Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 1195 – 1208
© TÜBİTAK
doi:10.3906/mat-2101-20

Fekete-Szegö problem for a new subclass of analytic functions satisfying subordinate condition associated with Chebyshev polynomials

Muhammet KAMALİ[1], **Murat ÇAĞLAR**[2,*], **Erhan DENİZ**[2], **Mirzaolim TURABAEV**[1]
[1]Department of Mathematics, Faculty of Sciences, Kyrgyz-Turkish Manas University,
Chyngz Aitmatov Avenue, Bishkek, Kyrgyz Republic
[2]Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey

Received: 06.01.2021 • **Accepted/Published Online:** 26.03.2021 • **Final Version:** 20.05.2021

Abstract: In this paper,we define a class of analytic functions $F_{(\beta,\lambda)}(H,\alpha,\delta,\mu)$, satisfying the following subordinate condition associated with Chebyshev polynomials

$$\left\{\alpha\left[\frac{zG^{'}(z)}{G(z)}\right]^{\delta}+(1-\alpha)\left[\frac{zG^{'}(z)}{G(z)}\right]^{\mu}\left[1+\frac{zG^{''}(z)}{G^{'}(z)}\right]^{1-\mu}\right\}\prec H(z,t),$$

where $G(z)=\lambda\beta z^{2}f^{''}(z)+(\lambda-\beta)zf^{'}(z)+(1-\lambda+\beta)f(z)$, $0\leq\alpha\leq 1$, $1\leq\delta\leq 2$, $0\leq\mu\leq 1$, $0\leq\beta\leq\lambda\leq 1$ and $t\in\left(\frac{1}{2},1\right]$. We obtain initial coefficients $|a_2|$ and $|a_3|$ for this subclass by means of Chebyshev polynomials expansions of analytic functions in $\mathcal{D}$. Furthermore, we solve Fekete-Szegö problem for functions in this subclass.We also provide relevant connections of our results with those considered in earlier investigations. The results presented in this paper improve the earlier investigations.

Key words: Analytic and univalent functions, subordination, Chebyshev polynomials, coefficient estimates, Fekete–Szegö inequality

1. Introduction

Let $\mathcal{D}$ be the open unit disc $\mathcal{D}=\{z\in\mathbb{C}:|z|<1\}$ and $\mathcal{A}$ be the class of functions analytic in $\mathcal{D}$, satisfying the conditions $f(0)=0$ and $f'(0)=1$.

Then each functions f in $\mathcal{A}$ has the following Taylor expansion

$$f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n}. \tag{1.1}$$

Furthermore, by $\mathcal{S}$ we shall denote the class of all functions $\mathcal{A}$ that are univalent in $\mathcal{D}$.

Let f and g be analytic functions in $\mathcal{D}$. We define that the function f is subordinate to g in $\mathcal{D}$ and denoted by

$$f(z)\prec g(z)\quad(z\in\mathcal{D}),$$

*Correspondence: mcaglar25@gmail.com
2010 *AMS Mathematics Subject Classification:* 30C45, 30C50

if there exists a Schwarz function ω, which is analytic in $\mathcal{D}$ with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in\mathcal{D})$ such that

$$f(z)=g\left(\omega(z)\right)\quad (z\in\mathcal{D})\,. \tag{1.2}$$

If g is a univalent function in $\mathcal{D}$, then

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0) \text{ and } f(\mathcal{D})\subset g(\mathcal{D}).$$

A function $f\in\mathcal{A}$ maps $\mathcal{D}$ onto a starlike domain with respect to $w_0=0$ if and only if

$$\frac{zf'(z)}{f(z)}\prec\frac{1-z}{1+z}\quad (z\in\mathcal{D})\,. \tag{1.3}$$

A function $f\in\mathcal{A}$ maps $\mathcal{D}$ onto a convex domain if and only if

$$1+\frac{zf''(z)}{f'(z)}\prec\frac{1-z}{1+z}\quad (z\in\mathcal{D})\,. \tag{1.4}$$

It is well known that if a function $f\in\mathcal{S}$ satisfies (1.3) and (1.4), then f is starlike and convex in $\mathcal{D}$, respectively. Let $\beta\in[0,1)$. A function $f\in\mathcal{A}$ is said to be starlike of order β and convex of order β, respectively, if

$$\frac{zf'(z)}{f(z)}\prec\frac{1-(1-2\beta)\,z}{1+z}\quad (z\in\mathcal{D})$$

and

$$1+\frac{zf''(z)}{f'(z)}\prec\frac{1-(1-2\beta)\,z}{1+z}\quad (z\in\mathcal{D}) \tag{1.5}$$

are satisfied.

The aritmetic means of some functions and expressions is very frequently used in mathematics, specially in geometric functions theory. Making use of the arithmetic means Mocanu [14] introduced the class of $\alpha-$convex $(0\leq\alpha\leq1)$ functions as follows

$$M_\alpha=\left\{f\in\mathcal{A}:\Re\left[(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>0,\ z\in\mathcal{D}\right\},$$

which, in some special cases, reduced the class of starlike and convex functions. In general, the class of $\alpha-$convex functions determines the arithmetic bridge between starlikeness and convexity.

Using the geometric means, Lewandowski et al. [12] defined the class of $\mu-$starlike functions $(0\leq\mu\leq1)$ consisting of the functions $f\in\mathcal{A}$ that satisfy the inequality to

$$\Re\left[\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\mu}\right]>0\quad (z\in\mathcal{D})\,.$$

We note that the class $\mu-$starlike functions constitutes the geometric bridge between starlikeness and convexity.

There is a close relationship between the above classes. For example, $0-$convex (or $1-$starlike) and $1-$convex (or $0-$starlike) functions are, respectively, starlike and convex functions.

In 1933, Fekete and Szegö [9] obtained a sharp bound of the functional $\left|a_3 - \mu a_2^2\right|$, with real μ $(0 \leq \mu \leq 1)$ for a univalent function f. Since then, the problem of finding the sharp bounds for this functional of any compact family of functions or $f \in \mathcal{A}$ with any complex μ is known as the classical Fekete–Szegö problem or inequality.

Chebyshev polynomials have greater importance in numerical analysis and, more generally, in applications of mathematics. There are four kinds of Chebyshev polynomials. The majority of books and research papers dealing with specific ortogonal polynomials of the Chebyshev family contain mainly results of Chebyshev polynomials of the first and second kinds $T_n(t)$, $U_n(t)$ and their numerous uses in different applications; see, for example, Doha [7] and Mason [13].

The Chebyshev polynomials of the first and second kinds are well known. In the case of a real variable t on $(-1, 1)$, they are defined by

$$T_n(t) = \cos n\varphi,$$

$$U_n(t) = \frac{\sin(n+1)\varphi}{\sin\varphi}$$

where n denotes the polynomial degree and $t = \cos\varphi$. For a brief history of Chebyshev polynomials of the first kind $T_n(t)$, the second kind $U_n(t)$ and their applications one can refer ([1]-[5], [7], [8], [18]-[20]).

We consider that if $t = \cos\varphi$ $\left(\frac{-\pi}{3} < \varphi < \frac{\pi}{3}\right)$, then

$$\begin{aligned} H(z,t) \quad : &= \frac{1}{1-2tz+z^2} = \frac{1}{1-2\cos\varphi z + z^2} \\ &= 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\varphi}{\sin\varphi} z^n \qquad (z \in \mathcal{D}). \end{aligned}$$

Thus, we have

$$H(z,t) = 1 + 2\cos\varphi z + \left(3\cos^2\varphi - \sin^2\varphi\right) z^2 + \cdots \quad (z \in \mathcal{D}).$$

So, according to [20], we write the Chebyshev polynomials of the second kind as following:

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots \quad (-1 < t < 1,\ z \in \mathcal{D})$$

where $U_{n-1}(t) = \frac{\sin(n \arccos t)}{\sqrt{1-t^2}}$ $(n \in \mathbb{N})$ and we have

$$\begin{aligned} U_n(t) &= 2tU_{n-1}(t) - U_{n-2}(t), \\ U_0(t) &= 1, \\ U_1(t) &= 2t, \\ U_2(t) &= 4t^2 - 1, \\ U_3(t) &= 8t^3 - 4t, \\ U_4(t) &= 16t^4 - 12t^2 + 1. \end{aligned} \tag{1.6}$$

The Chebyshev polynomials $T_n(t)$, $t \in [-1,1]$ of the first kind have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(t) z^n = \frac{1-tz}{1-2tz+z^2} \quad (z \in \mathcal{D}).$$

There are the following connections by the Chebyshev polynomials of the first kind $T_n(t)$ and the second kind $U_n(t)$:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t), \qquad T_n(t) = U_n(t) - tU_{n-1}(t), \qquad 2T_n(t) = U_n(t) - U_{n-2}(t).$$

Now, we define a subclass of analytic functions in $\mathcal{D}$ with the following subordination condition:

Definition 1.1 *Let $0 \leq \alpha \leq 1,\ 1 \leq \delta \leq 2,\ 0 \leq \mu \leq 1,\ 0 \leq \beta \leq \lambda \leq 1$ and $t \in \left(\frac{1}{2}, 1\right]$. We say that $f \in \mathcal{A}$ of the form (1.1) belong to $f \in F_{(\beta,\lambda)}\left(H, \alpha, \delta, \mu\right)$ if*

$$\alpha \left[\frac{zG'\left(z\right)}{G\left(z\right)}\right]^{\delta} + (1-\alpha)\left[\frac{zG'\left(z\right)}{G\left(z\right)}\right]^{\mu}\left[1 + \frac{zG''\left(z\right)}{G'\left(z\right)}\right]^{1-\mu} \prec H\left(z,t\right) = \frac{1}{1-2tz+z^2} \tag{1.7}$$

where $z \in \mathcal{D}$ and $G\left(z\right) = \lambda\beta z^2 f''\left(z\right) + \left(\lambda - \beta\right) z f'\left(z\right) + \left(1 - \lambda + \beta\right) f\left(z\right)$.

Taking $\alpha = \delta = t = 1,\ \beta = \lambda = 0$ and $\omega(z) = z$ (in (1.2)) in Definition 1.1, we obtain the following example.

Example 1.2 *The function*

$$f(z) = \frac{z}{1-z} e^{\frac{z}{1-z}}$$

with the series expansion $f(z) = z + 2z^2 + \frac{7}{2}z^3 + \cdots$ belongs to $F_{(0,0)}\left(H, 1, 1, \mu\right)$.

Remark 1.3 *Note that for restricted values of the parameters involved in the class $F_{(\beta,\lambda)}\left(H, \alpha, \delta, \mu\right)$ gives the following special subclasses:*

i) A function $f \in \mathcal{A}$ is said to be in the class $F_{(\beta,\lambda)}\left(H, 1, 1, \mu\right) = N\left(\lambda, \beta, t\right),\ 0 \leq \beta \leq \lambda \leq 1,\ t \in \left(\frac{1}{2}, 1\right]$, if the following subordination holds:

$$\frac{\lambda\beta z^3 f'''\left(z\right) + \left(2\lambda\beta + \lambda - \beta\right) z^2 f''\left(z\right) + z f'\left(z\right)}{\lambda\beta z^2 f''\left(z\right) + \left(\lambda - \beta\right) z f'\left(z\right) + \left(1 - \lambda + \beta\right) f\left(z\right)} \prec H\left(z,t\right) \qquad \left(z \in \mathcal{D}\right).$$

This class was introduced and studied by Çağlar et al. [5].

ii) A function $f \in \mathcal{A}$ is said to be in the class $F_{(0,0)}\left(H, \alpha, \delta, \mu\right) = F\left(H, \alpha, \delta, \mu\right),\ 0 \leq \alpha \leq 1,\ 1 \leq \delta \leq 2,\ 0 \leq \mu \leq 1,\ t \in \left(\frac{1}{2}, 1\right]$, if the following subordination holds:

$$\alpha\left(\frac{zf'\left(z\right)}{f\left(z\right)}\right)^{\delta} + (1-\alpha)\left(\frac{zf'\left(z\right)}{f\left(z\right)}\right)^{\mu}\left(1 + \frac{zf''\left(z\right)}{f'\left(z\right)}\right)^{1-\mu} \prec H\left(z,t\right) \qquad \left(z \in \mathcal{D}\right).$$

This class was introduced and studied by Szatmari and Yalçın [19].

iii) A function $f \in \mathcal{A}$ is said to be in the class $F_{(0,0)}\left(H, 0, \delta, \mu\right) = L\left(\mu, t\right),\ \mu \geq 0,\ t \in \left(\frac{1}{2}, 1\right]$, if the following subordination holds:

$$\left(\frac{zf'\left(z\right)}{f\left(z\right)}\right)^{\mu}\left(1 + \frac{zf''\left(z\right)}{f'\left(z\right)}\right)^{1-\mu} \prec H\left(z,t\right) \qquad \left(z \in \mathcal{D}\right).$$

This class was introduced and studied by Altınkaya and Yalçın [2].

iv) A function $f \in \mathcal{A}$ is said to be in the class $F_{(0,0)}(H, 1-\eta, 1, 0) = K(\eta, t)$, $\eta \geq 0$, $t \in \left(\frac{1}{2}, 1\right]$, if the following subordination holds:

$$(1-\eta)\frac{zf'(z)}{f(z)} + \eta\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec H(z,t) \qquad (z \in \mathcal{D}).$$

This class was introduced and studied by Altınkaya and Yalçın [1].

v) A function $f \in \mathcal{A}$ is said to be in the class $F_{(0,\lambda)}(H, 1-\eta, 1, 0) = G_\lambda^\eta(t)$, $\eta \geq 0$, $0 \leq \lambda \leq 1$, $t \in \left(\frac{1}{2}, 1\right]$, if the following subordination holds:

$$(1-\eta)\left(\frac{zG'(z)}{G(z)}\right) + \eta\left(1 + \frac{zG''(z)}{G'(z)}\right) \prec H(z,t) \qquad (z \in \mathcal{D}).$$

where $G(z) = \lambda z f'(z) + (1-\lambda) f(z)$. This class was introduced and studied by Bulut et al. [3].

vi) A function $f \in \mathcal{A}$ is said to be in the class $F_{(0,0)}(H, 0, \delta, 0) = H(t)$, $t \in \left(\frac{1}{2}, 1\right]$, if the following subordination holds:

$$1 + \frac{zf''(z)}{f'(z)} \prec H(z,t).$$

This class was introduced and studied by Dziok et al. [8].

Many studies have been conducted on different classes defined by many mathematicians on different dates and various results have been obtained ([4], [6], [10], [15]-[18]).

In this paper, we obtain initial coefficients $|a_2|$ and $|a_3|$ for subclass $F_{(\beta,\lambda)}(H, \alpha, \delta, \mu)$ by means of Chebyshev polynomials expansions of analytic functions in $\mathcal{D}$. Also, we solve Fekete–Szegö problem for functions in this subclass.

2. Coefficient bounds for the function class $F_{(\beta,\lambda)}(H, \alpha, \delta, \mu)$

We begin with the following result involving initial coefficient bounds for the function class $F_{(\beta,\lambda)}(H, \alpha, \delta, \mu)$.

Theorem 2.1 *Let the function $f(z)$ given by (1.1) be in the class $F_{(\beta,\lambda)}(H, \alpha, \delta, \mu)$. Then,*

$$|a_2| \leq \frac{2t}{[\alpha\delta + (1-\alpha)(2-\mu)](2\lambda\beta + \lambda - \beta + 1)}$$

and

$$\begin{aligned} |a_3| \quad \leq \quad & \frac{1}{2[\alpha\delta + (1-\alpha)(3-2\mu)][2(3\lambda\beta + \lambda - \beta) + 1]} \\ & \times \left\{ \frac{2[\alpha\delta + (1-\alpha)(2-\mu)]^2 - \alpha\delta(\delta-3) - (1-\alpha)\left(\mu^2 + 5\mu - 8\right)}{2[\alpha\delta + (1-\alpha)(2-\mu)]^2} 4t^2 - 1 \right\}. \end{aligned}$$

Proof Let the function $f(z)$ given by (1.1) be in the class $F_{(\beta,\lambda)}(H,\alpha,\delta,\mu)$. From (1.7), we have

$$\begin{aligned} &\alpha\left[\frac{zG'(z)}{G(z)}\right]^{\delta}+(1-\alpha)\left[\frac{zG'(z)}{G(z)}\right]^{\mu}\left[1+\frac{zG''(z)}{G'(z)}\right]^{1-\mu} \\ = &\ 1+U_1(t)\,p(z)+U_2(t)\,p^2(z)+\cdots \end{aligned} \tag{2.1}$$

for some analytic functions

$$p(z)=c_1z+c_2z^2+c_3z^3+\cdots \qquad (z\in\mathcal{D}), \tag{2.2}$$

such that $p(0)=0,\ |p(z)|<1\ (z\in\mathcal{D})$. For such functions, it is well known that (see [11])

$$|c_j|\le 1 \quad (j\in\mathbb{N}) \tag{2.3}$$

and for all $\nu\in\mathbb{C}$

$$\left|c_2-\nu c_1^2\right|\le \max\{1,|\nu|\}. \tag{2.4}$$

Therefore from (2.1) and (2.2) we have

$$\begin{aligned} &\alpha\left[\frac{zG'(z)}{G(z)}\right]^{\delta}+(1-\alpha)\left[\frac{zG'(z)}{G(z)}\right]^{\mu}\left[1+\frac{zG''(z)}{G'(z)}\right]^{1-\mu} \\ = &\ 1+U_1(t)\,c_1z+\left[U_1(t)\,c_2+U_2(t)\,c_1^2\right]z^2+\cdots \end{aligned} \tag{2.5}$$

where $G(z)=\lambda\beta z^2f''(z)+(\lambda-\beta)\,zf'(z)+(1-\lambda+\beta)\,f(z)$. It follows from (2.5) that

$$[\alpha\delta+(1-\alpha)(2-\mu)](2\lambda\beta+\lambda-\beta+1)\,a_2=U_1(t)\,c_1 \tag{2.6}$$

and

$$\begin{aligned} &\frac{1}{2}\left[\alpha\delta(\delta-3)+(1-\alpha)\left(\mu^2+5\mu-8\right)\right](2\lambda\beta+\lambda-\beta+1)^2a_2^2 \\ &+2\left[\alpha\delta+(1-\alpha)(3-2\mu)\right]\left[2(3\lambda\beta+\lambda-\beta)+1\right]a_3 \\ = &\ U_1(t)\,c_2+U_2(t)\,c_1^2. \end{aligned} \tag{2.7}$$

From (1.6), (2.3) and (2.6), we have

$$|a_2|\le\frac{2t}{\{\alpha\delta+(1-\alpha)(2-\mu)\}(2\lambda\beta+\lambda-\beta+1)}. \tag{2.8}$$

By using (2.6), we can rewrite the equaliy (2.7) as follows

$$\begin{aligned} &2\left[\alpha\delta+(1-\alpha)(3-2\mu)\right]\left[2(3\lambda\beta+\lambda-\beta)+1\right]a_3 \\ = &\ U_1(t)\,c_2+U_2(t)\,c_1^2-\frac{1}{2}\left[\alpha\delta(\delta-3)+(1-\alpha)\left(\mu^2+5\mu-8\right)\right] \\ &\times(2\lambda\beta+\lambda-\beta+1)^2\left[\frac{U_1(t)\,c_1}{\{\alpha\delta+(1-\alpha)(2-\mu)\}(2\lambda\beta+\lambda-\beta+1)}\right]^2. \end{aligned}$$

If we consider (1.6) and (2.3) in last equality, we obtain

$$\begin{aligned} & 2\left[\alpha\delta+(1-\alpha)(3-2\mu)\right]\left[2\left(3\lambda\beta+\lambda-\beta\right)+1\right]a_3 \\ = \ & 2tc_2+\left\{\frac{2\left[\alpha\delta+(1-\alpha)(2-\mu)\right]^2-\alpha\delta(\delta-3)-(1-\alpha)\left(\mu^2+5\mu-8\right)}{2\left[\alpha\delta+(1-\alpha)(2-\mu)\right]^2}4t^2-1\right\}c_1^2 \\ = \ & 2t\left\{c_2-\frac{1}{2t}\left(1-\frac{2\left[\alpha\delta+(1-\alpha)(2-\mu)\right]^2-\alpha\delta(\delta-3)-(1-\alpha)\left(\mu^2+5\mu-8\right)}{2\left[\alpha\delta+(1-\alpha)(2-\mu)\right]^2}4t^2\right)c_1^2\right\}. \end{aligned}$$

Thus, from (2.4), we have

$$\begin{aligned} |a_3| \leq \ & \frac{2t}{2\left[\alpha\delta+(1-\alpha)(3-2\mu)\right]\left[2\left(3\lambda\beta+\lambda-\beta\right)+1\right]} \\ & \times\max\left\{1,\frac{1}{2t}\left|\frac{2\left[\alpha\delta+(1-\alpha)(2-\mu)\right]^2-\alpha\delta(\delta-3)-(1-\alpha)\left(\mu^2+5\mu-8\right)}{2\left[\alpha\delta+(1-\alpha)(2-\mu)\right]^2}4t^2-1\right|\right\}. \end{aligned}$$

By using Mathematica (version 8.0), we find that

$$\frac{2\left[\alpha\delta+(1-\alpha)(2-\mu)\right]^2-\alpha\delta(\delta-3)-(1-\alpha)\left(\mu^2+5\mu-8\right)}{2\left[\alpha\delta+(1-\alpha)(2-\mu)\right]^2}\geq 1$$

for $0\leq\alpha\leq 1,\ 1\leq\delta\leq 2$ and $0\leq\mu\leq 1$.

Consequently, we obtain

$$\begin{aligned} |a_3| \leq \ & \frac{1}{2\left[\alpha\delta+(1-\alpha)(3-2\mu)\right]\left[2\left(3\lambda\beta+\lambda-\beta\right)+1\right]} \\ & \times\left\{\frac{2\left[\alpha\delta+(1-\alpha)(2-\mu)\right]^2-\alpha\delta(\delta-3)-(1-\alpha)\left(\mu^2+5\mu-8\right)}{2\left[\alpha\delta+(1-\alpha)(2-\mu)\right]^2}4t^2-1\right\}. \end{aligned}$$

The proof of Theorem 2.1 is completed. □

Taking $\lambda=0,\ \beta=0$ in Theorem 2.1, we obtain the following Corollary 2.2.

Corollary 2.2 *Let the function $f(z)$ given by (1.1) be in the class $F(H,\alpha,\delta,\mu)$. Then,*

$$|a_2|\leq\frac{2t}{\alpha\delta+(1-\alpha)(2-\mu)}$$

and

$$\begin{aligned} |a_3| \leq \ & \frac{1}{2\left[\alpha\delta+(1-\alpha)(3-2\mu)\right]} \\ & \times\left\{\frac{2\left[\alpha\delta+(1-\alpha)(2-\mu)\right]^2-\alpha\delta(\delta-3)-(1-\alpha)\left(\mu^2+5\mu-8\right)}{2\left[\alpha\delta+(1-\alpha)(2-\mu)\right]^2}4t^2-1\right\}. \end{aligned}$$

Remark 2.3 *The estimate of $|a_3|$ which obtained in Corollary 2.2 is better than the corresponding estimate in Szatmari and Altınkaya [19].*

Taking $\alpha = 0,\ \lambda = 0$ and $\beta = 0$ in Theorem 2.1, we obtain the following Corollary 2.4.

Corollary 2.4 *Let the function $f(z)$ given by (1.1) be in the class $L(\mu, t)$. Then,*

$$|a_2| \leq \frac{2t}{2-\mu}$$

and

$$|a_3| \leq \frac{\left(16 - 13\mu + \mu^2\right) t^2}{(3-2\mu)(2-\mu)^2} - \frac{1}{2(3-2\mu)}.$$

Remark 2.5 *The estimate of $|a_3|$ which obtained in Corollary 2.4 is better than the corresponding estimate in Altınkaya and Yalçın [2].*

Taking $\alpha = 1-\eta,\ \delta = 1,\ \mu = 0,\ \beta = 0$ in Theorem 2.1, we obtain the following Corollary 2.6.

Corollary 2.6 *Let the function $f(z)$ given by (1.1) be in the class $G_\lambda^\eta(t)$. Then,*

$$|a_2| \leq \frac{2t}{(1+\eta)(1+\lambda)}$$

and

$$|a_3| \leq \frac{1}{2(1+2\eta)(1+2\lambda)} \left[\frac{4t^2(\eta^2 + 5\eta + 2)}{(1+\eta)^2} - 1 \right].$$

Remark 2.7 *The estimate of $|a_3|$ which obtained in Corollary 2.6 is better than the corresponding estimate in Bulut et al. [3].*

Taking $\alpha = 1-\eta,\ \delta = 1,\ \mu = 0,\ \lambda = 0,\ \beta = 0$ in Theorem 2.1, we obtain the following Corollary 2.8.

Corollary 2.8 *Let the function $f(z)$ given by (1.1) be in the class $K(\eta, t)$. Then,*

$$|a_2| \leq \frac{2t}{1+\eta}$$

and

$$|a_3| \leq \frac{1}{2(1+2\eta)} \left[\frac{4t^2(\eta^2 + 5\eta + 2)}{(1+\eta)^2} - 1 \right].$$

Remark 2.9 *The estimate of $|a_3|$ which obtained in Corollary 2.8 is better than the corresponding estimate in Altınkaya and Yalçın [1].*

Taking $\alpha = 1,\ \delta = 1$ in Theorem 2.1, we obtain result of Çağlar et al. [5] the following Corollary 2.10.

Corollary 2.10 *Let the function $f(z)$ given by (1.1) be in the class $N(\lambda, \beta, t)$. Then*

$$|a_2| \leq \frac{2t}{2\lambda\beta + \lambda - \beta + 1}$$

and

$$|a_3| \leq \frac{8t^2-1}{2\left[6\lambda\beta+2\lambda-2\beta+1\right]}.$$

Taking $\mu=0,\ \alpha=0,\ \lambda=0,\ \beta=0$ in Theorem 2.1, we obtain result of Dziok et al. [8] the following Corollary 2.11.

Corollary 2.11 *Let the function $f(z)$ given by (1.1) be in the class $H(t)$. Then,*

$$|a_2| \leq t$$

and

$$|a_3| \leq \frac{4t^2}{3}-\frac{1}{6}.$$

Taking $\alpha=0,\ \delta=1,\ \mu=1,\ \lambda=1,\ \beta=1$ in Theorem 2.1, we obtain the following Corollary 2.12.

Corollary 2.12 *Let the function $f(z)$ given by (1.1) be in the class $F_{(1,1)}(H,0,1,1)$. Then,*

$$|a_2| \leq \frac{2t}{3}$$

and

$$|a_3| \leq \frac{4t^2}{7}-\frac{1}{14}.$$

3. Fekete–Szegö inequality for the function class $F_{(\beta,\lambda)}(H,\alpha,\delta,\mu)$

Now, we are ready to find the sharp bounds of Fekete–Szegö functional $\left|a_3-\xi a_2^2\right|$ defined for $f\in F_{(\beta,\lambda)}(H,\alpha,\delta,\mu)$ given by (1.1).

Theorem 3.1 *Let the function $f(z)$ given by (1.1) be in the class $F_{(\beta,\lambda)}(H,\alpha,\delta,\mu)$. Then for some $\xi\in\mathbb{R}$,*

$$\left|a_3-\xi a_2^2\right| \leq \begin{cases} \frac{2t}{K} & for\quad \xi\in[\xi_1,\xi_2] \\ \frac{2t}{K}\left|\frac{4t^2-1}{2t}-\frac{Rt}{B}-\xi\frac{2tK}{B(2\lambda\beta+\lambda-\beta+1)^2}\right| & for\quad \xi\notin[\xi_1,\xi_2] \end{cases}, \tag{3.1}$$

where $\xi_1=\left\{\frac{2(2B-R)t^2-(1+2t)B}{4t^2K}\right\}(2\lambda\beta+\lambda-\beta+1)^2$, $\xi_2=\left\{\frac{2(2B-R)t^2-(1-2t)B}{4t^2K}\right\}(2\lambda\beta+\lambda-\beta+1)^2$ *such that*

$$\begin{aligned} B &= \left[\alpha\delta+(1-\alpha)(2-\mu)\right]^2, \\ K &= 2\left[\alpha\delta+(1-\alpha)(3-2\mu)\right]\left[2(3\lambda\beta+\lambda-\beta)+1\right], \\ R &= \alpha\delta(\delta-3)+(1-\alpha)\left(\mu^2+5\mu-8\right). \end{aligned}$$

Proof Let $f\in F_{(\beta,\lambda)}(H,\alpha,\delta,\mu)$ and

$$\begin{aligned} B &= \left[\alpha\delta+(1-\alpha)(2-\mu)\right]^2, \\ K &= 2\left[\alpha\delta+(1-\alpha)(3-2\mu)\right]\left[2(3\lambda\beta+\lambda-\beta)+1\right], \\ R &= \alpha\delta(\delta-3)+(1-\alpha)\left(\mu^2+5\mu-8\right). \end{aligned}$$

From (2.6) and (2.7) for some $\xi \in \mathbb{R}$, we can easily see that

$$\left|a_3 - \xi a_2^2\right| = \frac{U_1(t)}{K}\left|c_2 + \left\{\frac{U_2(t)}{U_1(t)} - \frac{R}{2B}U_1(t) - \xi\frac{U_1(t)K}{B(2\lambda\beta+\lambda-\beta+1)^2}\right\}c_1^2\right|.$$

Then, in view of (2.4), we conclude that

$$\left|a_3 - \xi a_2^2\right| \le \frac{U_1(t)}{K}\max\left\{1, \left|\frac{U_2(t)}{U_1(t)} - \frac{R}{2B}U_1(t) - \xi\frac{U_1(t)K}{B(2\lambda\beta+\lambda-\beta+1)^2}\right|\right\}. \tag{3.2}$$

Finally, by using (1.6) in (3.2), we get

$$\left|a_3 - \xi a_2^2\right| \le \frac{2t}{K}\max\left\{1, \left|\frac{4t^2-1}{2t} - \frac{Rt}{B} - \xi\frac{2tK}{B(2\lambda\beta+\lambda-\beta+1)^2}\right|\right\}.$$

Because $t > 0$, we have

$$\begin{aligned}
&\left|\tfrac{4t^2-1}{2t} - \tfrac{Rt}{B} - \xi\tfrac{2tK}{B(2\lambda\beta+\lambda-\beta+1)^2}\right| \le 1\\
&\Leftrightarrow \left\{\tfrac{2(2B-R)t^2-(1+2t)B}{4t^2K}\right\}(2\lambda\beta+\lambda-\beta+1)^2 \le \xi \le \left\{\tfrac{2(2B-R)t^2-(1-2t)B}{4t^2K}\right\}(2\lambda\beta+\lambda-\beta+1)^2\\
&\Leftrightarrow \xi_1 \le \xi \le \xi_2.
\end{aligned}$$

Thus, the proof of theorem is completed. □

Taking $\alpha = 1,\ \delta = 1$ in Theorem 3.1, we obtain result of Çağlar et al. [5] the following Corollary 3.2.

Corollary 3.2 *Let the function $f(z)$ given by (1.1) be in the class $N(\lambda,\beta,t)$. Then for some $\xi \in \mathbb{R}$,*

$$\left|a_3 - \xi a_2^2\right| \le \begin{cases} \dfrac{t}{6\lambda\beta+2\lambda-2\beta+1} & \text{for} \quad \xi \in [\xi_1,\xi_2] \\ \dfrac{t}{6\lambda\beta+2\lambda-2\beta+1}\left|\dfrac{8t^2-1}{2t} - \xi\dfrac{4t(6\lambda\beta+2\lambda-2\beta+1)}{(2\lambda\beta+\lambda-\beta+1)^2}\right| & \text{for} \quad \xi \notin [\xi_1,\xi_2] \end{cases},$$

where

$$\xi_1 = \frac{\left(8t^2-2t-1\right)(2\lambda\beta+\lambda-\beta+1)^2}{8t^2(6\lambda\beta+2\lambda-2\beta+1)}$$

and

$$\xi_2 = \frac{\left(8t^2+2t-1\right)(2\lambda\beta+\lambda-\beta+1)^2}{8t^2(6\lambda\beta+2\lambda-2\beta+1)}.$$

Taking $\lambda = 0,\ \beta = 0$ in (3.2), we obtain result of Szatmari and Altınkaya [19] the following Corollary 3.3.

Corollary 3.3 *Let the function $f(z)$ given by (1.1) be in the class $F(H,\alpha,\delta,\mu)$. Then for some $\xi \in \mathbb{C}$,*

$$\begin{aligned}
&\left|a_3 - \xi a_2^2\right| \le \tfrac{t}{\alpha\delta+(1-\alpha)(3-2\mu)}\\
&\times\max\left\{1, \left|2t\left(\tfrac{2\xi[\alpha\delta+(1-\alpha)(3-2\mu)]}{[\alpha\delta+(1-\alpha)(2-\mu)]^2} - \tfrac{3\alpha\delta+(1-\alpha)(8-5\mu)-\alpha\left(\delta^2-\mu^2\right)-\mu^2}{2[\alpha\delta+(1-\alpha)(2-\mu)]^2}\right) - \tfrac{4t^2-1}{2t}\right|\right\}
\end{aligned}$$

Taking $\alpha=1-\eta,\ \delta=1,\ \mu=0,\ \beta=0$ in Theorem 3.1, we obtain result of Bulut et al. [3] the following Corollary 3.4.

Corollary 3.4 *Let the function $f(z)$ given by (1.1) be in the class $G_{\lambda}^{\eta}(t)$. Then for some $\xi\in\mathbb{R}$,*

$$\left|a_3-\xi a_2^2\right| \leq \begin{cases} \frac{t}{(1+2\eta)(1+2\lambda)} & \text{for} \quad \xi\in[\xi_1,\xi_2] \\ \frac{t}{(1+2\eta)(1+2\lambda)}\left|\frac{4t^2-1}{2t}+\frac{2(1+3\eta)t}{(1+\eta)^2}-\xi\frac{4t(1+2\eta)(1+2\lambda)}{(1+\eta)^2(1+\lambda)^2}\right| & \text{for} \quad \xi\notin[\xi_1,\xi_2] \end{cases},$$

where

$$\xi_1=\left\{\frac{4\left(\eta^2+5\eta+2\right)t^2-(1+2t)\left(1+\eta\right)^2}{8\left(1+2\eta\right)\left(1+2\lambda\right)t^2}\right\}\left(1+\lambda\right)^2$$

and

$$\xi_2=\left\{\frac{4\left(\eta^2+5\eta+2\right)t^2-(1-2t)\left(1+\eta\right)^2}{8\left(1+2\eta\right)\left(1+2\lambda\right)t^2}\right\}\left(1+\lambda\right)^2.$$

Taking $\alpha=1-\eta,\ \delta=1,\ \mu=0,\ \lambda=0,\ \beta=0$ in Theorem 3.1, we obtain result of Altınkaya and Yalçın [1] the following Corollary 3.5.

Corollary 3.5 *Let the function $f(z)$ given by (1.1) be in the class $K(\eta,t)$. Then for some $\xi\in\mathbb{R}$,*

$$\left|a_3-\xi a_2^2\right| \leq \begin{cases} \frac{t}{1+2\eta} & \text{for} \quad \xi\in[\xi_1,\xi_2] \\ \frac{t}{1+2\eta}\left|\frac{4t^2-1}{2t}+\frac{2(1+3\eta)t}{(1+\eta)^2}-\xi\frac{4t(1+2\eta)}{(1+\eta)^2}\right| & \text{for} \quad \xi\notin[\xi_1,\xi_2] \end{cases},$$

where

$$\xi_1=\frac{4\left(\eta^2+5\eta+2\right)t^2-(1+2t)\left(1+\eta\right)^2}{8\left(1+2\eta\right)t^2}$$

and

$$\xi_2=\frac{4\left(\eta^2+5\eta+2\right)t^2-(1-2t)\left(1+\eta\right)^2}{8\left(1+2\eta\right)t^2}$$

Taking $\alpha=0,\ \lambda=0$ and $\beta=0$ in Theorem 3.1, we obtain result of Altınkaya and Yalçın [2] the following Corollary 3.6.

Corollary 3.6 *Let the function $f(z)$ given by (1.1) be in the class $L\left(\mu,t\right)$. Then, for some $\xi\in\mathbb{R}$,*

$$\left|a_3-\xi a_2^2\right| \leq \begin{cases} \frac{t}{3-2\mu} & \text{for} \quad \xi\in[\xi_1,\xi_2] \\ \frac{t}{3-2\mu}\left|\frac{4t^2-1}{2t}-\frac{\left(\mu^2+5\mu-8\right)t}{(2-\mu)^2}-\xi\frac{4t(3-2\mu)}{(2-\mu)^2}\right| & \text{for} \quad \xi\notin[\xi_1,\xi_2] \end{cases},$$

where

$$\xi_1=\frac{2\left(\mu^2-13\mu+16\right)t^2-\left(2-\mu\right)^2(1+2t)}{8\left(3-2\mu\right)t^2}$$

and

$$\xi_2 = \frac{2\left(\mu^2 - 13\mu + 16\right)t^2 - (2-\mu)^2(1-2t)}{8(3-2\mu)t^2}.$$

Taking $\alpha = 0$ in Theorem 3.1, we obtain following Corollary 3.7.

Corollary 3.7 *Let the function $f(z)$ given by (1.1) be in the class $F_{(\beta,\lambda)}(H, 0, \delta, \mu)$. Then for some $\xi \in \mathbb{R}$,*

$$\left|a_3 - \xi a_2^2\right| \leq \left\{ \begin{array}{ll} \frac{t}{(3-2\mu)(6\lambda\beta+2\lambda-2\beta+1)} & for \quad \xi \in [\xi_1, \xi_2] \\ \frac{t}{(3-2\mu)(6\lambda\beta+2\lambda-2\beta+1)}\left|\frac{4t^2-1}{2t} - \frac{\left(\mu^2+5\mu-8\right)t}{(2-\mu)^2} - \xi\frac{4t(3-2\mu)(6\lambda\beta+2\lambda-2\beta+1)}{(2-\mu)^2(2\lambda\beta+\lambda-\beta+1)^2}\right| & for \quad \xi \notin [\xi_1, \xi_2] \end{array} \right. ,$$

where

$$\xi_1 = \left\{\frac{2\left(\mu^2 - 13\mu + 16\right)t^2 - (1+2t)(2-\mu)^2}{8(3-2\mu)(6\lambda\beta + 2\lambda - 2\beta + 1)t^2}\right\}(2\lambda\beta + \lambda - \beta + 1)^2$$

and

$$\xi_2 = \left\{\frac{2\left(\mu^2 - 13\mu + 16\right)t^2 - (1-2t)(2-\mu)^2}{8(3-2\mu)(6\lambda\beta + 2\lambda - 2\beta + 1)t^2}\right\}(2\lambda\beta + \lambda - \beta + 1)^2.$$

Taking $\alpha = 0,\ \lambda = 0,\ \beta = 0$ in Theorem 3.1, we obtain the following Corollary 3.8.

Corollary 3.8 *Let the function $f(z)$ given by (1.1) be in the class $F(H, 0, \delta, \mu)$. Then, for some $\xi \in \mathbb{R}$,*

$$\left|a_3 - \xi a_2^2\right| \leq \left\{ \begin{array}{ll} \frac{t}{3-2\mu} & for \quad \xi \in [\xi_1, \xi_2] \\ \frac{t}{(3-2\mu)}\left|\frac{4t^2-1}{2t} - \frac{\left(\mu^2+5\mu-8\right)t}{(2-\mu)^2} - \xi\frac{4(3-2\mu)}{(2-\mu)^2}\right| & for \quad \xi \notin [\xi_1, \xi_2] \end{array} \right. ,$$

where

$$\xi_1 = \left\{\frac{2\left(\mu^2 - 13\mu + 16\right)t^2 - (1+2t)(2-\mu)^2}{8(3-2\mu)t^2}\right\}$$

and

$$\xi_2 = \left\{\frac{2\left(\mu^2 - 13\mu + 16\right)t^2 - (1-2t)(2-\mu)^2}{8(3-2\mu)t^2}\right\}.$$

Taking $\mu = 0,\ \alpha = 0,\ \lambda = 0,\ \beta = 0$ in Theorem 3.1, we obtain result of Dziok et al. [8] the following Corollary 3.9.

Corollary 3.9 *Let the function $f(z)$ given by (1.1) be in the class $H(t)$. Then for some $\xi \in \mathbb{R}$,*

$$\left|a_3 - \xi a_2^2\right| \leq \left\{ \begin{array}{ll} \frac{t}{3} & for \quad \xi \in \left[\frac{8t^2-2t-1}{6t^2}, \frac{8t^2+2t-1}{6t^2}\right] \\ \left|\frac{8t^2-1}{6} - \xi t^2\right| & for \quad \xi \notin \left[\frac{8t^2-2t-1}{6t^2}, \frac{8t^2+2t-1}{6t^2}\right] \end{array} \right. .$$

Acknowledgment

The research was supported by the Commission for the Scientific Research Projects of Kyrgyz-Turkish Manas University, project number KTMU-BAP-2020.FB.04.

The authors are thankful to the referees for helpful suggestions for the improvement of this article.

References

[1] Altınkaya S, Yalçın S. On the Chebyshev polynomial bounds for classes of univalent functions. Khayyam Journal of Mathematics 2016; 2 (1): 1-5.

[2] Altınkaya S, Yalçın S. On the Chebyshev coefficients for a general subclass of univalent functions. Turkish Journal of Mathematics 2018; 42: 2885-2890.

[3] Bulut S, Magesh N, Balaji VK. Certain subclasses of analytic functions associated with the Chebyshev polynomials. Honam Mathematical Journal 2018; 40 (4): 611-619.

[4] Çağlar M, Deniz E, Kazımoğlu S. Fekete-Szegö problem for a subclass of analytic functions defined by Chebyshev polynomials. 3rd International Conference on Mathematical and Related Sciences: Current Trend and Developments; Turkey; 2020. pp. 114-120.

[5] Çağlar M, Orhan H, Kamali M. Fekete-Szegö problem for a subclass of analytic functions associated with Chebyshev polynomials. Boletim da Sociedade Paranaense de Matematica (BSPM), Accepted for publication, 2020.

[6] Deniz E, Orhan H. The Fekete-Szegö problem for a generalized subclass of analytic functions. Kyungpook Mathematical Journal 2010; 50 (1): 37-47.

[7] Doha EH. The first and second kind Chebyshev coefficients of the moments of the general-order derivative of an infinitely differentiable function. International Journal of Computer Mathematics 1994; 51: 21-35.

[8] Dziok J, Raina RK, Sokol J. Application of Chebyshev polynomials to classes of analytic functions. Comptes Rendus de l'Académie des Sciences Paris 2015; 353 (5): 433-438.

[9] Fekete M, Szegö G. Eine Bemerkung Über ungerade schlichte funktionen. Journal of London Mathematical Society 1933; 8: 85-89.

[10] Kazımoğlu S, Deniz E. Fekete-Szegö problem for generalized bi-subordinate functions of complex order. Hacettepe Journal of Mathematics and Statistics 2020; 49 (5): 1695-1705.

[11] Keogh FR, Merkes EP. A coefficient inequality for certain classes of analytic functions. Proceeding of American Mathematical Society 1969; 20 (1): 8-12.

[12] Lewandowski Z, Müller SS, Zlotkiewicz E. Generating functions for some classes of univalent functions. Proceeding of American Mathematical Society 1976; 56: 111-117.

[13] Mason JC. Chebyshev polynomial approximations for the L-membrane eigenvalue problem. SIAM Journal of Applied Mathematics 1967; 15: 172-186.

[14] Mocanu PT. Une propriete de convexite dans la theorie de representation conforme. Studia Universitatis Babeş-Bolyai Mathematica 1969; 34 (11): 127-134.

[15] Orhan H, Deniz E, Çağlar M. Fekete-Szegö problem for certain subclasses of analytic functions. Demonstratio Mathematica 2012; 45 (4): 835-846.

[16] Orhan H, Deniz E, Raducanu D. The Fekete-Szegö problem for subclasses of analytic functions defined by a differential operator related to conic domains. Computers and Mathematics With Applications 2010; 59 (1): 283-295.

[17] Orhan H, Yağmur N, Çağlar M. Fekete-Szegö problem for a generalized subclass of analytic functions. Kyungpook Mathematical Journal 2013; 53 (1): 13-23.

[18] Ramachandran C, Dhanalaksmi K. Fekete-Szegö inequality for the subclasses of analytic functions bounded by Chebyshev polynomial. Global Journal of Pure and Applied Mathematics 2017; 13 (9): 4953-4958.

[19] Szatmari E, Altınkaya Ş. Coefficient estimates and Fekete-Szegö inequality for a class of analytic functions satisfying subordinate condition associated with Chebyshev polynomials. Acta Universitatis Sapientiae Mathematica 2019; 11 (2), 430-436.

[20] Whittaker ET, Watson GN. A Course on Modern Analysis: An introduction to the general theory of infinite process of analytic functions with an account of the principal transcendental functions. Fourth Edition, Cambridge University Press, 1963.