z_n - z_2| \in V + V \subseteq U.$$

Therefore, similar to the first part of the proof, we have $|z_1 - z_2|^2 = 0$. Since E is semiprime f -algebra. Then, we get $|z_1 - z_2| = 0$, i.e., we have $z_1 = z_2$. □

The following result is the st_m -version of [6, Thm.11.].

Theorem 4.2 *Let (E, τ) be a locally solid f -algebra. Then the st_m -convergence has a unique limit if and only if E is semiprime.*

Proof Suppose E is semiprime. Assume the st_m -convergence is not uniquely determined in E . Then, from [6, Lem.1.1.], it follows that there exists a constant sequence $x_n = x$ in E such that $x_n \xrightarrow{stm} y \neq x$. Thus, for given zero neighborhood U , we have $\mu(K) = 0$ for the set $K = \{n \in \mathbb{N} : u \cdot |x - y| \notin U\}$ for all $u \in E_+$. By repeating the same argument as in the proof Theorem 4.1, we can get $u \cdot |x - y| = 0$ for each $u \in E_+$. Especially, take $u = |x - y|$. Then, we have $|x - y|^2 = 0$, i.e., we get $x = y$ because E is semiprime. But, it contradicts with $x \neq y$, and so, we obtain the desired result by the contradiction.

Conversely, suppose the st_m -convergence has a unique limit point. Assume there exists an element $0 \neq x \in E$ such that $x^2 = 0$. Hence, x is a nilpotent element in E ; see [17, Prop.10.2(i)]. Then, by applying [17, Prop.10.2(iii)], we get $u \cdot |x| = 0$ for all $u \in E_+$. Therefore, one can get $x \xrightarrow{stm} 0$. Then it follows from unique st_m -limit and [6, Lem.1.1.] that $x = 0$, that is, E is semiprime. □

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