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Mixed Lagrange function in minimax fractional programming problems

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Abstract: In this paper, the aim of this work to study mixed Lagrange function for the minimax fractional programming problem with nonsmooth exponential (p, r)-invex functions with respect to η. We introduced a new concept of saddle point for a mixed Lagrange function. We present the equivalence between a saddle point of the mixed Lagrange function and an optimal solution in the considered minimax fractional programming problem under appropriate nonsmooth exponential (p, r)-invexity hypotheses.

Key words: Minimax problem, exponential (p, r)-invex function, mixed Lagrange function

1. Introduction

In recent years, minimax programming problems have been the subjective of interest. Some basic results of minimax programming problems under differentiable functions have be presented. In [15], Rockafellar has pointed out that in many practical applications of applied mathematics the functions involved are not necessarily differentiable. Shortly after, Clarke [6] studied generalized gradient involving nondifferentiable functions. Many authors have discussed various concepts of generalized convexity/ invexity and have showed optimality conditions and duality theorems for fractional optimization problems, for instance, [1, 7–15, 18, 19], and others. In [9], Ho and Lai defined the concept of nonsmooth exponential (p, r)-invexity for a locally Lipschitz function, which is a generalization of the invexity concept; they presented necessary and sufficient optimality theorems for minimax fractional programming problems as well as weak, strong and strict converse duality theorems for the introducted parametric dual model under appropriate nonsmooth exponential (p, r)-invexity hypotheses. Later, Ho and Lai [10, 11] used their results to minimax fractional programming problems and established duality theorems with appropriate nonsmooth exponential (p, r)-invexity hypotheses.

In mathematical optimization, the method of classical Lagrange multipliers is well-known to solve standard nonlinear mathematical programming problems in which an objective function can be applied in different fields, different types and different constraints, for instance, [2–4, 8, 16, 17, 20], and others. In [4], Bector et al. introduced an incomplete Lagrange function and saddle point of nonlinear programming problems under differentiable generalized invexity suppositions. Antczak [2] introduced the η-Lagrange function and η-saddle point and established the equivalence between optima of a differentiable multiobjective programming problem and η-saddle points of its associated η-approximated vector optimization problem with differentiable invexity supposition. In [20], Zalmai presented four sets of parametric and nonparametric saddle-point-type necessary

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and sufficient optimality conditions for a discrete minimax fractional subset programming problem under suitable $(b, \varphi, \rho, \theta)$-convexity suppositions. Antczak [3] considered semi-infinite minimax fractional programming problems with both inequality and equality constraints and presented the optimal solution as a saddle point of the Lagrange function defined for the considered minimax problem under appropriate (Φ, ρ)-invexity hypotheses. Ho [8] studied a mixed Lagrange function to a system of nondifferentiable multiobjective nonlinear fractional programming problem and presented the equivalence between a saddle point of the aforesaid Lagrange function and an efficient solution of the considered multiobjective programming problem involving appropriate nonsmooth exponential (p, r)-invexity hypotheses.

In this paper, motivated by [3], [4], [8] and incomplete Lagrange technique, we introduce mixed Lagrange function for a minimax fractional programming problem. We present the characterization of an optimal solution as a saddle point of the mixed Lagrange function defined for the considered problem under appropriate nonsmooth exponential (p, r)-invexity hypotheses. The rest of the paper is written as follows: In Section2, we present some preliminary notation, definitions and results, which will be needed in the sequel. First of all, we give the definition of a nonsmooth exponential (p, r)-invex function and also we define the minimax fractional programming problem considered in the paper. For the aforesaid extremum problem, we define its associated nonfractional parametric optimization problem introduced by Bector et al. [5]. Then, we give some results which show the equivalence between the minimax fractional programming problem and its associated nonfractional parametric programming problem. At last, we re-call the parametric necessary optimality conditions for the considered minimax fractional programming problem established by Liu [14]. In Section 3, for nonfractional parametric optimization problem, we define the so-called mixed Lagrange function and its saddle point. Then, under appropriate nonsmooth exponential (p,r) -invexity hypotheses, we prove the equivalence between an optimal solution in the considered minimax fractional programming problem and a saddle point of the mixed Lagrange function.

2. Definitions and preliminaries

Throughout the paper, $\mathbb{R}^n$ is the n-dimensional Euclidean space and $\mathbb{R}^n_+$ denote the order cone. For the cone partial order, if $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$ in $\mathbb{R}^n$, we define:

(1) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \cdots, n$;

(2) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, \cdots, n$;

(3) $x \geqq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2, \cdots, n$;

(4) $x \geq y$ if and only if $x \geqq y$ and $x_i \neq y_i$ for some $i \in \{1, 2, \cdots, n\}$.

Let X be an open subset of $\mathbb{R}^n$. A function $f : X \longrightarrow \mathbb{R}$ is said to be locally Lipschitz around $x \in X$ if there exist a positive constant $c \in \mathbb{R}$ and a neighborhood Γ of $x \in X$ such that

$$|f(y) - f(z)| \leqq c\|y - z\| \qquad \text{for all} \quad z,\, y \in \Gamma.$$

where $\|\cdot\|$ stands for any norm of $\mathbb{R}^n$.

For any vector ν in $\mathbb{R}^n$, the generalized directional derivative of f at x in the direction $\nu \in \mathbb{R}^n$ in Clarke sense (see [6]) is given by

$$f^\circ(x;\nu) = \limsup_{\substack{y \longrightarrow x \\ \lambda \longrightarrow 0^+}} \frac{f(y+\lambda\nu) - f(y)}{\lambda}.$$

The generalized subdifferential of f at $x \in S$ is defined by the set

$$\partial^c f(x) = \{\xi \in \mathbb{R}^n : f^\circ(x;\nu) \geq \langle \xi, \nu \rangle \quad \text{for all} \quad \nu \in \mathbb{R}^n\}$$

where $\langle \xi, \nu \rangle$ denotes the inner product in $\mathbb{R}^n$.

Now, we introduce the following definitions on exponential (p,r)-invex function conditions by Ho and Lai [9].

Definition 2.1 *(cf. [9]) Let p, r be arbitrary real numbers. A locally Lipschitz function $f : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be exponential $(\mathbf{p},\mathbf{r})$-invexity (strictly) at $u \in X$ if there exists a function $\eta : X \times X \longrightarrow \mathbb{R}^n$ with property $\eta(x,u) = 0$ only if $u = x$ in X such that for each $x \in X$, the following inequalities hold for any $\xi \in \partial^c f(u)$*

$$\frac{1}{r}e^{rf(x)} \geq \frac{1}{r}e^{rf(u)}\left[1 + \frac{r}{p}\left\langle \xi \text{ , } (e^{p\eta(x,u)} - \mathbf{1})\right\rangle\right] \quad (> \text{ if } x \neq u) \text{ for } p \neq 0,\, r \neq 0, \tag{2.1}$$

$$e^{rf(x)} - e^{rf(u)} \geq re^{rf(u)}\langle \xi \text{ , } \eta(x,u)\rangle \quad (> \quad \text{if} \quad x \neq u) \text{ for } r \neq 0,\, p = 0, \tag{2.2}$$

$$f(x) - f(u) \geq \frac{1}{p}\left\langle \xi \text{ , } (e^{p\eta(x,u)} - \mathbf{1})\right\rangle \quad (> \quad \text{if} \quad x \neq u) \text{ for } p \neq 0,\, r = 0, \tag{2.3}$$

$$f(x) - f(u) \geq \langle \xi \text{ , } \eta(x,u)\rangle \quad (> \quad \text{if} \quad x \neq u) \text{ for } p = 0,\, r = 0, \tag{2.4}$$

where $\mathbf{1} = (1,1,\cdots,1) \in \mathbb{R}^n$, $(e^{p\eta(x,u)} - \mathbf{1})$ stands for the n-vector $(e^{p\eta_1(x,u)} - 1,\ e^{p\eta_2(x,u)} - 1, \cdots, e^{p\eta_n(x,u)} - 1)$, and $\langle \cdot \text{ , } \cdot \rangle$ stands for the inner product in $\mathbb{R}^n$ throughout this paper.

Remark 2.2 *All theorems in our work will be described only in the case of $p \neq 0$ and $r \neq 0$. We omit the proof of other cases like in (2.2), (2.3), and (2.4).*

In this paper, we consider the following minimax fractional programming problem as the primal problem:

$$(FP) \qquad v^* = \min_{x \in \mathfrak{F}} \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)}$$
$$\text{subject to} \quad \mathfrak{F} = \left\{x \in X \;\middle|\; h(x) \in -\mathbb{R}^m_+\right\},$$

where

1 $\mathfrak{F}$ is nonempty and compact set;

2 $f_i(x)$ and $g_i(x) : X \longrightarrow \mathbb{R}$, and $h(\cdot) : X \longrightarrow \mathbb{R}^m$ are locally Lipschitz functions;

3 Without loss of generality, we may assume that $g_i(x) > 0$ and $f_i(x) \geq 0$ for all $x \in \mathfrak{F}$.

In Bector et al. [5], we know that the primal problem (FP) is equivalent to a nonfractional parametric programming problem $(EP)_v$ for a given v:

$$
\begin{aligned}
(EP)_v \qquad & \min \quad q \\
& \text{subject to} \quad f_i(x) - v g_i(x) \leq q, \qquad i = 1, 2, \cdots, p, \\
& h(x) \in -\mathbb{R}^m_+.
\end{aligned}
$$

The precise relationship linking (FP) and $(EP)_v$ that will be useful for our present purposes is stated in the following lemmas:

Lemma 2.3 *(cf. [5]) If (x, v, q) is a feasible solution to the nonfractional parametric programming problem $(EP)_v$, then x is a feasible solution to the primal problem (FP). If x is a feasible solution to the primal problem (FP), then there exist v and q such that (x, v, q) is a feasible solution to the nonfractional parametric programming problem $(EP)_v$.*

Lemma 2.4 *(cf. [5]) x^* is an optimal solution to the primal problem (FP) with corresponding optimal value of the (FP)-objective equal to v^* iff (x^*, v^*, q^*) is an optimal solution to the nonfractional parametric programming problem $(EP)_v$ with corresponding optimal value of the $(EP)_v$-objective equal to zero, i.e., $q^* = 0$.*

For the primal problem (FP), we give the parametric necessary optimality conditions presented by Liu [14].

Theorem 2.5 *(cf. [14]) Let x^* be an optimal solution to the primal problem (FP) with optimal value equal to v^* and an appropriate constraint qualification [7] holds for $(EP)_{v^*}$. Then, there exist $y^* \in \mathbb{R}^p$, $q^* \in \mathbb{R}$, and an m-vector Lagrange multiplier $z^* \in \mathbb{R}^m$ such that*

$$0 \in \sum_{i=1}^{p} y_i^* \left\{ \partial^c f_i(x^*) - v^* \partial^c g_i(x^*) \right\} + \sum_{j=1}^{m} z_j^* \partial^c h_j(x^*), \tag{2.5}$$

$$y_i^* [f_i(x^*) - v^* g_i(x^*)] = 0, \qquad i = 1, 2, \cdots, p, \tag{2.6}$$

$$z_j^* h_j(x^*) = 0, \qquad j = 1, 2, \cdots, m, \tag{2.7}$$

$$q^* = 0, z^* \in \mathbb{R}^m_+, \, y^* \in \Im, \tag{2.8}$$

where $\Im = \{y^* \in \mathbb{R}^p_+ : y^* = (y_1^*, y_2^*, \cdots, y_p^*)$ with $\sum_{i=1}^{p} y_i^* = 1\}$.

3. Mixed Lagrange function and saddle point theorems

We introduce a new mixed Lagrange function that associates with the considered minimax fractional programming problem (FP). For convenience, the symbols are stated as follows to define mixed Lagrange function. At first, we define the index sets $P = \{1, 2, \cdots, p\}$ and $M = \{1, 2, \cdots, m\}$. We partition the index set M of the constraint function $h = (h_1, h_2, \cdots, h_m) : X \to \mathbb{R}^m$ of the primal problem (FP) to be $J_0, J_1, \cdots, J_s$ $(s < m)$ with $\cup_{\alpha=0}^{s} J_\alpha = M$, $J_\alpha \cap J_\beta = \emptyset$ if $\alpha \neq \beta$, and $|J_\alpha|$ denotes the cardinality of the index set J_α, $\alpha = 0, 1, 2, \cdots, s$.

Now, we perform the mixed Lagrange function in the primal problem as the following form:

Definition 3.1 *The mixed Lagrange function associated with the constrained primal problem (FP) is the function $L : \mathfrak{F} \times \mathbb{R}_+^p \times \mathbb{R}_+ \times \mathbb{R}_+^{|J_0|} \to \mathbb{R}^p$ defined by*

$$L(x, y, v, z) \equiv (L_1(x, y, v, z),\, L_2(x, y, v, z), \cdots,\, L_p(x, y, v, z)),$$

where $L_i(x, y, v, z) \equiv y_i[f_i(x) - v g_i(x)] + \dfrac{1}{p} \sum_{j \in J_0} z_j h_j(x)$, $i \in P$, $x \in \mathfrak{F}$.

If the index set M of the constraints in the primal problem (FP) is separated into two parts J_0 and J_1, that is, $M = J_0 \cup J_1$ ($J_\alpha = \emptyset$ for $\alpha = 2, 3, \ldots, s$), we have

(i) If $J_0 = M$ and $J_1 = \emptyset$, then the mixed Lagrange function is the vector-valued Lagrange function without equality constraints in [3].

(ii) If J_0 and J_1 are nonempty, then the mixed Lagrange function is the imcomplete Lagrange function in [4].

Now, we give the definition of a saddle point of the mixed Lagrange function L in the considered primal problem (FP).

Definition 3.2 *A point $(\overline{x}, \overline{y}, \overline{v}, \overline{z}) \in \mathfrak{F} \times \mathbb{R}_+^p \times \mathbb{R}_+ \times \mathbb{R}_+^{|J_0|}$ is said to be a saddle point for the mixed Lagrange function L if*

(i) $L(\overline{x}, \overline{y}, \overline{v}, z) \leqq L(\overline{x}, \overline{y}, \overline{v}, \overline{z})$, *for all* $z \in \mathbb{R}_+^{|J_0|}$,

(ii) $L(\overline{x}, \overline{y}, \overline{v}, \overline{z}) \leqq L(x, \overline{y}, \overline{v}, \overline{z})$, *for all* $x \in \mathfrak{F}$.

At first, we show the saddle point related to an optimal solution of the primal problem (FP) as follows:

Theorem 3.3 *Let $(\overline{x}, \overline{y}, \overline{v}, \overline{z})$ be a saddle point of the mixed Lagrange function in the considered primal problem (FP). Then, $\overline{x}$ is an optimal solution of the primal problem (FP).*

Proof Since $(\overline{x}, \overline{y}, \overline{v}, \overline{z})$ is a saddle point of the mixed Lagrange function in the considered primal problem (FP), by Definition 3.2, the conditions (i) and (ii) are satisfied. From the condition (i), we have

$$\overline{y}_i[f_i(\overline{x}) - \overline{v} g_i(\overline{x})] + \frac{1}{p} \sum_{j \in J_0} z_j h_j(\overline{x}) \leq \overline{y}_i[f_i(\overline{x}) - \overline{v} g_i(\overline{x})] + \frac{1}{p} \sum_{j \in J_0} \overline{z}_j h_j(\overline{x}) \tag{3.1}$$

hold for all $z \in \mathbb{R}_+^{|J_0|}$.

It follows the relation (3.1) as well as p is a positive integer, we obtain

$$\sum_{j \in J_0} z_j h_j(\overline{x}) \leq \sum_{j \in J_0} \overline{z}_j h_j(\overline{x}) \quad \text{for all} \quad z \in \mathbb{R}_+^{|J_0|}. \tag{3.2}$$

From the inequality (i) in Definition 3.2 we have, by setting $z_j = 0$, $j \in J_0$, and the relation (3.2), that

$$\sum_{j\in J_0} \overline{z}_j h_j(\overline{x}) \geq 0. \tag{3.3}$$

$\overline{x} \in \mathfrak{F}$ along with $\overline{z}_j \geq 0$, $j \in J_0$ implies

$$\sum_{j\in J_0} \overline{z}_j h_j(\overline{x}) \leq 0. \tag{3.4}$$

Thus, by inequalities (3.3) and (3.4), it follows that

$$\sum_{j\in J_0} \overline{z}_j h_j(\overline{x}) = 0. \tag{3.5}$$

We proceed by contradiction. Suppose, contrary to the result, that $\overline{x} \in \mathfrak{F}$ is not an optimal solution in the considered primal problem (FP). Then, there exists $x \in \mathfrak{F}$ such that

$$\max_{1\leq i\leq p} \frac{f_i(x)}{g_i(x)} < \max_{1\leq i\leq p} \frac{f_i(\overline{x})}{g_i(\overline{x})} = \overline{v}$$

that is,

$$f_i(x) - \overline{v}g_i(x) < f_i(\overline{x}) - \overline{v}g_i(\overline{x}), \quad i \in P. \tag{3.6}$$

Multiplying the inequalities (3.6) by the associated Lagrange multiplier $\overline{y}_i$, $i = 1, 2, \cdots, p$, we obtain

$$\overline{y}_i[f_i(x) - \overline{v}g_i(x)] \leq \overline{y}_i[f_i(\overline{x}) - \overline{v}g_i(\overline{x})], \quad i \in P, \tag{3.7}$$

and

$$\overline{y}_i[f_i(x) - \overline{v}g_i(x)] < \overline{y}_i[f_i(\overline{x}) - \overline{v}g_i(\overline{x})], \quad \text{for at least one} \quad i \in P. \tag{3.8}$$

For x and $\overline{x}$ belong to $\mathfrak{F}$ and the equality (3.5), we get, respectively,

$$\begin{aligned} &\overline{y}_i[f_i(x) - \overline{v}g_i(x)] + \frac{1}{p}\sum_{j\in J_0} \overline{z}_j h_j(x) \\ &\leq \overline{y}_i[f_i(\overline{x}) - \overline{v}g_i(\overline{x})] + \frac{1}{p}\sum_{j\in J_0} \overline{z}_j h_j(\overline{x}), \quad i \in P, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\overline{y}_i[f_i(x) - \overline{v}g_i(x)] + \frac{1}{p}\sum_{j\in J_0} \overline{z}_j h_j(x) \\ &< \overline{y}_i[f_i(\overline{x}) - \overline{v}g_i(\overline{x})] + \frac{1}{p}\sum_{j\in J_0} \overline{z}_j h_j(\overline{x}), \quad \text{for at least one} \quad i \in P. \end{aligned} \tag{3.10}$$

By the definition of the mixed Lagrange function, (3.9) and (3.10) imply, respectively,

$$L_i(x, \overline{y}, \overline{v}, \overline{z}) \leq L_i(\overline{x}, \overline{y}, \overline{v}, \overline{z}), \quad i \in P, \tag{3.11}$$

and

$$L_i(x, \overline{y}, \overline{v}, \overline{z}) < L_i(\overline{x}, \overline{y}, \overline{v}, \overline{z}), \quad \text{for at least one} \quad i \in P, \tag{3.12}$$

contradicting the condition (ii) in the definition of a saddle point of the mixed Lagrange function L defined in the primal problem (FP). □

Suppose that $\overline{x}$ is an optimal solution of the primal problem (FP). Using $\overline{x}$, the optimality conditions of the primal problem (FP) and some reasonable conditions, we can find a saddle point of the mixed Lagrange function in the considered primal problem (FP) as the following theorem.

Theorem 3.4 *Let $\overline{x} \in \mathfrak{F}$ be an optimal solution to the considered primal problem (FP) such that there exist $\overline{y} \in \mathbb{R}^p_+$, $\overline{v} \in \mathbb{R}_+$ and $\overline{z} \in \mathbb{R}^m_+$ satisfying conditions from (2.5) $\sim$ (2.8). Denote*

$$A(\cdot) = \sum_{i=1}^{p} \overline{y}_i[f_i(\cdot) - \overline{v}g_i(\cdot)] + \sum_{j \in J_0} \overline{z}_j h_j(\cdot).$$

If any one of the following three conditions holds:

(1). $A(\cdot)$ is an exponential (p,r)-invexity and $\sum_{j \in J_\alpha} \overline{z}_j h_j(\cdot)$ for $\alpha = 1, 2, \cdots, s$ are exponential (p,r)-invexities with respect to η at $\overline{x}$.

(2). $A(\cdot)$ is a strictly exponential (p,r)-invexity and $\sum_{j \in J_\alpha} \overline{z}_j h_j(\cdot)$ for $\alpha = 1, 2, \cdots, s$ are exponential (p,r)-invexities with respect to η at $\overline{x}$.

(3). $A(\cdot)$ is a strictly exponential (p,r)-invexity and $\sum_{j \in J_\alpha} \overline{z}_j h_j(\cdot)$ for $\alpha = 1, 2, \cdots, s$ are strictly exponential (p,r)-invexities with respect to η at $\overline{x}$.

Then, $(\overline{x}, \overline{y}, \overline{v}, \overline{z})$ is a saddle point of the mixed Lagrange function in the considered primal problem (FP).

Proof From $\overline{x} \in \mathfrak{F}$, we obtain

$$\sum_{j \in J_0} z_j h_j(\overline{x}) \leq 0 \tag{3.13}$$

holds for all $z \in \mathbb{R}^{|J_0|}_+$.

By assumption, $\overline{x} \in \mathfrak{F}$ is an optimal solution to the primal problem (FP) and there exist $\overline{y} \in \mathbb{R}^p_+$, $\overline{v} \in \mathbb{R}$ and $\overline{z} \in \mathbb{R}^m_+$ such that conditions (2.5) $\sim$ (2.8) are fulfilled at this point. Then, we obtain

$$\sum_{j \in J_0} \overline{z}_j h_j(\overline{x}) = 0. \tag{3.14}$$

From the relations (3.13) and (3.14), it follows that

$$\sum_{j \in J_0} z_j h_j(\overline{x}) \leq 0 = \sum_{j \in J_0} \overline{z}_j h_j(\overline{x}). \tag{3.15}$$

For each $i \in P$, we obtain

$$\overline{y}_i[f(\overline{x}) - \overline{v}g_i(\overline{x})] + \frac{1}{p} \sum_{j \in J_0} z_j h_j(\overline{x}) \leq \overline{y}_i[f(\overline{x}) - \overline{v}g_i(\overline{x})] + \frac{1}{p} \sum_{j \in J_0} \overline{z}_j h_j(\overline{x}).$$

The above inequality implies

$$L_i(\overline{x}, \overline{y}, \overline{v}, z) \leq L_i(\overline{x}, \overline{y}, \overline{v}, \overline{z}), \quad i \in P \tag{3.16}$$

holds for all $z \in \mathbb{R}_+^{|J_0|}$. This means that the inequality (i) in the Definition 3.2 of a saddle point of the mixed Lagrange function is satisfied.

Now, we prove the inequality (ii) in Defintion 3.2. We proceed by contradication. Suppose, contrary to the result, that there exists $x \in \mathfrak{F}$ such that $L(\overline{x}, \overline{y}, \overline{v}, \overline{z}) \nleq L(x, \overline{y}, \overline{v}, \overline{z})$. This means that

$$L_i(x, \overline{y}, \overline{v}, \overline{z}) \leq L_i(\overline{x}, \overline{y}, \overline{v}, \overline{z}), \quad i \in P, \tag{3.17}$$

and

$$L_t(x, \overline{y}, \overline{v}, \overline{z}) < L_t(\overline{x}, \overline{y}, \overline{v}, \overline{z}), \quad \text{for some} \quad t \in P. \tag{3.18}$$

By the relations (3.17), (3.18) and Definition 3.1, we have

$$\begin{aligned}\overline{y}_i[f_i(x) - \overline{v}g_i(x)] \quad &+\frac{1}{p}\sum_{j\in J_0}\overline{z}_j h_j(x)\\ &\leq \overline{y}_i[f_i(\overline{x}) - \overline{v}g_i(\overline{x})] + \frac{1}{p}\sum_{j\in J_0}\overline{z}_j h_j(\overline{x}), \quad i \in P,\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\overline{y}_t[f_t(x) \quad &-\overline{v}g_t(x)] + \frac{1}{p}\sum_{j\in J_0}\overline{z}_j h_j(x)\\ &< \overline{y}_t[f_t(\overline{x}) - \overline{v}g_t(\overline{x})] + \frac{1}{p}\sum_{j\in J_0}\overline{z}_j h_j(\overline{x}), \quad \text{for some} \quad t \in P.\end{aligned} \tag{3.20}$$

Adding both sides of the inequalities (3.19) and (3.20), we obtain

$$\sum_{i=1}^{p}\overline{y}_i[f_i(x) - \overline{v}g_i(x)] + \sum_{j\in J_0}\overline{z}_j h_j(x) < \sum_{i=1}^{p}\overline{y}_i[f_i(\overline{x}) - \overline{v}g_i(\overline{x})] + \sum_{j\in J_0}\overline{z}_j h_j(\overline{x}),$$

that is,

$$A(x) < A(\overline{x}). \tag{3.21}$$

According to relation (2.5), there exist $\xi_i \in \partial^c f_i(\overline{x})$, $\rho_i \in \partial^c g_i(\overline{x})$ for $i \in P$, and $\zeta_j \in \partial^c h_j(\overline{x})$ for all $j \in M$, such that the vector

$$\sum_{i=1}^{p}\overline{y}_i[\xi_i - \overline{v}\rho_i] + \sum_{j=1}^{m}\overline{z}_j\zeta_j = 0 = \sum_{i=1}^{p}\overline{y}_i[\xi_i - \overline{v}\rho_i] + \sum_{j\in J_0}\overline{z}_j\zeta_j + \sum_{j\in M\setminus J_0}\overline{z}_j\zeta_j. \tag{3.22}$$

By the equality (3.22), it follows that

$$\frac{1}{p}\left\langle \sum_{i=1}^{p}\overline{y}_i[\xi_i - \overline{v}\rho_i] + \sum_{j\in J_0}\overline{z}_j\zeta_j + \sum_{j\in M\setminus J_0}\overline{z}_j\zeta_j \text{ , } (e^{p\eta(x,\overline{x})} - \mathbf{1})\right\rangle = 0. \tag{3.23}$$

For each $x \in \mathfrak{F}$, we get

$$h_j(x) \leq 0, \quad \text{for all} \quad j \in M. \tag{3.24}$$

Since $\overline{x}$ is an efficient solution and from the inequality (3.24) and $\overline{z} \in \mathbb{R}^m_+$, we have

$$\sum_{j\in J_\alpha} \overline{z}_j h_j(x) \leq 0 = \sum_{j\in J_\alpha} \overline{z}_j h_j(\overline{x}), \quad \text{for all} \quad \alpha = 1, 2, \cdots, s. \tag{3.25}$$

By assumption, $\sum_{j\in J_\alpha} \overline{z}_j h_j(\cdot)$ for $\alpha = 1, 2, \cdots, s$ are exponential (p, r)-invexity with respect to η at $\overline{x}$. Hence, by Definition 2.1, the following inequalities

$$\frac{1}{r} e^{r \sum_{j\in J_\alpha} \overline{z}_j h_j(x)} \geq \frac{1}{r} e^{r \sum_{j\in J_\alpha} \overline{z}_j h_j(\overline{x})} \left[1 + \frac{r}{p} \left\langle H_\alpha \text{ , } (e^{p\eta(x,\overline{x})} - \mathbf{1}) \right\rangle \right] \tag{3.26}$$

where $x \in \mathfrak{F}$ and $H_\alpha = \sum_{j\in J_\alpha} \overline{z}_j \zeta_j$ for $\alpha = 1, 2, \cdots, s$.

Using the inequality (3.25) together with (3.26), we have

$$\frac{1}{p} \left\langle H_\alpha \text{ , } (e^{p\eta(x,\overline{x})} - \mathbf{1}) \right\rangle \leq 0, \quad \alpha = 1, 2, \cdots, s. \tag{3.27}$$

From the inequality (3.27), we obtain

$$\frac{1}{p} \left\langle \sum_{\alpha=1}^{s} H_\alpha \text{ , } (e^{p\eta(x,\overline{x})} - \mathbf{1}) \right\rangle \leq 0,$$

that is,

$$\frac{1}{p} \left\langle \sum_{j\in M\setminus J_0} \overline{z}_j \zeta_j \text{ , } (e^{p\eta(x,\overline{x})} - \mathbf{1}) \right\rangle \leq 0. \tag{3.28}$$

Thus, from (3.23) and (3.28), it follows that

$$\frac{1}{p} \left\langle \sum_{i=1}^{p} \overline{y}_i [\xi_i - \overline{v} \rho_i] + \sum_{j\in J_0} \overline{z}_j \zeta_j \text{ , } (e^{p\eta(x,\overline{x})} - \mathbf{1}) \right\rangle \geq 0. \tag{3.29}$$

If hypothesis (1) holds, $A(\cdot)$ is an exponential (p, r)-invexity with respect to η at $\overline{x}$, we obtain

$$\frac{1}{r} e^{rA(x)} \geq \frac{1}{r} e^{rA(\overline{x})} \left[1 + \frac{r}{p} \left\langle \sum_{i=1}^{p} \overline{\lambda}_i [\xi_i - \overline{v}_i \rho_i] + \sum_{j\in J_0} \overline{z}_j \zeta_j \text{ , } (e^{p\eta(x,\overline{x})} - \mathbf{1}) \right\rangle \right]. \tag{3.30}$$

Using the inequality (3.29) together with (3.30), we have

$$\frac{1}{r} e^{rA(x)} - \frac{1}{r} e^{rA(\overline{x})} \geq 0,$$

that is,

$$A(x) \geq A(\overline{x}),$$

which contradicts inequality (3.21). This completes the proof of theorem under hypotheses (1).

Proof of the second inequality in Definition 3.2 under hypotheses (2) and (3) are similar to the proof under hypotheses (1).

We have established under each hypotheses that the inequality (ii) in Definition 3.2 is satisfied. This means that $(\overline{x}, \overline{\lambda}, \overline{v}, \overline{z})$ is a saddle point of the mixed Lagrange function in the considered primal problem (FP). □

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