



On Hilbert genus fields of imaginary cyclic quartic fields

Moulay Ahmed HAJJAMI¹ , Mohamed Mahmoud CHEMS-EDDIN^{2,*} 

¹Department of Mathematics, Faculty of Sciences and Technology, Moulay Ismail University of Meknes, Errachidia, Morocco

²Department of Mathematics, Faculty of Sciences, Mohammed First University, Oujda, Morocco

Received: 28.01.2021

Accepted/Published Online: 22.05.2021

Final Version: 27.07.2021

Abstract: Let p be a prime number such that $p = 2$ or $p \equiv 1 \pmod{4}$. Let ε_p denote the fundamental unit of $\mathbb{Q}(\sqrt{p})$ and let a be a positive square-free integer. The main aim of this paper is to determine explicitly the Hilbert genus field of the imaginary cyclic quartic fields of the form $\mathbb{Q}(\sqrt{-a\varepsilon_p\sqrt{p}})$.

Key words: Imaginary cyclic quartic fields, unramified extensions, Hilbert genus fields

1. Introduction

The study of the unramified extensions of a given number field k is of a huge interest in algebraic number theory. For instance, the importance of the Hilbert class field of k , denoted by $H(k)$, is represented in the fact that it is the maximal abelian unramified extension of k and its Galois group over k , i.e. $G := \text{Gal}(H(k)/k)$, is isomorphic to $Cl(k)$, the class group of k (cf. [14, p. 228]). Another important example of these unramified extensions is the genus field of k , which is defined as the maximal extension of k which is unramified at all finite and infinite primes of k of the form kk_1 , where k_1 is an abelian extension of \mathbb{Q} (cf. [13]). These two fields have been largely investigated in old and recent studies (e.g. [1, 2, 4–6, 15]).

Another very interesting example of unramified extensions of k is the Hilbert genus field of k which is the invariant field $E(k)$ of G^2 . Then, by Galois theory, we have:

$$Cl(k)/Cl(k)^2 \simeq G/G^2 \simeq \text{Gal}(E(k)/k),$$

and thus, $2\text{-rank}(Cl(k)) = 2\text{-rank}(\text{Gal}(E(k)/k))$. On the other hand, $E(k)/k$ is the maximal unramified Kummer extension of exponent 2. Thus, by Kummer theory (cf. [18, p. 14]), there exists a unique multiplicative group Δ such that

$$E(k) = H(k) \cap k(\sqrt{k^*}) = k(\sqrt{\Delta}) \text{ and } k^{*2} \subset \Delta \subset k^*.$$

Therefore, the question that arises is how to construct the Hilbert genus field of k , or equivalently, how to give a set of generators for the finite group Δ/k^{*2} . Note that many mathematicians have investigated this question for some biquadratic number fields. For example, Bae and Yue studied the Hilbert genus field of the fields

*Correspondence: 2m.chemseddin@gmail.com

2010 AMS Mathematics Subject Classification: 11R16, 11R29, 11R27, 11R04, 11R37

$\mathbb{Q}(\sqrt{p}, \sqrt{d})$, for a prime number p such that, $p = 2$ or $p \equiv 1 \pmod{4}$, and a positive square-free integer d (cf. [3]).

Recently, Ouang and Zhang have determined the Hilbert genus field of the imaginary biquadratic fields $\mathbb{Q}(\sqrt{\delta}, \sqrt{d})$, where $\delta = -1, -2$ or $-p$ with $p \equiv 3 \pmod{4}$ a prime number and d any square-free integer. Thereafter, they constructed the Hilbert genus field of real biquadratic fields $\mathbb{Q}(\sqrt{\delta}, \sqrt{d})$, for any positive square-free integer d , and $\delta = p, 2p$ or $p_1 p_2$ where p, p_1 , and p_2 are prime numbers congruent to $3 \pmod{4}$, such that the class number of $\mathbb{Q}(\sqrt{\delta})$ is odd (cf. [19, 20]). For more works on this problem, we refer the reader to the papers [10, 21, 23].

In the present work, using other easier techniques based on genus fields, we shall construct the Hilbert genus fields of imaginary cyclic quartic fields of the form $K = \mathbb{Q}(\sqrt{-a\varepsilon_p\sqrt{p}})$, for a prime number p such that $p = 2$ or $p \equiv 1 \pmod{4}$, and a positive square-free integer a relatively prime to p .

The plan of this paper is as follows. In Section 2, we shall collect some results of cyclic quartic fields theory and genus theory. Section 3 is dedicated to the investigation of the genus field of the imaginary cyclic quartic fields K . In Section 4, we will construct the Hilbert genus fields of the fields K . Therein, we give some numerical examples.

1.1. Notations

Let k be a number field. The next notations will be used for the rest of this article:

- \mathcal{O}_k : the ring of integers of k ,
- $Cl(k)$: the class group of k ,
- $h(k)$: the class number of k ,
- $N_{k/k'}$: the norm map of an extension k/k' ,
- E_k : the unit group of k ,
- k^* : the nonzero elements of k ,
- $k^{(*)}$: the genus field of k ,
- $E(k)$: the Hilbert genus field of k ,
- $H(k)$: the Hilbert class field of k ,
- δ_k : the absolute discriminant of k ,
- $\delta_{k/k'}$: the generator of the relative discriminant of an extension k/k' ,
- $r_2(A)$: the 2-rank of a finite abelian group A ,
- ε_d : the fundamental unit of $\mathbb{Q}(\sqrt{d})$, where d is a positive square-free integer,
- p : a prime number such that $p = 2$ or $p \equiv 1 \pmod{4}$,
- a : a positive square-free integer relatively prime to p ,

- $\delta = -a\varepsilon_p\sqrt{p}$,
- $k_0 = \mathbb{Q}(\sqrt{p})$,
- $K = k_0(\sqrt{\delta})$: an imaginary quartic cyclic number field,
- q_j : an odd prime integer,
- $\left(\frac{\cdot}{\cdot}\right)$: the Legendre symbol,
- $\left(\frac{-\cdot}{\mathfrak{p}_i}\right)$: the Hilbert symbol over k_0 .

For more notations, see the beginning of each section below.

2. Preliminary results

In this section, we start by recalling some results that we will need in what follows. Let L be a cyclic quartic extension of the rational number field \mathbb{Q} . It is known that L can be expressed uniquely in the form:

$$L = \mathbb{Q}(\sqrt{a(d + b\sqrt{d})}),$$

for some integers a, b, c , and d are such that $d = b^2 + c^2$ is square-free with $b > 0$ and $c > 0$, and a is an odd square-free integer relatively prime to d (cf. [9, 24]). Note that L possesses a unique quadratic subfield $k = \mathbb{Q}(\sqrt{d})$.

Lemma 2.1 ([12]) *Keep the above notations. We have:*

1. *The absolute discriminant of L is given by δ_L , where:*

$$\delta_L = \begin{cases} 2^8 a^2 d^3, & \text{if } d \equiv 0 \pmod{2}, \\ 2^4 a^2 d^3, & \text{if } d \equiv 1 \pmod{4}, b \equiv 0 \pmod{2}, a + b \equiv 3 \pmod{4}, \\ a^2 d^3, & \text{if } d \equiv 1 \pmod{4}, b \equiv 0 \pmod{2}, a + b \equiv 1 \pmod{4}, \\ 2^6 a^2 d^3, & \text{if } d \equiv 1 \pmod{4}, b \equiv 1 \pmod{2}. \end{cases}$$

2. *The relative discriminant of L/k is given by $\Delta_{L/k} = \delta_{L/k} \mathcal{O}_k$, where:*

$$\delta_{L/k} = \begin{cases} 4a\sqrt{d}, & \text{if } d \equiv 0 \pmod{2}, \\ 4a\sqrt{d}, & \text{if } d \equiv 1 \pmod{4}, b \equiv 0 \pmod{2}, a + b \equiv 3 \pmod{4}, \\ a\sqrt{d}, & \text{if } d \equiv 1 \pmod{4}, b \equiv 0 \pmod{2}, a + b \equiv 1 \pmod{4}, \\ 8a\sqrt{d}, & \text{if } d \equiv 1 \pmod{4}, b \equiv 1 \pmod{2}. \end{cases}$$

Lemma 2.2 ([12]) *Keep the above notations. If the class number of $k = \mathbb{Q}(\sqrt{d})$ is odd, then $L = \mathbb{Q}(\sqrt{a'\varepsilon_d\sqrt{d}})$, where*

$$a' = \begin{cases} 2a, & \text{if } d \equiv 1 \pmod{4} \text{ and } b \equiv 1 \pmod{2}, \\ a, & \text{otherwise.} \end{cases}$$

Proposition 2.3 ([13]) *Let L be an abelian extension of \mathbb{Q} of degree n . If $n = r^s$, where r is a prime number and s is a positive integer, then*

$$L^{(*)} = \left(\prod_{p/\delta_L, p \neq r} M_p \right) L,$$

where M_p is the unique subfield of degree e_p (the ramification index of p in L) over \mathbb{Q} of $\mathbb{Q}(\xi_p)$ the p -th cyclotomic field and δ_L is the discriminant of L .

Proposition 2.4 ([16], p. 160) *If p is a prime number such that $p \equiv 1 \pmod{4}$, then $\mathbb{Q}(\sqrt{\varepsilon_p^* \sqrt{p}})$ is the quartic subfield of $\mathbb{Q}(\xi_p)$, where $\varepsilon_p^* = \left(\frac{2}{p}\right) \varepsilon_p$ and $\mathbb{Q}(\sqrt{2 + \sqrt{2}})$ is the real quartic field of $\mathbb{Q}(\xi_{16})$.*

We close this section with the following two results.

Lemma 2.5 ([8]) *Let k/k' be a quadratic extension of number fields. If the class number of k' is odd, then the rank of the 2-class group of k is given by*

$$r_2(\text{Cl}(k)) = t - 1 - e,$$

where t is the number of ramified primes (finite or infinite) in the extension k/k' and e is defined by $2^e = [E_{k'} : E_{k'} \cap N_{k/k'}(k^*)]$.

Proposition 2.6 ([11]) *Let k/k' be a quadratic extension of number fields, and μ a number of k' , coprime with 2, such that $k = k'(\sqrt{\mu})$. The extension k/k' is unramified at all finite primes of k' if and only if the two following items hold:*

1. *the ideal generated by μ is the square of the fractional ideal of k' , and*
2. *there exists a nonzero number ξ of k' verifying $\mu \equiv \xi^2 \pmod{4}$.*

3. Genus fields of the imaginary cyclic quartic fields : $K = \mathbb{Q}(\sqrt{-a\varepsilon_p \sqrt{p}})$

Let p denote a prime number such that $p = 2$ or $p \equiv 1 \pmod{4}$ and a be a positive square-free integer coprime with p . In the present section, we shall investigate the genus field of the imaginary cyclic quartic fields $K = \mathbb{Q}(\sqrt{-a\varepsilon_p \sqrt{p}})$. Note that K is a CM-field with maximal real subfield $k_0 = \mathbb{Q}(\sqrt{p})$. For the construction of the genus field of K , we shall need the factorization of the integer a ; therefore, we put:

$$a = \prod_{i=1}^n q_i \text{ or } 2 \prod_{i=1}^n q_i, \tag{3.1}$$

where q_1, q_2, \dots, q_n are distinct odd primes. Assume that the Legendre symbols $\left(\frac{p}{q_j}\right) = 1$ for $1 \leq j \leq m$ ($m \leq n$, the case $m = 0$ is included here) and $\left(\frac{p}{q_j}\right) = -1$ for $m + 1 \leq j \leq n$.

Let e_ℓ denote the ramification index of a prime number ℓ in K/\mathbb{Q} and put $q_j^* = (-1)^{\frac{q_j-1}{2}} q_j$.

Proposition 3.1 *Assume that $p \equiv 5 \pmod{8}$, then*

1. *If $a = \left(\prod_{i=1}^n q_i\right) \equiv 1 \pmod{4}$, we have*

$$K^{(*)} = K(\sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*});$$

2. *If $a = \left(\prod_{i=1}^n q_i\right) \equiv 3 \pmod{4}$, we have*

$$K^{(*)} = K(\sqrt{-1}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*});$$

3. *If $a = 2 \prod_{i=1}^n q_i$ and $\left(\prod_{i=1}^n q_i\right) \equiv 1 \pmod{4}$, we have*

$$K^{(*)} = K(\sqrt{2}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*});$$

4. *If $a = 2 \prod_{i=1}^n q_i$ and $\left(\prod_{i=1}^n q_i\right) \equiv 3 \pmod{4}$, we have*

$$K^{(*)} = K(\sqrt{-2}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}).$$

Proof We have $p \equiv 5 \pmod{8}$, then by the expression of the absolute discriminant of K (cf. Lemma 2.1), the prime numbers of \mathbb{Q} that ramify in K are :

- the odd prime divisors of a (with ramification index equals 2),
- the prime p (with $e_p = 4$),
- and 2 if $a \not\equiv 1 \pmod{4}$ (with $e_2 = 2$).

On the other hand, we know that $\mathbb{Q}(\xi_p)/\mathbb{Q}$ is a Galois extension such that

$$\text{Gal}(\mathbb{Q}(\xi_p)/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^* \simeq \mathbb{Z}/(p-1)\mathbb{Z},$$

and if m divides $p-1$, then $\mathbb{Q}(\xi_p)$ contains a unique subfield of degree m over \mathbb{Q} . Since p is the unique prime which is ramified in $\mathbb{Q}(\xi_p)$, then $\mathbb{Q}(\sqrt{-\varepsilon_p \sqrt{p}})$ is the unique subfield of degree 4 of $\mathbb{Q}(\xi_p)$ (Proposition 2.4). Therefore, by Proposition 2.3, we get:

1. If $a = \prod_{i=1}^n q_i \equiv 1 \pmod{4}$, then:

$$K^{(*)} = \mathbb{Q}(\sqrt{q_1^*})\mathbb{Q}(\sqrt{q_2^*}) \dots \mathbb{Q}(\sqrt{q_n^*})\mathbb{Q}(\sqrt{-\varepsilon_p \sqrt{p}})K = K(\sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}).$$

2. If $a = \prod_{i=1}^n q_i \equiv 3 \pmod{4}$, then:

$$K^{(*)} = \mathbb{Q}(\sqrt{q_1^*})\mathbb{Q}(\sqrt{q_2^*}) \dots \mathbb{Q}(\sqrt{q_n^*})\mathbb{Q}(\sqrt{-\varepsilon_p \sqrt{p}})K = K(\sqrt{-1}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}).$$

3. If $a = 2 \prod_{i=1}^n q_i$ and $\prod_{i=1}^n q_i \equiv 1 \pmod{4}$, then:

$$K^{(*)} = \mathbb{Q}(\sqrt{q_1^*})\mathbb{Q}(\sqrt{q_2^*}) \dots \mathbb{Q}(\sqrt{q_n^*})\mathbb{Q}(\sqrt{-\varepsilon_p \sqrt{p}})K = K(\sqrt{2}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}).$$

4. If $a = 2 \prod_{i=1}^n q_i$ and $\prod_{i=1}^n q_i \equiv 3 \pmod{4}$, then:

$$K^{(*)} = \mathbb{Q}(\sqrt{q_1^*})\mathbb{Q}(\sqrt{q_2^*}) \dots \mathbb{Q}(\sqrt{q_n^*})\mathbb{Q}(\sqrt{-\varepsilon_p \sqrt{p}})K = K(\sqrt{-2}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}),$$

which completes the proof. □

We similarly prove the following two propositions.

Proposition 3.2 *Assume that $p \equiv 1 \pmod{8}$, then*

1. *If $a = \prod_{i=1}^n q_i \equiv 3 \pmod{4}$, then $K^{(*)} = K(\sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*})$.*

2. *If $a = \prod_{i=1}^n q_i \equiv 1 \pmod{4}$, then $K^{(*)} = K(\sqrt{-1}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*})$.*

3. *If $a = 2 \prod_{i=1}^n q_i$ and $\prod_{i=1}^n q_i \equiv 1 \pmod{4}$, then $K^{(*)} = K(\sqrt{-2}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*})$.*

4. *If $a = 2 \prod_{i=1}^n q_i$ and $\prod_{i=1}^n q_i \equiv 3 \pmod{4}$, then $K^{(*)} = K(\sqrt{2}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*})$.*

Proposition 3.3 *If $p = 2$, then $K^{(*)} = K(\sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*})$.*

4. Hilbert genus fields of the imaginary cyclic quartic fields : $K = \mathbb{Q}(\sqrt{-a\varepsilon_p \sqrt{p}})$

Keep the notations of the previous sections. At this stage, we can start the construction of the Hilbert genus field of the imaginary cyclic quartic field $K = \mathbb{Q}(\sqrt{-a\varepsilon_p \sqrt{p}})$. However, we must first expose some more ingredients of our proofs. We have the following lemmas.

Lemma 4.1 *Let k/k' be a quadratic extension of number fields such that the class number of k' is odd. Let Δ denote the multiplicative group such that $k^{*2} \subset \Delta \subset k^*$ and $k(\sqrt{\Delta})$ is the Hilbert genus field of k (cf. §1). Then*

$$r_2(\Delta/k^{*2}) = t - e - 1,$$

where t and e are defined in Lemma 2.5.

Proof Put $G = \text{Gal}(H(k)/k)$. By the definition of $E(k)$ (cf. §1) we have:

$$\text{Cl}(k)/\text{Cl}(k)^2 \simeq G/G^2 \simeq \text{Gal}(E(k)/k),$$

Since $E(k) = k(\sqrt{\Delta})$, then by class field theory:

$$r_2(\Delta/k^{*2}) = \log_2[k(\sqrt{\Delta}) : k] = \log_2[E(k) : k] = r_2(\text{Cl}(k)).$$

Thus, we get the result by the ambiguous class number formula (Lemma 2.5). □

Lemma 4.2 ([22]) *Let $p \equiv 1 \pmod{4}$ be a prime number and ε_p be the fundamental unit of $\mathbb{Q}(\sqrt{p})$. Then, there are two natural integers u and v such that:*

$$\varepsilon_p^\lambda = u + v\sqrt{p}, \quad u \equiv 0 \pmod{2}, \quad v \equiv 1 \pmod{4},$$

where $\lambda = \begin{cases} 1, & \text{if } p \equiv 1 \pmod{8}; \\ 3, & \text{if } p \equiv 5 \pmod{8}. \end{cases}$

Lemma 4.3 *Let p and q be two different prime numbers such that $p \equiv 1 \pmod{4}$, $q \equiv \pm 1 \pmod{4}$ and $\left(\frac{p}{q}\right) = 1$. Then, there exist two natural integers x and y such that:*

$$x^2 - py^2 = q^{\lambda h},$$

where h is the class number of $\mathbb{Q}(\sqrt{p})$ and $\lambda = \begin{cases} 1, & \text{if } p \equiv 1 \pmod{8}; \\ 3, & \text{if } p \equiv 5 \pmod{8}. \end{cases}$

Furthermore, we have:

$$\begin{cases} x \equiv 1 \pmod{2}, & y \equiv 0 \pmod{2}, & \text{if } q \equiv 1 \pmod{4}; \\ x \equiv 0 \pmod{2}, & y \equiv 1 \pmod{2}, & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

Proof Since $\left(\frac{p}{q}\right) = 1$, then q splits in $k_0 = \mathbb{Q}(\sqrt{p})$. Thus, there exist two prime ideals \mathcal{H}_1 and \mathcal{H}_2 of \mathcal{O}_{k_0} , such that $q\mathcal{O}_{k_0} = \mathcal{H}_1\mathcal{H}_2$ and $\sigma(\mathcal{H}_1) = \mathcal{H}_2$ (σ is the generator of Galois group of k_0). Thus, $q^h\mathcal{O}_{k_0} = \mathcal{H}_1^h\mathcal{H}_2^h$, where h is the class number of k_0 . Since \mathcal{H}_1^h and \mathcal{H}_2^h are two principal ideals, we can choose two natural integers x and y such that $\mathcal{H}_1^{\lambda h} = (x + \sqrt{p}y)$, $\mathcal{H}_2^{\lambda h} = (x - \sqrt{p}y)$ and $x^2 - py^2 > 0$. Then $q^{\lambda h}\mathcal{O}_{k_0} = (x^2 - py^2)$. Therefore, $x^2 - py^2 = \eta q^{\lambda h}$, for a certain unit $\eta \in E_{k_0}$. Thus, $\eta \in E_{k_0} \cap \mathbb{Q}$. Since $E_{k_0} = \langle -1, \varepsilon_p \rangle$ and $x^2 - py^2 > 0$, it follows that $\eta = 1$ and so $x^2 - py^2 = q^{\lambda h}$, which gives the first part of the lemma. If we

assume that $q \equiv 1 \pmod{4}$, we get $x^2 - y^2 \equiv 1 \pmod{4}$. By checking all the possibilities of $x^2 - y^2 \pmod{4}$ we deduce that $x \equiv 1 \pmod{2}$ and $y \equiv 0 \pmod{2}$. If $q \equiv 3 \pmod{4}$, we get $x^2 - y^2 \equiv 3 \pmod{4}$. Hence, $x \equiv 0 \pmod{2}$ and $y \equiv 1 \pmod{2}$, which completes the proof. \square

Lemma 4.4 *Let $k = k'(\sqrt{\mu})$ be a quadratic extension of number fields and $\beta \in k'$. Then, $\beta \in k^2$ if and only if $\beta \in k'^2$ or $\mu\beta \in k'^2$.*

Proof If $\beta \in k^2$, then $\beta = (a + b\sqrt{\mu})^2$, for some $a, b \in k'$, so $\beta = a^2 + \mu b^2 + 2ab\sqrt{\mu}$. Since $\beta \in k'$, we have $a = 0$ or $b = 0$. If $b = 0$, then $\beta = a^2 \in k'^2$. If $a = 0$, then $\beta = \mu b^2$. Therefore, $\mu\beta = (\mu b)^2 \in k'^2$. The reciprocal implication is evident. \square

Now, after all the above preparations, we can state our first main theorem. We keep the notations in the beginning of the previous section and we shall put $E = E(K)$.

Theorem 4.5 *Let $p \equiv 5 \pmod{8}$ be a prime number and denote by E the Hilbert genus field of $K = \mathbb{Q}(\sqrt{-a\varepsilon_p\sqrt{p}})$. We have:*

1. *If $a \equiv 1 \pmod{4}$, then:*

$$E = K(\sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*});$$

2. *If $a \equiv 3 \pmod{4}$, then:*

$$E = K(\sqrt{-1}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*});$$

3. *If $a = 2 \prod_{i=1}^n q_i$ and $\prod_{i=1}^n q_i \equiv 1 \pmod{4}$, then:*

$$E = K(\sqrt{2}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*});$$

4. *If $a = 2 \prod_{i=1}^n q_i$ and $\prod_{i=1}^n q_i \equiv 3 \pmod{4}$, then:*

$$E = K(\sqrt{-2}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*});$$

where:

$$\begin{cases} q_i^* = (-1)^{\frac{q_i-1}{2}} q_i, & (1 \leq i \leq n), \\ \alpha_j^* = \begin{cases} (-1)^{\frac{x_j+y_j-1}{2}} \alpha_j, & \text{if } q_j \equiv 1 \pmod{4}, \\ (-1)^{\frac{x_j+y_j-1}{2}} \alpha_j \sqrt{p}, & \text{if } q_j \equiv 3 \pmod{4}, \end{cases} & \text{such that } x_j^2 - py_j^2 = q_j^{\lambda_j}, \end{cases} & (1 \leq j \leq m). \end{cases}$$

Proof Note that K/k_0 is a quadratic extension with the class number of $k_0 = \mathbb{Q}(\sqrt{p})$ is odd. So we are in the conditions of Lemma 4.1. Let Δ be the multiplicative group such that $E = K(\sqrt{\Delta})$. Thus

$r_2(\Delta/K^{*2}) = t - e - 1$. Let p_∞ and \bar{p}_∞ denote the infinite primes of k_0 which are respectively corresponding to the \mathbb{Q} -embeddings:

$$i_{p_\infty} : k_0 \hookrightarrow \mathbb{R} \quad \text{and} \quad i_{\bar{p}_\infty} : k_0 \hookrightarrow \mathbb{R}$$

$$\sqrt{p} \mapsto -\sqrt{p} \quad \text{and} \quad \sqrt{p} \mapsto \sqrt{p}$$

Set $i_{p_\infty}(\varepsilon_p) = \bar{\varepsilon}_p$. Note that $N_{k_0/\mathbb{Q}}(\varepsilon_p) = \varepsilon_p \bar{\varepsilon}_p = -1$. By the definition of Hilbert symbol (cf. [7]), we have:

$$\begin{aligned} \left(\frac{-1, -a\varepsilon_p\sqrt{p}}{p_\infty}\right) &= i_{p_\infty}^{-1}((i_{p_\infty}(-1), i_{p_\infty}(-a\varepsilon_p\sqrt{p}))_{p_\infty}) \\ &= i_{p_\infty}^{-1}((-1, a\bar{\varepsilon}_p\sqrt{p})_{p_\infty}) \\ &= i_{p_\infty}^{-1}(-1) \\ &= -1, \end{aligned}$$

$$\begin{aligned} \left(\frac{\varepsilon_p, -a\varepsilon_p\sqrt{p}}{p_\infty}\right) &= i_{p_\infty}^{-1}((i_{p_\infty}(\varepsilon_p), i_{p_\infty}(-a\varepsilon_p\sqrt{p}))_{p_\infty}) \\ &= i_{p_\infty}^{-1}((\bar{\varepsilon}_p, a\bar{\varepsilon}_p\sqrt{p})_{p_\infty}) \\ &= i_{p_\infty}^{-1}(-1) \\ &= -1, \end{aligned}$$

$$\begin{aligned} \left(\frac{-\varepsilon_p, -a\varepsilon_p\sqrt{p}}{\bar{p}_\infty}\right) &= i_{\bar{p}_\infty}^{-1}((i_{\bar{p}_\infty}(-\varepsilon_p), i_{\bar{p}_\infty}(-a\varepsilon_p\sqrt{p}))_{\bar{p}_\infty}) \\ &= i_{\bar{p}_\infty}^{-1}((-\varepsilon_p, -a\varepsilon_p\sqrt{p})_{\bar{p}_\infty}) \\ &= i_{\bar{p}_\infty}^{-1}(-1) \\ &= -1. \end{aligned}$$

It follows, by the Hasse norm theorem, that -1 , ε_p and $-\varepsilon_p$ are not in $N_{K/k_0}(K^*)$. Thus, $e = 2$. Therefore, to compute $r_2(\Delta/K^{*2})$, it suffices to determine the number of prime ideals of k_0 which ramify in K .

Case 1. Assume that $a \equiv 1 \pmod{4}$:

In this case, we have $\delta_{K/k_0} = a\sqrt{p}$ (Lemma 2.1). Thus, the finite primes of k_0 which ramify in K are the prime divisors of a in k_0 and the prime ideal (\sqrt{p}) . Since $\left(\frac{p}{q_j}\right) = 1$, for $1 \leq j \leq m$, then q_j splits in k_0 . Thus, we have:

$$r_2(\Delta/K^{*2}) = (n + m + 1) + 2 - 2 - 1 = n + m.$$

On the other hand, to explicitly determine E , it suffices to determine the set of generators for the finite group (Δ/K^{*2}) . For this, we consider the set:

$$\mathbb{B} = \{q_1^*, q_2^*, \dots, q_n^*, \alpha_1^*, \alpha_2^*, \dots, \alpha_m^*\}.$$

Let us show that the elements of \mathbb{B} are linearly independent modulo K^{*2} (provided that the notion of linear independence is translated to a multiplicative setting: $\alpha_1, \dots, \alpha_s$ are multiplicatively independent if $\alpha_1^{m_1} \dots \alpha_s^{m_s} = 1$ implies that $m_i = 0$, for all i).

We consider the element $\beta = \left(\prod_{i=1}^n q_i^{*a_i}\right) \left(\prod_{j=1}^m \alpha_j^{*b_j}\right)$, where $a_i, b_j \in \{0, 1\}$ and are not all zero.

Suppose that $\beta \in K^{*2}$.

- If we assume that $\forall j \in \{1, 2, \dots, m\} \ b_j = 0$, we get $\beta = \left(\prod_{i=1}^n q_i^{*a_i}\right) \in K^{*2}$, which implies that $\sqrt{\beta} \in K^*$. Thus, $\mathbb{Q}(\sqrt{\beta})$ is a quadratic subfield of K , which is impossible, because K has only one quadratic subfield that is $k_0 = \mathbb{Q}(\sqrt{p})$.
- Assume that $\exists j \in \{1, \dots, m\}$ with $b_j \neq 0$. Note that $\beta \in k_0^*$ and $\beta \in K^{*2}$. By Lemma 4.4, we have $\beta \in k_0^{*2}$ or $\delta\beta \in k_0^{*2}$, where $\delta = -a\varepsilon_p\sqrt{p}$. We shall discuss each case.

Suppose that $\beta \in k_0^{*2}$. Then, we have $N_{k_0/\mathbb{Q}}(\beta) = \beta\beta^\sigma \in \mathbb{Q}^{*2}$ (with $\text{Gal}(k_0/\mathbb{Q}) = \{1, \sigma\}$). Put $N = N_{k_0/\mathbb{Q}}$. We have $\forall i \in \{1, 2, \dots, n\}$, $N(q_i^*) = q_i^2$ and $\forall j \in \{1, 2, \dots, m\}$:

$$N(\alpha_j) = x_j^2 - py_j^2 = q_j^{3h} = q_j z_j^2,$$

where $z_j = (q_j)^{\frac{3h-1}{2}}$ (where h is the class number of k_0 , note that here h is an odd number) and $N(\sqrt{p}) = -p$, so:

$$N(\alpha_j^*) = \begin{cases} q_j z_j^2, & \text{if } q_j \equiv 1 \pmod{4}, \\ -pq_j z_j^2, & \text{if } q_j \equiv 3 \pmod{4}. \end{cases}$$

Thus,

$$N(\beta) = (-p)^l \left(\prod_{i=1}^n q_i^{a_i}\right)^2 \left(\prod_{j=1}^m q_j^{b_j}\right) \left(\prod_{j=1}^m z_j^{b_j}\right)^2,$$

where l is the number of q_j such that $q_j \equiv 3 \pmod{4}$.

If l is even, $N(\beta) = \left(\prod_{j=1}^m q_j^{b_j}\right) Z^2 \in \mathbb{Q}^{*2}$ where $Z \in \mathbb{Q}$, then $\left(\prod_{j=1}^m q_j^{b_j}\right) \in \mathbb{Q}^{*2}$.

If l is odd, $N(\beta) = -p \left(\prod_{j=1}^m q_j^{b_j}\right) Z^2 \in \mathbb{Q}^{*2}$, so $-p \left(\prod_{j=1}^m q_j^{b_j}\right) \in \mathbb{Q}^{*2}$.

Both cases are not possible because p, q_1, q_2, \dots, q_m are distinct prime numbers.

If now $\delta\beta \in k_0^{*2}$, then $N(\delta\beta) = N(\beta)N(\delta) \in \mathbb{Q}^{*2}$. Since $N(\varepsilon_p) = -1$, we have $N(\delta) = N(-a)N(\varepsilon_p)N(\sqrt{p}) = pa^2$. Thus,

$$N(\delta\beta) = pa^2(-p)^l \left(\prod_{i=1}^n q_i^{a_i}\right)^2 \left(\prod_{j=1}^m q_j^{b_j}\right) \left(\prod_{j=1}^m z_j^{b_j}\right)^2,$$

if l is even, $N(\delta\beta) = p \left(\prod_{j=1}^m q_j^{b_j}\right) Y^2 \in \mathbb{Q}^{*2}$, where $Y \in \mathbb{Q}$, then $p \prod_{j=1}^m q_j^{b_j} \in \mathbb{Q}^{*2}$, if l is odd,

$N(\delta\beta) = - \left(\prod_{j=1}^m q_j^{b_j}\right) Y^2 \in \mathbb{Q}^{*2}$, then $-\prod_{j=1}^m q_j^{b_j} \in \mathbb{Q}^{*2}$. Both of these cases are impossible. Hence,

the elements of \mathbb{B} are linearly independent modulo K^{*2} .

On the other hand, according to Proposition 3.1, the genus field of K is $K^{(*)} = (\sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*})$. Thus, for $1 \leq i \leq n$, $K(\sqrt{q_i^*})/K$ is an unramified extension, and by Lemma 4.3, for $1 \leq j \leq m$,

$q_j \mathcal{O}_k = \mathcal{H}_1^j \mathcal{H}_2^j$, where \mathcal{H}_1^j and \mathcal{H}_2^j are the prime ideals of \mathcal{O}_{k_0} above q_j , since \mathcal{H}_1^j is ramified in K and $\mathcal{H}_1^{j^{3h}} = (x_j + y_j \sqrt{p}) = (\alpha_j)$, then (α_j) is the square of the fractional ideal of K . Note also that if $q_j \equiv 1 \pmod{4}$ for some $1 \leq j \leq m$, we have x_j is odd and y_j is even. Hence, $\alpha_j = x_j + y_j \sqrt{p} \equiv x_j + y_j + \frac{-1 + \sqrt{p}}{2} 2y_j \pmod{4} \equiv x_j + y_j \pmod{4} \equiv \pm 1 \pmod{4}$. Therefore, $\alpha_j^* \equiv 1 \pmod{4}$ and the ideal generated by α_j^* is the square of the fractional ideal of K . Thus, by Proposition 2.6, $K(\sqrt{\alpha_j^*})/K$ is also an unramified extension (the proof of the case $q_j \equiv 3 \pmod{4}$ is analogous). Therefore,

$$\mathbb{B} = \{q_1^*, q_2^*, \dots, q_n^*, \alpha_1^*, \alpha_2^*, \dots, \alpha_m^*\},$$

is a representative set of Δ/K^{*2} . Hence,

$$E = K(\sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*}),$$

is the Hilbert genus field of K .

Case 2. Case: $a \equiv 3 \pmod{4}$:

In this case, we have $\delta_{K/k_0} = 4a\sqrt{p}$, so the prime ideals of k_0 which ramify in K are the prime divisors of a in k_0 , the ideal (\sqrt{p}) and also the prime ideal (2) . Thus, $r_2(\Delta/K^{*2}) = n + m + 1$.

We consider the set:

$$\mathbb{B} = \{-1, q_1^*, q_2^*, \dots, q_n^*, \alpha_1^*, \alpha_2^*, \dots, \alpha_m^*\},$$

We proceed as in the first case and we show that the elements of \mathbb{B} are linearly independent modulo K^{*2} . Noting that the genus fields of K is:

$$K^{(*)} = K(\sqrt{-1}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}),$$

we deduce that the extensions $K(\sqrt{-1})/K$ and $K(\sqrt{q_i^*})/K$ (for $1 \leq i \leq n$) are unramified, and in a similar way as in the previous case, we prove that the extensions $K(\sqrt{\alpha_j^*})/K$ are also unramified.

Thus, \mathbb{B} is a representative set of Δ/K^{*2} . Therefore,

$$E = K(\sqrt{-1}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*}).$$

Case 3. $a = 2 \prod_{i=1}^n q_i$ and $\prod_{i=1}^n q_i \equiv 1 \pmod{4}$.

In this case, we have $\delta_{K/k_0} = 4a\sqrt{p}$, so the prime ideals of k_0 which ramify in K are the prime divisors of a in k_0 , the ideal (\sqrt{p}) , and the prime ideal (2) . Then $r_2(\Delta/K^{*2}) = n + m + 1$, and since the genus fields of K is:

$$K^{(*)} = K(\sqrt{2}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}),$$

so we shall consider the set:

$$\mathbb{B} = \{2, q_1^*, q_2^*, \dots, q_n^*, \alpha_1^*, \alpha_2^*, \dots, \alpha_m^*\}.$$

Similar to the previous cases, we show that \mathbb{B} is a representative set of Δ/K^{*2} , then:

$$E = K(\sqrt{2}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*}).$$

Using similar techniques as in the previous case, we prove the fourth item. Which completes the proof. \square

Theorem 4.6 *Let $p \equiv 1 \pmod{8}$ be a prime number and denote by E the Hilbert genus field of $K = \mathbb{Q}(\sqrt{-a\varepsilon_p\sqrt{p}})$. We have:*

1. *If $a \equiv 3 \pmod{4}$, then:*

$$E = K(\sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*});$$

2. *If $a \equiv 1 \pmod{4}$, then:*

$$E = K(\sqrt{-1}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*}, \sqrt{\varepsilon_p});$$

3. *If $a = 2 \prod_{i=1}^n q_i$ and $\prod_{i=1}^n q_i \equiv 1 \pmod{4}$, then:*

$$E = K(\sqrt{-2}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*}, \sqrt{\varepsilon_p});$$

4. *If $a = 2 \prod_{i=1}^n q_i$ and $\prod_{i=1}^n q_i \equiv 3 \pmod{4}$, then:*

$$E = K(\sqrt{2}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*}, \sqrt{\varepsilon_p}).$$

Proof The proof of the case $a \equiv 3 \pmod{4}$ is similar to that of the first item of Theorem 4.5, so we let it for the reader.

In the case where $a \equiv 1 \pmod{4}$, we have $\delta_{K/k_0} = 4a\sqrt{p}$, then the prime ideals of k_0 that ramify in K are the prime divisors of a in k_0 , the ideal (\sqrt{p}) , 2_1 and 2_2 , where 2_1 and 2_2 are the two prime ideals of k_0 above 2. Thus, $r_2(\Delta/K^{*2}) = n + m + 2$. On the other hand, the genus field of K in this case is $K^{(*)} = K(\sqrt{-1}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*})$. As in the proof of Theorem 4.5, we shall consider the set:

$$\mathbb{B} = \{-1, q_1^*, q_2^*, \dots, q_n^*, \alpha_1^*, \alpha_2^*, \dots, \alpha_m^*, \varepsilon_p\},$$

where ε_p is the fundamental unit of k_0 and we shall show that the elements of \mathbb{B} are linearly independent

modulo K^{*2} . Let $\beta = (-1)^a \varepsilon_p^b \left(\prod_{i=1}^n q_i^{*a_i} \right) \left(\prod_{j=1}^m \alpha_j^{*b_j} \right)$, where $a, b, a_i, b_j \in \{0, 1\}$ and they are not all zero.

Assume that $\beta \in K^{*2}$.

- If $b = 0$, then $\beta = (-1)^a \left(\prod_{i=1}^n q_i^{*a_i} \right) \left(\prod_{j=1}^m \alpha_j^{*b_j} \right)$, following the same reasoning in the proof of Theorem 4.5,

we get a contradiction.

- If $b = 1$, we will distinguish two cases:

1. Assume that $\forall j \in \{1, 2, \dots, m\}, b_j = 0, \beta = (-1)^a \varepsilon_p \left(\prod_{i=1}^n q_i^{*a_i} \right)$, then by Lemma 4.4, $\beta \in k_0^{*2}$ or $\delta\beta \in k_0^{*2}$. If $\beta \in k_0^{*2}$, then $N(\beta) = - \left(\prod_{i=1}^n q_i^{*a_i} \right)^2 \in \mathbb{Q}^2$ (in fact, $N(\varepsilon_p) = -1$). Thus, -1 is a square in \mathbb{Q} , which is impossible. If $\delta\beta \in k_0^{*2}$, $N(\delta\beta) = -pa^2 \left(\prod_{i=1}^n q_i^{*a_i} \right)^2 \in \mathbb{Q}^2$, then $-p$ is a square in \mathbb{Q} , which is also impossible.
2. Assume that $\exists j \in \{1, 2, \dots, m\}, b_j \neq 0$, then $\beta = (-1)^a \varepsilon_p \left(\prod_{i=1}^n q_i^{*a_i} \right) \left(\prod_{j=1}^m \alpha_j^{*b_j} \right)$. If $\beta \in k_0^{*2}$, then by keeping the same previous notations we obtain:

$$N(\beta) = -(-p)^l \left(\prod_{i=1}^n q_i^{a_i} \right)^2 \left(\prod_{j=1}^m q_j^{b_j} \right) \left(\prod_{j=1}^m z_j^{b_j} \right)^2,$$

Thus,

$$N(\beta) = \begin{cases} - \left(\prod_{j=1}^m q_j^{b_j} \right) Z^2 \in \mathbb{Q}^{*2}, & \text{if } l \text{ is even,} \\ -p \left(\prod_{j=1}^m q_j^{b_j} \right) Z^2 \in \mathbb{Q}^{*2}, & \text{if } l \text{ is odd,} \end{cases}$$

where $Z \in \mathbb{Q}$, so $- \left(\prod_{j=1}^m q_j^{b_j} \right) \in \mathbb{Q}^{*2}$ or $-p \left(\prod_{j=1}^m q_j^{b_j} \right) \in \mathbb{Q}^{*2}$, which is impossible.

If now $\delta\beta \in k_0^{*2}$, we get:

$$N(\delta\beta) = \begin{cases} p \left(\prod_{j=1}^m q_j^{b_j} \right) Y^2 \in \mathbb{Q}^{*2}, & \text{if } l \text{ is even,} \\ - \left(\prod_{j=1}^m q_j^{b_j} \right) Y^2 \in \mathbb{Q}^{*2}, & \text{if } l \text{ is odd,} \end{cases}$$

where $Y \in \mathbb{Q}$, so $- \left(\prod_{j=1}^m q_j^{b_j} \right) \in \mathbb{Q}^{*2}$ or $p \left(\prod_{j=1}^m q_j^{b_j} \right) \in \mathbb{Q}^{*2}$, which is also impossible. Therefore, the elements of \mathbb{B} are linearly independent modulo K^{*2} . It is easy seen by [6, p. 67] that the extension $K(\sqrt{\varepsilon_p})/K$ is unramified. Thus, by the above discussion, \mathbb{B} is a representative set of Δ/K^{*2} and the Hilbert genus field of K is:

$$E = K \left(\sqrt{-1}, \sqrt{q_1^*}, \sqrt{q_2^*}, \dots, \sqrt{q_n^*}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*}, \sqrt{\varepsilon_p} \right).$$

We similarly prove the rest. □

Now we shall construct the Hilbert genus field of $K = \mathbb{Q}(\sqrt{-a\varepsilon_p\sqrt{p}})$, with $p = 2$. Let us now assume that $a = \prod_{i=1}^n q_i$ is an odd positive square-free integer such that $q_j \equiv \pm 1 \pmod{8}$ for all $1 \leq j \leq m$ ($m \leq n$, the case $m = 0$ is possible here).

Theorem 4.7 *Let a be as above. For any j such that $1 \leq j \leq m$, let x_j and y_j be the two positive integers such that $q_j = x_j^2 - 2y_j^2$. Put $\alpha_j = x_j + y_j\sqrt{2}$, then the Hilbert genus field of $K = \mathbb{Q}(\sqrt{-a\varepsilon_2\sqrt{2}})$ is:*

$$E = K(\sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*}),$$

where α_j^* is defined as follows:

- If $q_j \equiv 1 \pmod{8}$, $\alpha_j^* := (-1)^{\frac{x_j-1}{2}}\alpha_j$,
- If $q_j \equiv -1 \pmod{8}$, then x_j and y_j are odd and we put

$$\alpha_j^* := \begin{cases} -\varepsilon_2\alpha_j, & \text{if } x_j \equiv \pm 1 \pmod{4} \text{ and } y_j \equiv -1 \pmod{4}, \\ \varepsilon_2\alpha_j, & \text{else.} \end{cases}$$

Proof It is known that for a prime number $q \equiv \pm 1 \pmod{8}$, there are positive integers x, y such that $q = x^2 - 2y^2$, and if $q \equiv 1 \pmod{8}$, the integers x and y may be chosen such that $x \equiv 1 \pmod{2}$ and $y \equiv 0 \pmod{4}$ (cf. [17]).

- Assume that $q_j \equiv 1 \pmod{8}$, for some $1 \leq j \leq m$. We have $(\alpha_j) = (x_j + y_j\sqrt{2})$ and the prime ideals of k_0 above (q_j) , are ramified in K . Then the ideal (α_j) is the square of a fractional ideal of K . By the definition of α_j^* , we find $\alpha_j^* \equiv 1 \pmod{4}$ and the ideal (α_j^*) is the square of a fractional ideal of K for $1 \leq j \leq m$. Hence, by Proposition 2.6, $K(\sqrt{\alpha_j^*})/K$ is an unramified extension.
- Assume that $q_j \equiv -1 \pmod{8}$, for some $1 \leq j \leq m$. Let us firstly show that y_j is odd. It is clear that x_j is odd. We have, $2y_j^2 \equiv x^2 - q \equiv 2 \pmod{8}$. By easy calculus one can check that the classes \bar{y} of $\mathbb{Z}/8\mathbb{Z}$ such that $2\bar{y}^2 = \bar{2}$ are exactly the classes of odd integers y . Thus, y_j is odd. If $x_j \equiv y_j \equiv -1 \pmod{4}$, then we have $\alpha_j^* = -(1+\sqrt{2})(x_j+y_j\sqrt{2}) = -(x_j+2y_j) - (x_j+y_j)\sqrt{2}$. Thus, $\alpha_j^* \equiv -1-2\sqrt{2} \equiv (3-2\sqrt{2}) \equiv (1-\sqrt{2})^2 \pmod{4}$. Since q_j ramify in K/k_0 , (α_j) is the square of an ideal of K . Therefore, by Proposition 2.6, $K(\sqrt{\alpha_j^*})/K$ is an unramified extension. We similarly proceed for the other cases of x_j and y_j .

Note that, by Lemma 2.1, $\delta_{K/k_0} = 4a\sqrt{2}$, so the prime ideals of k_0 which ramify in K are the prime divisors of a in k_0 and the prime ideal $(\sqrt{2})$. Therefore, $r_2(\Delta/K^{*2}) = n + m$. Hence, we show as in the proofs of the previous theorems that the set:

$$\mathbb{B} = \{q_1^*, q_2^*, \dots, q_n^*, \alpha_1^*, \alpha_2^*, \dots, \alpha_m^*\}$$

is a representation of Δ/K^{*2} . Finally, we get:

$$E = K(\sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \dots, \sqrt{\alpha_m^*}),$$

which completes the proof. □

We close this section with some numerical examples.

Examples

1. Theorem 4.5: (the case $a \equiv 3 \pmod{4}$). Let $K = \mathbb{Q}(\sqrt{-42427(5 + 2\sqrt{5})}) = \mathbb{Q}(\sqrt{-42427(2 + \sqrt{5})}\sqrt{5})$.

We have $p = 5 \equiv 5 \pmod{8}$, $\varepsilon_5 = 2 + \sqrt{5}$ is the fundamental unit of $\mathbb{Q}(\sqrt{5})$, $42427 = 7 \times 11 \times 19 \times 29 \equiv 3 \pmod{4}$ and $(\frac{5}{7}) = -1$, $(\frac{5}{11}) = (\frac{5}{19}) = (\frac{5}{29}) = 1$, then $n = 4$ and $m = 3$. Thus, $r_2(\Delta/K^{*2}) = n + m + 1 = 8$. As $11 = 4^2 - 5$, $19 = 8^2 - 5 \times 3^2$ and $29 = 7^2 - 5 \times 2^2$, it follows that $\alpha_1 = 4 + \sqrt{5}$, $\alpha_2 = 8 + 3\sqrt{5}$ and $\alpha_3 = 7 + 2\sqrt{5}$, so $\alpha_1^* = 5 + 4\sqrt{5}$, $\alpha_2^* = -(15 + 8\sqrt{5})$ and $\alpha_3^* = 7 + 2\sqrt{5}$. Hence,

$$E = K(\sqrt{-3}, \sqrt{-7}, \sqrt{-11}, \sqrt{-19}, \sqrt{29}, \sqrt{\alpha_1^*}, \sqrt{\alpha_2^*}, \sqrt{\alpha_3^*})$$

is the Hilbert genus fields of K .

2. Theorem 4.6: (the case $a \equiv 1 \pmod{4}$). Let $K = \mathbb{Q}(\sqrt{-4199\varepsilon_{73}\sqrt{73}})$, we have: $73 \equiv 1 \pmod{8}$, $4199 = 13 \times 17 \times 19 \equiv 1 \pmod{4}$, $(\frac{73}{13}) = (\frac{73}{17}) = -1$ and $(\frac{73}{19}) = 1$. Thus, $n = 3, m = 1$. Therefore, $r_2(\Delta/K^{*2}) = n + m + 2 = 6$.

As $19 = 26^2 + 3\sqrt{73}$, then $\alpha_1 = 26 + 3\sqrt{73}$ and $\alpha_1^* = 219 + 26\sqrt{73}$. Hence, the Hilbert genus fields of K is:

$$E = K(\sqrt{-1}, \sqrt{13}, \sqrt{-17}, \sqrt{-19}, \sqrt{\varepsilon_{73}}, \sqrt{219 + 26\sqrt{73}}),$$

where ε_{73} is the fundamental unit of $\mathbb{Q}(\sqrt{73})$.

3. Theorem 4.7: (the case $p = 2$). Let $K = \mathbb{Q}(\sqrt{595(1 + \sqrt{2})\sqrt{2}})$. We have $595 = 7 \times 17 \times 5$, $(\frac{2}{5}) = -1$ and $(\frac{2}{7}) = (\frac{2}{17}) = 1$. Then $r_2(\Delta/K^{*2}) = n + m = 3 + 2 = 5$, by $7 \equiv 7 \pmod{8}$, $17 \equiv 1 \pmod{8}$, $7 = 3^2 - 2 \times 1^2$ and $17 = 7^2 - 2 \times 4^2$. So $\alpha_1 = 3 + 1\sqrt{2}$, $\alpha_2 = 7 + 4\sqrt{2}$, $\alpha_1^* = (1 + \sqrt{2})(3 + \sqrt{2}) = 5 + 4\sqrt{2}$ and $\alpha_2^* = -(7 + 4\sqrt{2})$. Hence, the Hilbert genus field of K is:

$$E = K(\sqrt{5}, \sqrt{-7}, \sqrt{17}, \sqrt{5 + 4\sqrt{2}}, \sqrt{-(7 + 4\sqrt{2})}).$$

Acknowledgment

The authors are so grateful to Professor Mohammed Taous for reading the preliminary versions of this paper as well for many helpful suggestions and discussions.

References

[1] Azizi A, Talbi M. Structure du groupe de Galois pour certains corps de classes. International Journal of Algebra 2010; 4 (23): 1127-1136 (in French).

- [2] Azizi A, Taous M. Capitulation des 2-classes d'idéaux de $k = \mathbb{Q}(\sqrt{2p}, i)$. *Acta Arithmetica* 2008; 131: 103-123 (in French).
- [3] Bae S, Yue Q. Hilbert genus fields of real biquadratic fields. *The Ramanujan Journal* 2011; 24 (2): 161-181.
- [4] Chems-Eddin MM, Zekhnini A, Azizi A. Units and 2-class field towers of some multiquadratic number fields. *Turkish Journal of Mathematics* 2020; 44 (4): 1466-1483.
- [5] Chems-Eddin MM, Azizi A, Zekhnini A, Jerrari I. On the Hilbert 2-class field tower of some imaginary biquadratic number fields. *Czechoslovak Mathematical Journal* 2021; 71 (1): 269-281.
- [6] Cohn H. The explicit Hilbert 2-cyclic class field for $\mathbb{Q}(\sqrt{-p})$. *Journal Für Mathematik*, 1980; 321: 64-77.
- [7] Gras G. *Class Field Theory, From Theory to Practice*. Heidelberg, Germany: Springer, 2003.
- [8] Gras G. Sur les l-classes d'idéaux dans les extensions cycliques relatives de degré premier l. *Annales de l'institut Fourier (Grenoble)* 1973; 23: 1-48 (in French).
- [9] Hardy H, Hudson RH, Richman D, Williams KS, Holtz NM. Calculation of the class numbers of imaginary cyclic quartic fields. *Mathematics of Computation* 1987; 49 (180): 615-620.
- [10] Hergoltz G. Über einen Dirichletschen Satz. *Mathematische Zeitschrift* 1922; 12: 225-261 (in German).
- [11] Hilbert D. Über die theorie der relativquadratischen Zahlkörper. *Mathematische Annalen* 1899; 51: 1-127 (in German).
- [12] Huard JG, Spearman BK, Williams KS. Integral bases for quartic fields with quadratic subfields. *Journal of Number Theory* 1995; 51 (1): 87-102.
- [13] Ishida M. The genus fields of algebraic number fields. *Lecture Notes in Mathematics* 555. Heidelberg, Germany: Springer-Verlag, 1976.
- [14] Janusz G. *Algebraic Number Fields*. San Diego, CA, USA: Academic press, 1973.
- [15] Kisilevsky H. Number fields with class number congruent to 4 (mod 8) and Hilbert's theorem 94. *Journal of Number Theory* 1976; 8 (3): 271-279.
- [16] Lemmermeyer F. *Reciprocity laws, from Euler to Eisenstein*. Springer Monographs in Mathematics. Berlin, Germany: Springer-Verlag, 2000.
- [17] Leonard PA, Williams KS. On the divisibility of the class numbers of $\mathbb{Q}(\sqrt{-p})$ and $Q(\sqrt{-2p})$ by 16. *Canadian Mathematical Bulletin* 1982; 25 (2): 200-206.
- [18] Neukirch J. *Class Field Theory*. Berlin, Germany: Springer, 1990.
- [19] Ouyang Y, Zhang Z. Hilbert genus fields of biquadratic fields. *Science China Mathematics* 2014; 57 (10): 2111-2122.
- [20] Ouyang Y, Zhang Z. Hilbert genus fields of biquadratic fields. *The Ramanujan Journal* 2015; 37 (2): 345-363.
- [21] Sime PJ. Hilbert Class Fields of Real Biquadratic Fields. *Journal of Number Theory* 1995; 50 (1): 154-166.
- [22] Williams KS. On Scholz's reciprocity law. *Proceedings of the American Mathematical Society* 1977; 64: 45-46.
- [23] Yue Q. Genus fields of real biquadratic fields. *The Ramanujan Journal* 2010; 21 (1): 17-25.
- [24] Zitouni M. Quelques propriétés des corps cycliques de degré 4. *Seminaire Delange-Pisot-Poitou, théorie des nombres, Fasc. 1, Exp. No. 4, 8 pp. Secrétariat Mathématique, Paris, 1973 (in French)*.